COMPOSITIO MATHEMATICA

On semi-infinite cohomology of finite-dimensional graded algebras

Roman Bezrukavnikov and Leonid Positselski

Compositio Math. **146** (2010), 480–496.

doi:10.1112/S0010437X09004382

FOUNDATION COMPOSITIO MATHEMATICA

The London Mathematical Society

https://doi.org/10.1112/S0010437X09004382 Published online by Cambridge University Press

Compositio Math. 146 (2010) 480–496
doi:10.1112/S0010437X09004382

On semi-infinite cohomology of finite-dimensional graded algebras

Roman Bezrukavnikov and Leonid Positselski

Abstract

We describe a general setting for the definition of semi-infinite cohomology of finite-dimensional graded algebras, and provide an interpretation of such cohomology in terms of derived categories. We apply this interpretation to compute semi-infinite cohomology of some modules over the small quantum group at a root of unity, generalizing an earlier result of Arkhipov (posed as a conjecture by B. Feigin).

1. Introduction

Semi-infinite cohomology of associative algebras was studied by Arkhipov in [Ark97a, Ark97b, Ark98a]; see also [Sev01]. (These works are partly based on an earlier paper by Voronov [Vor93], where the corresponding constructions were introduced in the context of Lie algebras.)

Recall that the definition of semi-infinite cohomology (see, e.g., [Ark97a, Definition 3.3.6]) applies in the following set-up. Suppose we are given an associative graded algebra A and two subalgebras $N, B \subset A$, such that $A = N \otimes B$ as a vector space, and satisfy some additional assumptions are satisfied. In this situation, the space of semi-infinite extensions, $\mathrm{Ext}^{\infty/2+\bullet}(X, Y)$, is defined for X and Y in the appropriate derived categories. The definition makes use of explicit complexes (a version of the bar resolution). The aim of this article is to show that, at least under certain simplifying assumptions, $\mathrm{Ext}^{\infty/2+\bullet}(X, Y)$ is a particular case of a general categorical construction.

To describe the situation in more detail, recall that starting from an algebra $A = N \otimes B$ as above, one can define another algebra $A^{\#}$, which also contains subalgebras that can be identified with N and B so that $A^{\#} = B \otimes N$. The semi-infinite Ext functor, $\mathrm{Ext}^{\infty/2+\bullet}(X, Y)$, is then defined for $X \in D(A^{\#}\text{-mod})$ and $Y \in D(A\text{-mod})$, where $D(A^{\#}\text{-mod})$ and $D(A\text{-mod})$ are derived categories of modules with certain restrictions on the grading.

Our categorical interpretation relies on the following construction. Given small categories $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{B}$ with functors $\Phi : \mathcal{B} \to \mathcal{A}$ and $\Phi' : \mathcal{B} \to \mathcal{A}'$, one can define for $X \in \mathcal{A}$ and $Y \in \mathcal{A}'$ the set of *morphisms from X to Y through $\mathcal{B}$*; we denote this set by $\mathrm{Hom}_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X, Y)$. We then show that if $\mathcal{A} = D^b(A^{\#}\text{-mod})$, $\mathcal{A}' = D^b(A\text{-mod})$, and $\mathcal{B}$ is the full triangulated subcategory in $\mathcal{A}$ generated by N-injective $A^{\#}$-modules, then $\mathcal{B}$ is identified with the full subcategory in $\mathcal{A}'$ generated by

Received 2 April 2008, accepted in final form 16 December 2008, published online 23 February 2010.
2000 Mathematics Subject Classification 16E30 (primary), 17B37, 81R50 (secondary).
Keywords: semi-infinite cohomology, small quantum groups.
The first author was partially supported by NSF grant DMS-0625234 and DARPA grant HR0011-04-1-0031; he worked on this paper while visiting the Institute for Advanced Study in Princeton, with the stay funded through grants from Bell Companies, the Oswald Veblen Fund, the James D. Wolfenson Fund and the Ambrose Monell Foundation. The second author gratefully acknowledges financial support from CRDF, INTAS and P. Deligne's 2004 Balzan prize.
This journal is © Foundation Compositio Mathematica 2010.

https://doi.org/10.1112/S0010437X09004382 Published online by Cambridge University Press

N-projective A-modules and, under certain assumptions, one has

$$\mathrm{Ext}^{\infty/2+i}(X, Y) = \mathrm{Hom}_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X, Y[i]). \tag{1}$$

Notice that description (1) of $\mathrm{Ext}^{\infty/2+i}(X, Y)$ is 'internal' in the derived category, i.e. it refers only to the derived categories and their full subcategories rather than to a particular category of complexes.

An example of the situation considered in this paper is provided by a small quantum group at a root of unity [Lus90] or by the restricted enveloping algebra of a simple Lie algebra in positive characteristic. The computation of semi-infinite cohomology in the former case is due to Arkhipov [Ark97a, Ark98b] (with the answer posed as a conjecture by B. Feigin). An attempt to find a natural interpretation of this answer was the starting point for the present work. In § 6 we sketch a generalization of Arkhipov's theorem based on our description of semi-infinite cohomology and the results of [ABG04, BL07]. Similarly, the main result of [Bez06] yields a description of semi-infinite cohomology of tilting modules over the 'big' quantum group restricted to the small quantum group as cohomology with support of coherent IC sheaves on the nilpotent cone [AB10, Bez00a].

It should be noted that some definitions of semi-infinite cohomology found in the literature apply in a more general (or different) situation than the one considered in the present paper. An important example is supplied by affine Lie algebras; in fact, semi-infinite cohomology was first defined in that context, in relation to the physical notion of BRST reduction. We hope that our approach can be extended to a more general setting such as this. Some of the ingredients needed for the generalization are provided by [Pos07].

The paper is organized as follows. Section 2 is devoted to basic general facts about 'Hom through a category'. Section 3 contains the definition of the algebra $A^{\#}$ and a summary of its properties. In § 4 we recall the definition of semi-infinite cohomology in the present context, and in § 5 we prove the main result linking that definition to the general categorical construction of § 2. In § 6 we discuss the example of a small quantum group.

2. Morphisms through a category

2.1 Preliminaries

Let $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{B}$ be small categories, and let $\Phi : \mathcal{B} \to \mathcal{A}$ and $\Phi' : \mathcal{B} \to \mathcal{A}'$ be functors. Fix $X \in \mathrm{Ob}(\mathcal{A})$ and $Y \in \mathrm{Ob}(\mathcal{A}')$. We define the set of *morphisms from* X *to* Y *through* $\mathcal{B}$ as π_0 of the category of diagrams

$$X \xrightarrow{a} \Phi(Z); \quad \Phi'(Z) \xrightarrow{a'} Y, \quad Z \in \mathcal{B}. \tag{2}$$

This set will be denoted by $\mathrm{Hom}_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X, Y)$. Thus elements of $\mathrm{Hom}_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X, Y)$ are diagrams of the form (2), with two diagrams (a_1, Z_1, a'_1) and (a_2, Z_2, a'_2) being identified if the objects Z_1 and Z_2 can be connected by a chain of morphisms in category $\mathcal{B}$ satisfying the natural compatibility condition. Specifically, a morphism $f : Z \to W$ in $\mathcal{B}$ identifies two diagrams (a, Z, a') and (c, W, c') whenever $c = \Phi(f) \circ a$ and $a' = c' \circ \Phi'(f)$.

If the categories and the functors are additive (respectively, R-linear for a commutative ring R), then $\mathrm{Hom}_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X, Y)$ is an abelian group (respectively, an R-module); to add two diagrams of the form (2), one sets $Z = Z_1 \oplus Z_2$ with the obvious arrows.

https://doi.org/10.1112/S0010437X09004382 Published online by Cambridge University Press

We have the composition map

$$\mathrm{Hom}_{\mathcal{A}}(X', X) \times \mathrm{Hom}_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X, Y) \times \mathrm{Hom}_{\mathcal{A}'}(Y, Y') \to \mathrm{Hom}_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X', Y');$$

in particular, in the additive setting, $\mathrm{Hom}_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X, Y)$ is an $\mathrm{End}(Y)$–$\mathrm{End}(X)$-bimodule.

2.2 Pro/Ind representable case

If the left adjoint functor Φ_L to Φ is defined on X, then

$$\mathrm{Hom}_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X, Y) = \mathrm{Hom}_{\mathcal{A}'}(\Phi'(\Phi_L(X)), Y),$$

because in this case the above category contracts to the subcategory of diagrams of the form

$$X \xrightarrow{\text{can}} \Phi(\Phi_L(X)); \quad \Phi'(\Phi_L(X)) \to Y,$$

where 'can' stands for the adjunction morphism. If the right adjoint functor Φ'_R is defined on Y, then

$$\mathrm{Hom}_{\mathcal{A}_{\mathcal{B}}\mathcal{A}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(X, \Phi(\Phi'_R(Y)))$$

for similar reasons.

More generally, we have the following theorem.

Proposition 1. *Fix $X \in \mathcal{A}$ and $Y \in \mathcal{A}'$. Assume that the functor $F_X : \mathcal{B} \to \mathrm{Sets}$, $Z \mapsto \mathrm{Hom}_{\mathcal{A}}(X, \Phi(Z))$ can be represented as a filtered inductive limit of representable functors $Z \mapsto \mathrm{Hom}_{\mathcal{B}}(\iota(S), Z)$ where $S \in \mathcal{I}$ and $\iota : \mathcal{I} \to \mathcal{B}^{\mathrm{op}}$ is a functor between small categories. Then we have*

$$\mathrm{Hom}_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X, Y) = \varinjlim_{S \in \mathcal{I}} \mathrm{Hom}_{\mathcal{A}'}(\Phi'\iota(S), Y).$$

Alternatively, assume that the functor $F_Y : \mathcal{B}^{\mathrm{op}} \to \mathrm{Sets}$, $Z \mapsto \mathrm{Hom}(\Phi'(Z), Y)$ can be represented as a filtered inductive limit of representable functors $Z \mapsto \mathrm{Hom}_{\mathcal{B}}(Z, \iota(S))$ where $S \in \mathcal{I}$ and $\iota : \mathcal{I} \to \mathcal{B}$. Then

$$\mathrm{Hom}_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X, Y) = \varinjlim_{S \in \mathcal{I}} \mathrm{Hom}_{\mathcal{A}}(X, \Phi\iota(S)).$$

Remark 1. The assumptions of the proposition can be rephrased by saying in the first case that the functor F_X is represented by the pro-object $\varprojlim \iota$, and in the second case that the functor F_Y is represented by the ind-object $\varinjlim \iota$.

Remark 2. The results of the proposition can be further generalized as follows. Fix $X \in \mathcal{A}$ and $Y \in \mathcal{A}'$; let $\iota : \mathcal{B}' \to \mathcal{B}$ be a functor between small categories. Assume either that for any morphism $X \to \Phi(Z)$ the category of pairs of morphisms $X \to \Phi\iota(S)$, $\iota(S) \to Z$ making the triangle $X \to \Phi\iota(S) \to \Phi(Z)$ commutative is non-empty and connected, or that for any morphism $\Phi'(Z) \to Y$ the category of pairs of morphisms $Z \to \iota(S)$, $\Phi'\iota(S) \to Y$ making the triangle $\Phi'(Z) \to \Phi'\iota(S) \to Y$ commutative is non-empty and connected. Then the natural map $\mathrm{Hom}_{\mathcal{A}_{\mathcal{B}'}\mathcal{A}'}(X, Y) \to \mathrm{Hom}_{\mathcal{A}_{\mathcal{B}}\mathcal{A}'}(X, Y)$ is an isomorphism.

Example 1. Let M be a Noetherian scheme, and let $\mathcal{A} = \mathcal{A}' = D^b(\mathrm{Coh}_M)$ be the bounded derived category of coherent sheaves on M; let $\Phi = \Phi' : \mathcal{B} \hookrightarrow \mathcal{A}$ be the full embedding of the subcategory of complexes whose cohomology is supported on a closed subset $i : N \hookrightarrow M$. Then the right adjoint functor $i_* \circ i^!$ is well-defined as a functor to a 'larger' derived category of quasi-coherent sheaves, while the left adjoint functor $i_* \circ i^*$ is a well-defined functor to the Grothendieck–Serre

https://doi.org/10.1112/S0010437X09004382 Published online by Cambridge University Press

dual category, the derived category of pro-coherent sheaves (introduced in Deligne's appendix to [Har66]).

Let $C^\bullet$ be a complex of coherent sheaves representing the object $X \in D^b(\mathrm{Coh}_M)$. Let X_n be the object in the derived category represented by the complex $C^i_n = C^i \otimes \mathcal{O}_M/\mathcal{J}^n_N$ (the non-derived tensor product), where $\mathcal{J}_N$ is the ideal sheaf of N. For $\mathcal{F} \in \mathcal{B}$ we have $\varinjlim \mathrm{Hom}(X_n, \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}(X, \mathcal{F})$. Thus, upon applying Proposition 1 to $\iota : \mathbb{Z}_+ \to \mathcal{B}$ given by $\iota : n \mapsto X_n$, we get

$$\mathrm{Hom}_{\mathcal{A}\mathcal{B}\mathcal{A}}(X, Y) = \varinjlim \mathrm{Hom}(X_n, Y) = \mathrm{Hom}(i_*(i^*(X)), Y) = \mathrm{Hom}(X, i_*(i^!(Y))).$$

In particular, if $X = \mathcal{O}_M$ is the structure sheaf, we get

$$\mathrm{Hom}_{\mathcal{A}\mathcal{B}\mathcal{A}}(\mathcal{O}_M, Y[i]) = H^i_N(Y), \tag{3}$$

where $H^\bullet_N(Y)$ stands for cohomology with support on N (see, e.g., [Har66]).

2.3 Triangulated full embeddings

In all of the examples below, $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{B}$ will be triangulated, and Φ and Φ' will be full embeddings of a thick subcategory. Assume that this is the case and that, moreover, $\mathcal{A} = \mathcal{A}'$ and $\Phi = \Phi'$.

Proposition 2. *We have a long exact sequence*

$$\mathrm{Hom}_{\mathcal{A}\mathcal{B}\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}\mathcal{B}\mathcal{A}}(X, Y[1]).$$

Proof. The connecting homomorphism $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}\mathcal{B}\mathcal{A}}(X, Y[1])$ is constructed as follows. Let $X \leftarrow X' \to Y$ be a fraction of morphisms in $\mathcal{A}$ representing a morphism $X \to Y$ in $\mathcal{A}/\mathcal{B}$; the cone K of the morphism $X' \to X$ belongs to $\mathcal{B}$. Assign to this fraction the diagram $X \to K; K \to Y[1]$, where the morphism $K \to Y[1]$ is defined as the composition $K \to X'[1] \to Y[1]$.

All the required verifications are straightforward; the hardest one to check is that the sequence is exact at the term $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y)$. Here one shows that for any two diagrams $X \to K'; K' \to Y[1]$ and $X \to K''; K'' \to Y[1]$ connected by a morphism $K' \to K''$ which makes the two triangles commute, and for any two fractions $X \leftarrow X' \to Y$ and $X \leftarrow X'' \to Y$ to which the connecting homomorphism assigns the respective diagrams, one can construct a morphism $X' \to X''$ that makes the triangle formed by X', X'' and X commutative; the triangle formed by X', X'' and Y will then commute up to a morphism $X \to Y$. $\square$

3. The algebra $A^\#$ and modules over it

All algebras below will be associative unital algebras over a field $\mathbf{k}$.

3.1 The set-up

We make the following assumptions. A $\mathbb{Z}$-graded finite-dimensional algebra A and graded subalgebras $K = A^0$, $B = A^{\leqslant 0}$ and $N = A^{\geqslant 0} \subset A$ are fixed and satisfy the following three conditions.

(i) $B = A^{\leqslant 0}$ and $N = A^{\geqslant 0}$ are graded by, respectively, $\mathbb{Z}^{\leqslant 0}$ and $\mathbb{Z}^{\geqslant 0}$, and $K = B \cap N$ is the component of degree zero in N. Notice that we do not assume $A^{\leqslant 0}$ and $A^{\geqslant 0}$ to be the maximal non-positively graded and maximal non-negatively graded subalgebras of A,

https://doi.org/10.1112/S0010437X09004382 Published online by Cambridge University Press

nor do we assume A^0 to be the whole zero-degree component of A. The grading in graded algebras and modules is denoted by lower indices.

(ii) $K = A^0$ is semisimple and the map $N \otimes_K B \to A$ provided by the multiplication map is an isomorphism.

(iii) Consider the K–N-bimodule $N^\vee = \mathrm{Hom}_{K^{\mathrm{op}}}(N, K)$. We require that the tensor product $S = N^\vee \otimes_N A$ be an injective right N-module.

3.2 N-modules, $N^\vee$-comodules and $N^\#$-modules

By a 'module' we shall mean a finite-dimensional graded left module, unless stated otherwise (though all the results of this section are also applicable to ungraded or infinite-dimensional modules).

Since N is a finitely generated projective right K-module, the K-bimodule $N^\vee$ has the natural structure of a coring, i.e. there exist a comultiplication map $N^\vee \to N^\vee \otimes_K N^\vee$ and a counit map $N^\vee \to K$ satisfying the usual coassociativity and counity conditions. Consequently, there is a natural algebra structure on $N^\# = \mathrm{Hom}_{K^{\mathrm{op}}}(N^\vee, K)$ and an injective morphism of algebras $K \to N^\#$. The category of right N-modules is isomorphic to the category of right $N^\vee$-comodules, and the category of left $N^\#$-modules is isomorphic to the category of left $N^\vee$-comodules. In particular, $N^\vee$ is an $N^\#$–N-bimodule.

Recall that the cotensor product $P \,\square_{N^\vee}\, Q$ of a right $N^\vee$-comodule P and a left $N^\vee$-comodule Q is defined as the kernel of the pair of maps $P \otimes_K Q \rightrightarrows P \otimes_K N^\vee \otimes_K Q$, one of which is induced by the coaction map $P \to P \otimes_K N^\vee$ and the other by the coaction map $Q \to N^\vee \otimes_K Q$. There are natural isomorphisms $P \,\square_{N^\vee}\, N^\vee \cong P$ and $N^\vee \,\square_{N^\vee}\, Q \cong Q$.

Proposition 3.

(a) (i) *For any right N-module P and any left N-module Q, there is a natural map of* $\mathbf{k}$*-vector spaces $P \otimes_N Q \to P \,\square_{N^\vee}\, (N^\vee \otimes_N Q)$, which is an isomorphism, at least, when P is injective or Q is projective.*

(ii) *For any right N-module P and any left $N^\#$-module Q, there is a natural map of* $\mathbf{k}$*-vector spaces $P \otimes_N (N \,\square_{N^\vee}\, Q) \to P \,\square_{N^\vee}\, Q$, which is an isomorphism, at least, when P is projective or Q is injective.*

(b) *The functors $P \mapsto N^\vee \otimes_N P$ and $M \mapsto N \,\square_{N^\vee}\, M$ are mutually inverse equivalences between the categories of projective left N-modules and injective left $N^\#$-modules.*

(c) *The functors $P \mapsto N^\vee \otimes_N P$ and $M \mapsto N \,\square_{N^\vee}\, M$ are mutually inverse tensor equivalences between the tensor category of N-bimodules that are projective left N-modules with the operation of tensor product over N and the tensor category of $N^\#$–N-bimodules that are injective left $N^\#$-modules with the operation of cotensor product over $N^\vee$.*

Proof. Both statements of (a) assert the existence of associativity (iso)morphisms connecting the tensor and cotensor products. In particular, in (i) we have to construct a natural map $(P \,\square_{N^\vee}\, N^\vee) \otimes_N Q \to P \,\square_{N^\vee}\, (N^\vee \otimes_N Q)$. More generally, let us consider an arbitrary $N^\#$–N-bimodule R and construct a natural map $(P \,\square_{N^\vee}\, R) \otimes_N Q \to P \,\square_{N^\vee}\, (R \otimes_N Q)$. This map can be defined in two equivalent ways. The first approach is to take the tensor product of the exact sequence of right N-modules $0 \to P \,\square_{N^\vee}\, R \to P \otimes_K R \to P \otimes_K N^\vee \otimes_K R$ with the left N-module Q. Since the resulting sequence is a complex, there exists a unique map $(P \,\square_{N^\vee}\, R) \otimes_N Q \to P \,\square_{N^\vee}\, (R \otimes_N Q)$ making a commutative triangle with the natural maps of $(P \,\square_{N^\vee}\, R) \otimes_N Q$ and $P \,\square_{N^\vee}\, (R \otimes_N Q)$ into $P \otimes_K R \otimes_N Q$. It is clear that this map is

https://doi.org/10.1112/S0010437X09004382 Published online by Cambridge University Press

an isomorphism whenever Q is a flat N-module. Analogously, for any P, Q and R there is a natural isomorphism $(P \mathbin{\square}_{N^\vee} R) \otimes_K Q \cong P \mathbin{\square}_{N^\vee} (R \otimes_K Q)$, since K is semisimple. The second approach is to take the cotensor product of the exact sequence of left $N^\vee$-comodules $R \otimes_K N \otimes_K Q \to R \otimes_K Q \to R \otimes_N Q \to 0$ with the right $N^\vee$-comodule P. Again, since the resulting sequence is a complex, there exists a unique map $(P \mathbin{\square}_{N^\vee} R) \otimes_N Q \to P \mathbin{\square}_{N^\vee} (R \otimes_N Q)$ making a commutative triangle with the natural maps from $P \mathbin{\square}_{N^\vee} R \otimes_K Q$ to $(P \mathbin{\square}_{N^\vee} R) \otimes_N Q$ and $P \mathbin{\square}_{N^\vee} (R \otimes_N Q)$. Clearly, this map is an isomorphism whenever P is a coflat $N^\vee$-comodule (i.e. the cotensor product with P preserves exactness). Now, any injective right N-module is a coflat right $N^\vee$-comodule, since it is a direct summand of a direct sum of copies of $N^\vee$. The two associativity maps that we have constructed coincide, since the relevant square diagram commutes. The proof of (ii) is analogous.

To prove (b), note the isomorphisms $N \mathbin{\square}_{N^\vee} (N^\vee \otimes_N P) \cong N \otimes_N P \cong P$ and $N^\vee \otimes_N (N \mathbin{\square}_{N^\vee} M) \cong N^\vee \mathbin{\square}_{N^\vee} M \cong M$ for a projective left N-module P and an injective left $N^\#$-module M. Since a projective left N-module is a direct summand of an N-module of the form $N \otimes_K V$ and an injective left $N^\#$-module is a direct summand of an $N^\#$-module of the form $\mathrm{Hom}_K(N^\#, V) \cong N^\vee \otimes_K V$ for a K-module V, the functors in question transform projective N-modules to injective $N^\#$-modules and vice versa.

To deduce (c), note the isomorphism $(N^\vee \otimes_N P) \mathbin{\square}_{N^\vee} (N^\vee \otimes_N Q) \cong N^\vee \otimes_N P \otimes_N Q$ for a N-bimodule P and a projective left N-module Q. It is straightforward to check that this isomorphism preserves the associativity constraints. □

3.3 Definition of $A^\#$

It follows from condition (ii) of § 3.1 that A is a projective left N-module. By Proposition 3(c), the tensor product $S = N^\vee \otimes_N A$ is a ring object in the tensor category of $N^\vee$-bicomodules with respect to the cotensor product over $N^\vee$. By condition (iii) of § 3.1 and the right analogue of Proposition 3(c), the cotensor product $A^\# = S \mathbin{\square}_{N^\vee} N^\#$ is a ring object in the tensor category of $N^\#$-bimodules with respect to the tensor product over $N^\#$. The embedding $N \to A$ induces injective maps $N^\vee \to S$ and $N^\# \to A^\#$; these are unit morphisms of the ring objects in the corresponding tensor categories. So $A^\#$ has a natural associative algebra structure and $N^\#$ is identified with a subalgebra in $A^\#$. Notice that $A^\#$ is a projective right $N^\#$-module by the definition.

Proposition 4. *There is a natural isomorphism between the $N^\#$–A-bimodule $S = N^\vee \otimes_N A$ and the $A^\#$–N-bimodule $S^\# = A^\# \otimes_{N^\#} N^\vee$, making S an $A^\#$–A-bimodule. Moreover, there are isomorphisms*

$$A^\# \cong \mathrm{End}_{A^{\mathrm{op}}}(S), \quad A^{\mathrm{op}} \cong \mathrm{End}_{A^\#}(S^\#).$$

Proof. By the definition, we have $S^\# = (S \mathbin{\square}_{N^\vee} N^\#) \otimes_{N^\#} N^\vee \cong S \mathbin{\square}_{N^\vee} N^\vee \cong S$, since S is an injective right N-module. Let us show that the right A-module and left $A^\#$-module structures on $S \cong S^\#$ commute. The isomorphism $S \otimes_N A \cong S \otimes_N (N \mathbin{\square}_{N^\vee} S) \cong S \mathbin{\square}_{N^\vee} S$ transforms the right action map $S \otimes_N A \to S$ into the map $S \mathbin{\square}_{N^\vee} S \to S$ defining the structure of a ring object in the tensor category of $N^\vee$-bicomodules on S. Analogously, the isomorphisms $A^\# \otimes_{N^\#} S^\# \cong (S \mathbin{\square}_{N^\vee} N^\#) \otimes_{N^\#} S \cong S \mathbin{\square}_{N^\vee} S$ and $S^\# \cong S$ transform the left action map $A^\# \otimes_{N^\#} S^\# \to S^\#$ into the same map $S \mathbin{\square}_{N^\vee} S \to S$. Finally, there is an isomorphism $A^\# \otimes_{N^\#} S \otimes_N A \cong (S \mathbin{\square}_{N^\vee} N^\#) \otimes_{N^\#} S \otimes_N (N \mathbin{\square}_{N^\vee} S) \cong S \mathbin{\square}_{N^\vee} S \mathbin{\square}_{N^\vee} S$; so the right and left actions commute since S is

https://doi.org/10.1112/S0010437X09004382 Published online by Cambridge University Press

an associative ring object in the tensor category of $N^\vee$-bicomodules. Now we have

$$\mathrm{Hom}_{A^{\mathrm{op}}}(N^\vee \otimes_N A, N^\vee \otimes_N A) \cong \mathrm{Hom}_{N^{\mathrm{op}}}(N^\vee, N^\vee \otimes_N A) \cong (N^\vee \otimes_N A) \,\square_{N^\vee}\, N^\# = A^\#,$$

$$\mathrm{Hom}_{A^\#}(A^\# \otimes_{N^\#} N^\vee, A^\# \otimes_{N^\#} N^\vee) \cong \mathrm{Hom}_{N^\#}(N^\vee, A^\# \otimes_{N^\#} N^\vee) \cong N \,\square_{N^\vee}\, (A^\# \otimes_{N^\#} N^\vee) \cong A,$$

and thus the proof is complete. □

3.4 *N*-projective (injective) modules

We shall denote the category of (graded finite-dimensional) left A-modules by A-mod.

Consider the full subcategory $A\text{-mod}_{N\text{-proj}} \subset A\text{-mod}$, consisting of modules whose restriction to N is projective, and the full subcategory $A^\#\text{-mod}_{N^\#\text{-inj}} \subset A^\#\text{-mod}$, consisting of modules whose restriction to $N^\#$ is injective.

We abbreviate $D^b(A\text{-mod})$ by $D(A)$ and $D^b(A^\#\text{-mod})$ by $D(A^\#)$, and we let $D_{\infty/2}(A) \subset D(A)$ and $D_{\infty/2}(A^\#) \subset D(A^\#)$ be the full triangulated subcategories generated by, respectively, $A\text{-mod}_{N\text{-proj}}$ and $A^\#\text{-mod}_{N^\#\text{-inj}}$.

Proposition 5. *We have the canonical equivalences*

$$A\text{-mod}_{N\text{-proj}} \cong A^\#\text{-mod}_{N^\#\text{-inj}} \quad \textit{and} \quad D_{\infty/2}(A) \cong D_{\infty/2}(A^\#).$$

Proof. Let us show that the adjoint functors $P \mapsto S \otimes_A P$ and $M \mapsto \mathrm{Hom}_{A^\#}(S, M)$ between the categories A-mod and $A^\#$-mod induce an equivalence between their full subcategories $A\text{-mod}_{N\text{-proj}}$ and $A^\#\text{-mod}_{N^\#\text{-inj}}$. It suffices to check that the adjunction morphisms $P \to \mathrm{Hom}_{A^\#}(S, S \otimes_A P)$ and $S \otimes_A \mathrm{Hom}_{A^\#}(S, M) \to M$ are isomorphisms when an A-module P is projective over N and an $A^\#$-module M is injective over $N^\#$. There are natural isomorphisms $S \otimes_A P \cong N^\vee \otimes_N P$ and $\mathrm{Hom}_{A^\#}(S, M) \cong \mathrm{Hom}_{N^\#}(N^\vee, M) \cong N \,\square_{N^\vee}\, M$, so it remains to apply Proposition 3(b). To obtain the equivalence of categories $D_{\infty/2}(A) \cong D_{\infty/2}(A^\#)$, it suffices to check that $D_{\infty/2}(A)$ is equivalent to the bounded derived category of the exact category $A\text{-mod}_{N\text{-proj}}$ and that $D_{\infty/2}(A^\#)$ is equivalent to the bounded derived category of the exact category $A^\#\text{-mod}_{N^\#\text{-inj}}$. Let us prove the former; the proof of the latter is analogous. It suffices to check that for any bounded complex of N-projective A-modules P and any bounded complex of A-modules X together with a quasi-isomorphism $X \to P$, there exists a bounded complex of N-projective A-modules Q together with a quasi-isomorphism $Q \to X$. Let Q' be a bounded-above complex of projective A-modules mapping quasi-isomorphically into X; then the canonical truncation $Q'_{\geqslant -n}$ for large enough n provides the desired complex Q. □

3.5 The case of an invertible entwining map

Consider the multiplication map $\phi : B \otimes_K N \to A \cong N \otimes_K B$. It yields a map $\psi : N^\vee \otimes_K B \to \mathrm{Hom}_{K^{\mathrm{op}}}(N, B) \cong B \otimes_K N^\vee$. Assume that the map ψ is an isomorphism and consider the inverse map $\psi^{-1} : B \otimes_K N^\vee \to N^\vee \otimes_K B$. By the analogous 'lowering of indices', we obtain from it a map $N^\# \otimes_K B \to \mathrm{Hom}_{K^{\mathrm{op}}}(N^\vee, B) = B \otimes_K N^\#$ that will be denoted by $\phi^\#$.

We can then also define the algebra $A^\#$ as the unique associative algebra with fixed embeddings of $N^\#$ and B into $A^\#$ such that:

(i) the embeddings $N^\# \to A^\#$ and $B \to A^\#$ form a commutative square with the embeddings $K \to N^\#$ and $K \to B$;

(ii) the multiplication map induces an isomorphism $B \otimes_K N^\# \to A^\#$;

https://doi.org/10.1112/S0010437X09004382 Published online by Cambridge University Press

(iii) the map induced by the multiplication map $N^{\#} \otimes_K B \to A^{\#} \cong B \otimes_K N^{\#}$ coincides with $\phi^{\#}$.

Indeed, the existence of an algebra A with subalgebras N and B in terms of which the map ϕ can be defined is easily seen to be equivalent to the map ψ satisfying the equations of a right entwining structure for the coring $N^{\vee}$ and the algebra B (see [BW03] or [Pos07] for the definition). When ψ is invertible, it is a right entwining structure if and only if ψ^{-1} is a left entwining structure, and the latter is equivalent to the existence of an algebra $A^{\#}$ satisfying (i)–(iii) above.

To show that the two definitions of $A^{\#}$ are equivalent, it suffices to check that the ring object S in the tensor category of $N^{\vee}$-bicomodules can be constructed in terms of the entwining structure ψ in the manner explained in [Brz02] or [Pos07].

3.6 The case of a self-injective N

Assume that N is self-injective. In this case $A^{\#}$ is canonically Morita-equivalent to A; the equivalence is defined by the $A^{\#}$–A-bimodule S, so it sends $A\text{-mod}_{N\text{-proj}} = A\text{-mod}_{N\text{-inj}}$ to $A^{\#}\text{-mod}_{N^{\#}\text{-proj}} = A^{\#}\text{-mod}_{N^{\#}\text{-inj}}$.

Indeed, $N^{\vee}$ is obviously an injective generator of the category of right N-modules. Since every injective N-module is projective, $N^{\vee}$ is a projective right N-module. Since N is an injective right N-module, it is a direct summand of a finite direct sum of copies of $N^{\vee}$. So $N^{\vee}$ is a projective generator of the category of right N-modules; hence $S = N^{\vee} \otimes_N A$ is a projective generator of the category of right A-modules. Now it remains to use Proposition 4. Analogously, $N^{\#}$ is Morita-equivalent to N; hence $N^{\#}$ is also self-injective.

If N is Frobenius, $N^{\#}$ is isomorphic to N and $A^{\#}$ is isomorphic to A. Indeed, K is also Frobenius. Choose a Frobenius linear function $K \to \mathbf{k}$; then the right K-module $\mathrm{Hom}_{\mathbf{k}}(K, \mathbf{k})$ is isomorphic to K. Hence a Frobenius linear function $N \to \mathbf{k}$ lifts to a right K-module map $N \to K$. Now the composition $N \otimes_K N \to N \to K$ of the multiplication map $N \otimes_K N \to N$ and the right K-module map $N \to K$ defines an isomorphism of right N-modules $N \to \mathrm{Hom}_{K^{\mathrm{op}}}(N, K) = N^{\vee}$. By Proposition 4, this leads to the isomorphism $A^{\#} \cong A$ and, analogously, to the isomorphism $N^{\#} \cong N$; these isomorphisms are compatible with the embeddings $N \to A$ and $N^{\#} \to A^{\#}$ but not with the embeddings of K to N and $N^{\#}$, in general.

4. Definitions of $\mathrm{Ext}^{\infty/2}$ by explicit complexes

4.1 Concave and convex resolutions

A complex of graded modules will be called *convex* if the grading 'goes down', i.e. for any $n \in \mathbb{Z}$ the sum of graded components of degree greater than n is finite-dimensional; it will be called *non-strictly convex* if the grading 'does not go up', i.e. the graded components of high enough degree all vanish. In a similar sense, a complex of graded modules will be called *concave* (respectively, *non-strictly concave*) if the grading 'goes up' (respectively, 'does not go down').

An $A^{\#}$-module M will be called *weakly projective relative to* $N^{\#}$ if for any $A^{\#}$-module J which is injective as an $N^{\#}$-module one has $\mathrm{Ext}^i_{A^{\#}}(M, J) = 0$ for all $i \neq 0$. Analogously, we can define A-modules that are *weakly injective relative to* N. Notice that any $A^{\#}$-module induced from an $N^{\#}$-module is weakly projective relative to $N^{\#}$. The class of $A^{\#}$-modules that are weakly projective relative to $N^{\#}$ is closed under extensions and kernels of surjective morphisms.

https://doi.org/10.1112/S0010437X09004382 Published online by Cambridge University Press

Lemma 1.

(i) *Any A-module admits a left concave resolution by A-modules which are projective as N-modules. Any $A^{\#}$-module admits a left non-strictly convex resolution by $A^{\#}$-modules which are weakly projective relative to $N^{\#}$.*

(ii) *Any finite complex of A-modules is a quasi-isomorphic quotient of a bounded-above concave complex of N-projective A-modules. Any finite complex of $A^{\#}$-modules is a quasi-isomorphic quotient of a bounded-above non-strictly convex complex of $A^{\#}$-modules that are weakly projective relative to $N^{\#}$.*

Proof. To deduce (ii) from (i), choose a quasi-isomorphic surjection onto a given complex $C^\bullet \in \mathrm{Com}^b(A\text{-mod})$ from a complex of A-projective modules $P^\bullet \in \mathrm{Com}^-(A\text{-mod})$ (notice that condition (ii) of § 3.1 implies that an A-projective module is also N-projective) and apply (i) to the module of cocycles $Z^n = P^{-n}/d(P^{-n-1})$ for large n.

To check (i), it suffices to find for any $M \in A$-mod a surjection $P \twoheadrightarrow M$ where P is N-projective and, if n is such that all graded components M_i for $i < n$ vanish, then $P_i = 0$ for $i < n$ and $P_n \xrightarrow{\sim} M_n$. It suffices to take $P = \mathrm{Ind}_B^A(\mathrm{Res}_B^A(M))$. This is indeed N-projective because of the equality

$$\mathrm{Res}_N^A(\mathrm{Ind}_B^A(M)) = \mathrm{Ind}_K^N(\mathrm{Res}_K^B(M)), \tag{4}$$

which is a consequence of assumption (ii) of § 3.1.

The second assertions in (i) and (ii) can be proven in an analogous way, except that one would use the induction from $N^{\#}$ (which is even simpler, as weak relative projectivity of the relevant modules is obvious). □

4.2 Definition of semi-infinite Ext functors

Definition 1 (cf. [BFS98, § 2.4]). Suppose the assumptions (i)–(iii) of § 3.1 are enforced. Let $X \in D(A^{\#})$ and $Y \in D(A)$. Let $P^X_{\swarrow}$ be a non-strictly convex bounded-above complex of $A^{\#}$-modules weakly projective relative to $N^{\#}$ that is quasi-isomorphic to X, and let $P^Y_{\nwarrow}$ be a concave bounded-above complex of N-injective A-modules that is quasi-isomorphic to Y. Then we set

$$\mathrm{Ext}^{\infty/2+i}(X, Y) = H^i(\mathrm{Hom}^\bullet_{A^{\#}}(P^X_{\swarrow}, S \otimes_A P^Y_{\nwarrow})). \tag{5}$$

The right-hand side of (5) is independent of the choice of $P^X_{\swarrow}$ and $P^Y_{\nwarrow}$; this follows from Theorem 1 below.

Remark 3. Notice that Hom in the right-hand side of (5) is Hom in the category of graded modules. As usual, it is convenient to denote by $\mathrm{Ext}^{\infty/2+i}(X, Y)$ the graded space which, with the present notation, is written as $\bigoplus_n \mathrm{Ext}^{\infty/2+i}(X, Y(n))$, where (n) refers to the shift of grading by $-n$.

Remark 4. Definition 1 is compatible with [Ark97a, Definition 3.3.6] in the sense explained below. In this remark we will use freely the notation of [Ark97a].

For a finite-dimensional algebra A, the definition of the algebra $A^{\#}$ given in [Ark97a, 3.3.2] reduces to $A^{\#} = \mathrm{End}_{A^{\mathrm{op}}}(S)$, where S is defined by $S = \mathrm{Hom}_{\mathbf{k}}(N, \mathbf{k}) \otimes_N A$. So, according to Proposition 4, this agrees with our definition (see also § 3.5). Notice that in [Ark97a] it is presumed that $K = \mathbf{k}$, so one has $N^{\#} = N$.

Let $L \in \mathrm{Com}^b(A^{\#}\text{-mod})$ and $M \in \mathrm{Com}^b(A\text{-mod})$. Then the restricted Bar-resolution $\mathrm{Bar}^\bullet(A^{\#}, N^{\#}, L)$ is a non-strictly convex bounded-above resolution of L by $A^{\#}$-modules weakly

https://doi.org/10.1112/S0010437X09004382 Published online by Cambridge University Press

projective relative to $N^\#$, and $\mathrm{Bar}^\bullet(A, B, M)$ is a concave bounded-above resolution of M by N-projective A-modules. Thus the definition of semi-infinite cohomology

$$\mathrm{Ext}^{\infty/2+i}(L, M) = \mathrm{Hom}^\bullet_{A^\#}(\mathrm{Bar}^\bullet(A^\#, N^\#, L), S \otimes_A \mathrm{Bar}^\bullet(A, B, M))$$

from [Ark97a] is a particular case of our definition in any situation where both are applicable.

4.3 Alternative assumptions

The conditions on the resolutions $P^X_\swarrow$ and $P^Y_\nwarrow$ used in (5) were formulated in terms of the subalgebras $N \subset A$ and $N^\# \subset A^\#$; the subalgebra $B \subset A$ is not mentioned there (and the left-hand side of (9) in Theorem 1 below does not depend on it either). However, existence of a 'complemental' subalgebra B is used in the construction of a resolution $P^Y_\nwarrow$ with the required properties. Moreover, the next lemma shows that conditions on the resolutions $P^X_\swarrow$ and $P^Y_\nwarrow$ can be rephrased in terms of the subalgebra B and any non-positively graded subalgebra $B^\# \subset A^\#$ such that $B^\# \otimes_K N^\# \cong A^\#$, provided that such a subalgebra exists (for example, in the assumptions of § 3.5 or when N is Frobenius and $B \otimes_K N \cong A$).

Lemma 2.

(i) *An A-module is N-projective if and only if it has a filtration with subquotients of the form $\mathrm{Ind}_B^A(M)$, $M \in B$-mod.*

(ii) *Assume that $B^\# \subset A^\#$ is a subalgebra graded by non-positive integers such that $K \subset B^\#$ and the multiplication map induces an isomorphism $B^\# \otimes_K N^\# \to A^\#$. Then an $A^\#$-module is $N^\#$-injective if and only if it has a filtration with subquotients of the form $\mathrm{CoInd}_{B^\#}^{A^\#}(M)$, $M \in B^\#$-mod. Consequently, an $A^\#$-module is weakly projective relative to $N^\#$ if and only if it is $B^\#$-projective.*

Proof. The 'if' direction follows from the semisimplicity of K and equality (4) above. To show the 'only if' part, let M be a projective N-module and let M^- be its graded component of minimal degree; then the canonical morphism

$$\mathrm{Ind}_K^N M^- \to M \tag{6}$$

is injective and its cokernel is again a projective N-module. If M is actually an A-module, then the injection $M^- \to M$ is an embedding of B-modules and hence yields a morphism of A-modules

$$\mathrm{Ind}_B^A M^- \to M. \tag{7}$$

Relation (4) shows that Res_N^A sends (7) into (6); in particular, (7) is injective and has an N-projective cokernel. Thus the bottom submodule of the required filtration is constructed, and the proof is finished by induction. The proof of (ii) is analogous. □

Remark 5. By replacing the assumption of existence of a subalgebra $B \subset A$ (assuming only that A is a projective left N-module) with the assumption of existence of a non-positively graded subalgebra $B^\# \subset A^\#$ such that $B^\# \otimes_K N^\# \cong A^\#$, one can define $\mathrm{Ext}^{\infty/2+i}(X, Y)$ in terms of injective resolutions rather than projective ones. Specifically, for $X \in D(A^\#)$ and $Y \in D(A)$, let $J^X_\searrow$ be a convex bounded-below complex of $N^\#$-injective modules that are quasi-isomorphic to X, and let $J^Y_\nearrow$ be a non-strictly concave bounded-below complex of A-modules that are weakly injective relative to N. Then set

$$\mathrm{Ext}^{\infty/2+i}(X, Y) = H^i(\mathrm{Hom}^\bullet(\mathrm{Hom}_{A^\#}(S, J^X_\searrow), J^Y_\nearrow)).$$

https://doi.org/10.1112/S0010437X09004382 Published online by Cambridge University Press

An analogue of Theorem 1 below holds for this definition as well; hence the two definitions are equivalent whenever both are applicable.

4.4 Comparison with ordinary Ext and Tor

In four special cases, $\mathrm{Ext}^{\infty/2+i}(X,Y)$ coincides with a combination of traditional derived functors. First, suppose that $\mathrm{Res}^A_N(Y)$ has finite projective dimension; then one can use a finite complex $P^Y_\nwarrow$ in (5) above. It follows immediately that, in this case,

$$\mathrm{Ext}^{\infty/2+i}(X,Y) \cong \mathrm{Hom}_{D(A^\#)}(X, S \overset{L}{\otimes}_A Y[i]).$$

Analogously, in the assumptions of Remark 5 above, whenever $\mathrm{Res}^{A^\#}_{N^\#}(X)$ has finite injective dimension, one has

$$\mathrm{Ext}^{\infty/2+i}(X,Y) \cong \mathrm{Hom}_{D(A)}(R\mathrm{Hom}_{A^\#}(S,X), Y[i]).$$

On the other hand, suppose that the complex $P^X_\swarrow$ in (5) can be chosen to be a finite complex of $A^\#$-modules whose terms have filtrations with subquotients being $A^\#$-modules induced from $N^\#$-modules. We claim that in this case we have

$$\mathrm{Ext}^{\infty/2+i}(X,Y) \cong H^i(R\mathrm{Hom}_{A^\#}(X,S) \overset{L}{\otimes}_A Y).$$

This isomorphism is an immediate consequence of the next lemma. Analogously, in the situation of Remark 5, whenever $J^Y_\nearrow$ can be chosen to be a finite complex of A-modules whose terms have filtrations with subquotients being A-modules coinduced from N-modules, one has

$$\mathrm{Ext}^{\infty/2+i}(X,Y) \cong H^i(X^* \overset{L}{\otimes}_{A^\#} R\mathrm{Hom}_A(S^*, Y)).$$

Here we denote by $V \mapsto V^*$ the passage to the dual vector space, $V^* = \mathrm{Hom}_{\mathbf{k}}(V, \mathbf{k})$, and the corresponding functor on the level of derived categories.

Lemma 3. *Let $L \in A^\#\text{-mod}$ and $M \in A\text{-mod}$ be such that L has a filtration with subquotients being $A^\#$-modules induced from $N^\#$-modules, while M is N-projective. Then the following hold.*

(a) (i) $\mathrm{Ext}^i_{A^\#}(L,S) = 0$ *and* $\mathrm{Tor}^A_i(\mathrm{Hom}_{A^\#}(L,S), M) = 0$ *for* $i \neq 0$.
  (ii) $\mathrm{Tor}^A_i(S,M) = 0$ *and* $\mathrm{Ext}^i_{A^\#}(L, S \otimes_A M) = 0$ *for* $i \neq 0$.

(b) *The natural map*

$$\mathrm{Hom}_{A^\#}(L,S) \otimes_A M \longrightarrow \mathrm{Hom}_{A^\#}(L, S \otimes_A M) \tag{8}$$

is an isomorphism.

Proof. The first equality in (i) holds because S is an injective $N^\#$-module. To check the second one, notice that if $L = \mathrm{Ind}^A_{N^\#} L_0$, then $\mathrm{Hom}_{A^\#}(L,S) \cong \mathrm{Hom}_{N^\#}(L_0, N^\vee \otimes_N A) \cong \mathrm{Hom}_{N^\#}(L_0, N^\vee) \otimes_N A$ is a right A-module induced from a right N-module. The first equality in (ii) holds because the right A-module S is induced from a right N-module, and the second one is verified because $S \otimes_A M$ is $N^\#$-injective. Let us now deduce (b) from (a). Observe that (a) implies that both sides of (8) are exact in M (and also in L), i.e. they send exact sequences $0 \to M' \to M \to M'' \to 0$, with M' and M'' being N-projective, into exact sequences. Also, (8) is evidently an isomorphism for $M = A$. For any N-projective M, there exists an exact sequence

$$A^n \xrightarrow{\phi} A^m \to M \to 0$$

with the image and kernel of ϕ being N-projective. Thus both sides of (8) turn into exact sequences, which shows that (8) is an isomorphism for any N-projective M. □

https://doi.org/10.1112/S0010437X09004382 Published online by Cambridge University Press

5. Main result

Theorem 1. *Let $D_{\infty/2} \subset D(A^{\#})$, $D_{\infty/2} \subset D(A)$, be the full triangulated subcategory of $D(A^{\#})$ generated by $N^{\#}$-injective modules, which is equivalent to the full triangulated subcategory of $D(A)$ generated by N-projective modules. For $X \in D^b(A^{\#}\text{-mod})$ and $Y \in D^b(A\text{-mod})$, we have a natural isomorphism*

$$\mathrm{Hom}_{D(A^{\#})_{D_{\infty/2}} D(A)}(X, Y[i]) \cong \mathrm{Ext}^{\infty/2+i}(X, Y). \tag{9}$$

Example 2. Assume that $A = N$ is a Frobenius algebra and $K = \mathbf{k}$. Then $A^{\#} \cong A$ and, according to § 4.4, we have $\mathrm{Ext}^{\infty/2+i}(X, Y) = \mathrm{Tor}^A_{-i}(X^*, Y)$. In this case, we can identify $\mathcal{A}'$ with $\mathcal{A}$, so that $\Phi' = \Phi$ is the embedding of the category of perfect complexes. The long exact sequence of Proposition 2 becomes a standard sequence linking Ext, Tor and Hom in the stable category $A/\mathcal{B}$; in particular, for modules over a finite group, we recover the description of Tate cohomology as Hom functors in the stable category. (Thanks are due to A. Beilinson for suggesting this example.)

Remark 6. Notice that the definition of the left-hand side in (9) applies also to non-graded algebras and modules. Thus the theorem allows one to extend the definition of semi-infinite cohomology to non-graded algebras. Another definition of the semi-infinite cohomology of non-graded algebras was given in [Pos07].

Let us point out that these two definitions are *not* equivalent: for example, when $\mathbf{k}$ is a finite or a countable field, the left-hand side of (9) in the non-graded case is no more than countable, while the semi-infinite cohomology defined in [Pos07] can have the cardinality of continuum.

The proof of Theorem 1 is based on the following lemma.

Lemma 4.

(i) *Every N-projective A-module admits a non-strictly convex left resolution consisting of A-projective modules.*

(ii) *A finite complex of N-projective A-modules is quasi-isomorphic to a non-strictly convex bounded-above complex of A-projective modules.*

Proof. Assertion (ii) follows from assertion (i) as in the proof of Lemma 1. (Recall that, according to a well-known argument due to Hilbert, if a bounded-above complex of projectives represents an object of the derived category which has finite projective dimension, then for large negative n the module of cocycles is projective.)

To prove (i), it is enough to find for any N-projective module M a surjection $Q \twoheadrightarrow M$, where Q is A-projective and $Q_n = 0$ for $i > n$ provided that $M_i = 0$ for $i > n$. (Notice that the kernel of such a surjection is N-projective, because Q is N-projective by condition (ii) in § 3.1.) We can take Q to be $\mathrm{Ind}^A_N(\mathrm{Res}^A_N(M))$, and the condition on grading is then clearly satisfied. □

Proposition 6. *Let $P_{\nwarrow}$ be a concave bounded-above complex of A-modules representing an object $Y \in D^-(A\text{-mod})$. Let $P^n_{\nwarrow}$ be the $(-n)$th stupid truncation of $P_{\nwarrow}$ (thus $P^n_{\nwarrow}$ is a subcomplex of $P_{\nwarrow}$).*

Let Z be a finite complex of N-projective A-modules. Then

$$\mathrm{Hom}_{D^-(A\text{-mod})}(Z, Y) \xrightarrow{\sim} \varinjlim \mathrm{Hom}_{D(A)}(Z, P^n_{\nwarrow}). \tag{10}$$

In fact, for n large enough, we have

$$\mathrm{Hom}_{D^-(A\text{-mod})}(Z, Y) \xrightarrow{\sim} \mathrm{Hom}_{D(A)}(Z, P^n_{\nwarrow}).$$

https://doi.org/10.1112/S0010437X09004382 Published online by Cambridge University Press

Proof. Let $Q_\swarrow$ be a non-strictly convex bounded-above complex of A-projective modules that are quasi-isomorphic to Z (which exists by Lemma 4(ii)). Then the left-hand side of (10) equals $\mathrm{Hom}_{\mathrm{Hot}}(Q_\swarrow, P_\nwarrow)$, where Hot stands for the homotopy category of complexes of A-modules. The conditions on gradings of our complexes ensure that there will be only finitely many degrees for which the corresponding graded components in both $Q_\swarrow$ and $P_\nwarrow$ are non-zero; thus any morphism between the graded vector spaces $Q_\swarrow$ and $P_\nwarrow$ factors through the finite-dimensional sum of the corresponding graded components. In particular, $\mathrm{Hom}^\bullet(Q_\swarrow, P_\nwarrow^n) \xrightarrow{\sim} \mathrm{Hom}^\bullet(Q_\swarrow, P_\nwarrow)$ for large n, and hence

$$\mathrm{Hom}_{D(A)}(Z, P_\nwarrow^n) = \mathrm{Hom}_{\mathrm{Hot}}(Q_\swarrow, P_\nwarrow^n) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Hot}}(Q_\swarrow, P_\nwarrow)$$

for large n. □

Corollary 1. *Let $P_\nwarrow$ be a concave bounded-above complex of N-projective A-modules, and let X be the corresponding object of $D^-(A\text{-mod})$. Then the functor on $D_{\infty/2}$ given by $Z \mapsto \mathrm{Hom}_{D^-(A\text{-mod})}(Z, Y)$ is represented by the ind-object $\varinjlim P_\nwarrow^n$.*

Proof of Theorem 1. We keep the notation of Definition 1. It follows from Proposition 6 that

$$\mathrm{Hom}_{D(A^\#)_{D_{\infty/2}}D(A)}(X, Y[i]) = \varinjlim_n \mathrm{Hom}_{D(A^\#)}(X, S \otimes_A (P_\nwarrow^Y)^n[i]).$$

The right-hand side of (9) (defined in (5)) equals $H^i(\mathrm{Hom}^\bullet_{A^\#}(P_\swarrow^X, S \otimes_A P_\nwarrow^Y))$. The conditions on gradings of $P_\swarrow^X$ and $P_\nwarrow^Y$ show that for large n, we have

$$\mathrm{Hom}^\bullet_{A^\#}(P_\swarrow^X, S \otimes_A (P_\nwarrow^Y)^n) \xrightarrow{\sim} \mathrm{Hom}^\bullet_{A^\#}(P_\swarrow^X, S \otimes_A P_\nwarrow^Y).$$

Since $\mathrm{Ext}^i_{A^\#}(L, S \otimes_A M) = 0$ for $i > 0$ if L is weakly projective relative to $N^\#$ and M is N-projective, we have

$$\mathrm{Hom}_{D(A^\#)}(X, S \otimes_A (P_\nwarrow^Y)^n[i]) = H^i(\mathrm{Hom}^\bullet_{A^\#}(P_\swarrow^X, S \otimes_A (P_\nwarrow^Y)^n)),$$

and the theorem is proved. □

Remark 7. There is a version of Theorem 1 that is applicable in the situation where the condition that K be the component of degree zero of N in assumption (i) of § 3.1 is replaced by the condition that K be the component of degree zero of B. One just needs to change the conditions on the complexes $P_\swarrow^X$ and $P_\nwarrow^Y$ in Definition 1, requiring that $P_\swarrow^X$ be convex and $P_\nwarrow^Y$ be non-strictly concave, and make the corresponding changes in the proof.

6. Semi-infinite cohomology of the small quantum group

This section concerns the example provided by a small quantum group. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}^-$. Let $q \in \mathbb{C}$ be a root of unity of order l, and let $A = u_q = u_q(\mathfrak{g})$ be the corresponding small quantum group [Lus90]. We assume that l is large enough (larger than the Coxeter number) and is prime to twice the maximal multiplicity of an edge in the Dynkin diagram of $\mathfrak{g}$.

Let $A^{\geqslant 0} = u_q^+ \subset u_q$ and $A^{\leqslant 0} = u_q^- \subset u_q$ be, respectively, the upper and lower triangular subalgebras (small quantum Borel subalgebras). The algebra u_q carries a canonical grading by the weight lattice. We fix an arbitrary element in the dual coweight lattice which is a dominant coweight; thus we obtain a $\mathbb{Z}$-grading on u_q. Then conditions (i)–(iii) of § 3.1 are satisfied.

For an augmented $\mathbf{k}$-algebra R, we write $H^\bullet(R)$ for $\mathrm{Ext}_R(\mathbf{k}, \mathbf{k})$; we abbreviate $H^\bullet(u_q)$ by $H^\bullet$.

https://doi.org/10.1112/S0010437X09004382 Published online by Cambridge University Press

The cohomology algebra $H^\bullet$ and the semi-infinite cohomology $\mathrm{Ext}^{\infty/2+\bullet}(\mathbf{k},\mathbf{k})$ were computed, respectively, in [GK93] and in [Ark97a, Ark98b]. Let us recall the results of these computations. In what follows, by 'Hom' we will mean graded Hom, as in Remark 3 above.

Let $\mathcal{N}\subset\mathfrak{g}$ be the cone of nilpotent elements, and let $\mathfrak{n}\subset\mathcal{N}$ be a maximal nilpotent subalgebra. Then a theorem of Ginzburg and Kumar asserts the existence of canonical isomorphisms

$$H^\bullet\cong\mathcal{O}(\mathcal{N}), \tag{11}$$

$$H^\bullet(u_q^+)=\mathcal{O}(\mathfrak{n}) \tag{12}$$

such that the restriction map $\mathcal{O}(\mathcal{N})\to\mathcal{O}(\mathfrak{n})$ coincides with the map arising from functoriality of cohomology with respect to maps of augmented algebras.

Moreover, a conjecture of Feigin proved by Arkhipov [Ark97a, Ark98b] asserts that

$$\mathrm{Ext}^{\infty/2+\bullet}(\mathbf{k},\mathbf{k})\cong H^d_{\mathfrak{n}^-}(\mathcal{N},\mathcal{O}), \tag{13}$$

where d is the dimension of $\mathfrak{n}^-$ and $H_{\mathfrak{n}^-}$ denotes cohomology with support in $\mathfrak{n}^-$; one also has $H^i_{\mathfrak{n}^-}(\mathcal{N},\mathcal{O})=0$ for $i\neq d$.

The aim of this section is to show how (a generalization of) this isomorphism follows from Theorem 1.

6.1 $D_{\infty/2}$ and cohomological support

Let $D^\bullet$ denote the category defined by

$$\mathrm{Ob}(D^\bullet)=\mathrm{Ob}(D),\quad \mathrm{Hom}_{D^\bullet}(X,Y)=\mathrm{Hom}^\bullet(X,Y)=\bigoplus_i\mathrm{Hom}(X,Y[i]).$$

Then $D^\bullet$ is an $HH^\bullet$-linear category, i.e. we have a canonical homomorphism $HH^\bullet\to\mathrm{End}(\mathrm{Id}_{D^\bullet})$, where $HH^\bullet$ denotes the Hochschild cohomology of u_q. Since u_q is a Hopf algebra, we have a canonical homomorphism $H^\bullet\to HH^\bullet$; thus $D^\bullet$ is an $H^\bullet$-linear category. For an object $X\in D^\bullet$, its *cohomological support* $\mathrm{supp}(X)\subset\mathrm{Spec}(H^\bullet)$ is the set-theoretic support of the $H^\bullet$-module $\mathrm{End}^\bullet(X)$.

Proposition 7. *For an object $X\in D$, we have*

$$X\in D_{\infty/2}\iff\mathrm{supp}(X)\subseteq\mathfrak{n}.$$

Proof. It is well known that $\mathrm{supp}(X)\subset\mathrm{supp}(Y)\cup\mathrm{supp}(Z)$ provided that there exists a distinguished triangle $Y\to X\to Z\to Y[1]$; thus the set of objects with cohomological support contained in $\mathfrak{n}$ forms a full triangulated subcategory. In view of Lemma 2, to check the implication $\Rightarrow$ it suffices to check that $\mathrm{supp}(X)\subset\mathfrak{n}$ if $X=\mathrm{CoInd}_{u_q^+}^{u_q}(M)$ for some M. For such X we have $\mathrm{Ext}^\bullet_{u_q}(X,X)=\mathrm{Ext}^\bullet_{u_q^+}(X,M)$. Moreover, it is not hard to check that this isomorphism is compatible with the $H^\bullet$ action, with the action on the right-hand side obtained as the composition $H^\bullet\to H^\bullet(u_q^+)\to\mathrm{Ext}^\bullet_{u_q^+}(M,M)$ and the canonical right action of $\mathrm{Ext}^\bullet_{u_q^+}(M,M)$. Thus in this case $\mathrm{Ext}^\bullet_{u_q}(X,X)$ is set-theoretically supported on $\mathfrak{n}$.

Assume now that $X\in D$ is such that $\mathrm{supp}(X)\subseteq\mathfrak{n}$. To check that $X\in D_{\infty/2}$, it suffices to show that $\mathrm{Ext}^\bullet_{u_q^-}(M,X)$ is finite-dimensional for any $M\in D^b(u_q^-\text{-mod})$. It is a standard fact that $\mathrm{Ext}^\bullet(M_1,M_2)$ is a finitely generated $H^\bullet(u_q^-)$-module for any $M_1,M_2\in D^b(u_q^-\text{-mod})$. Therefore it suffices to show that the $H^\bullet(u_q^-)$-module $\mathrm{Ext}^\bullet_{u_q^-}(M,X)$ is supported at $\{0\}\subset\mathfrak{n}^-=\mathrm{Spec}(H^\bullet(u_q^-))$; but this is clear, since viewed as a $H^\bullet$-module it is supported on $\mathfrak{n}$. □

https://doi.org/10.1112/S0010437X09004382 Published online by Cambridge University Press

6.2 A description of the derived u_q-modules category via coherent sheaves

Let $\tilde{\mathcal{N}} = T^*(\mathcal{B}) = \{(\mathfrak{b}, x) \mid \mathfrak{b} \in \mathcal{B},\ x \in \mathrm{rad}(\mathfrak{b})\}$, where $\mathcal{B} = G/B$ is the flag variety of G identified with the set of Borel subalgebras in $\mathfrak{g}$ and 'rad' stands for the nil-radical. Let $\pi : \tilde{\mathcal{N}} \to \mathcal{N}$ be the Springer map, $\pi : (\mathfrak{b}, x) \mapsto x$.

The result of [BL07] (based on [ABG04]) yields a triangulated functor $\Psi : D^b(\mathrm{Coh}^{\mathbb{G}_m}(\tilde{\mathcal{N}})) \to D^b(u_q\text{-Mod})$, where $\mathbb{G}_m$ acts on $\tilde{\mathcal{N}}$ by $t : (\mathfrak{b}, x) \mapsto (\mathfrak{b}, t^2 x)$ and u_q-Mod stands for the category of finite-dimensional modules. Notice that, in contrast to the definition of u_q-mod, the modules in u_q-Mod *do not* carry a grading.[1]

The functor satisfies the following properties:

$$\Psi(\mathcal{F}(1)) \cong \Psi(\mathcal{F})[1], \tag{14}$$

where $\mathcal{F}(1)$ is the twist of $\mathcal{F}$ by the tautological character of $\mathbb{G}_m$;

$$\Psi : \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(\mathcal{F}, \mathcal{G}[n](n)) \xrightarrow{\sim} \mathrm{Hom}(\Psi(\mathcal{F}), \Psi(\mathcal{G})); \tag{15}$$

$$\langle \mathrm{Im}(\Psi) \rangle = D^b(u_q\text{-Mod}_0), \tag{16}$$

where $\langle \mathrm{Im}(\Psi) \rangle$ denotes the full triangulated subcategory generated by objects of the form $\Psi(\mathcal{F})$ and $u_q\text{-Mod}_0$ is the block (direct summand) of the category u_q-Mod which contains the trivial representation; and

$$\Psi(\mathcal{O}_{\tilde{\mathcal{N}}}) = \mathbf{k}. \tag{17}$$

The following slight generalization of this result is proved by a straightforward modification of the argument in [BL07].

Proposition 8. *Let C be a subtorus in the maximal torus T, and let $u_q\text{-mod}^C$ be the category of u_q-modules carrying a compatible grading by weights of C. There exists a functor $\Psi^C : D^b(\mathrm{Coh}^{C \times \mathbb{G}_m}(\tilde{\mathcal{N}})) \to D^b(u_q\text{-mod}^C)$ that satisfies properties (14)–(17) above.*

6.3 Semi-infinite cohomology as cohomology with support

From now on, we fix C to be a copy of the multiplicative group corresponding to the coweight used to define the grading on u_q (see the beginning of this section); thus we have $u_q\text{-mod}^C = u_q\text{-mod}$.

Theorem 2. *For $\mathcal{F} \in D^b(\mathrm{Coh}^{C \times \mathbb{G}_m})$, we have a canonical isomorphism*

$$\mathrm{Ext}^{\infty/2+i}_{u_q}(\mathbf{k}, \Psi(\mathcal{F})) \cong R\Gamma^i_{\mathfrak{n}}(\pi_*(\mathcal{F})).$$

Proof. We have

$$R\Gamma^\bullet_{\mathfrak{n}}(\pi_*(\mathcal{F})) \cong \varinjlim \mathrm{Ext}^\bullet(\mathcal{O}_{\tilde{\mathcal{N}}}/\pi^*(I), \mathcal{F}),$$

where I runs over $C \times G_m$-invariant ideals in $\mathcal{O}_{\mathcal{N}}$ with support on $\mathfrak{n}$. We have a canonical arrow $\Psi(\mathcal{O}_{\tilde{\mathcal{N}}}) \to \Psi(\mathcal{O}_{\tilde{\mathcal{N}}}/\pi^*(I))$, and in view of Proposition 7 we have $\Psi(\mathcal{O}_{\tilde{\mathcal{N}}}/\pi^*(I)) \subset D_{\infty/2}$. Thus, by Theorem 1, we have a natural map

$$R\Gamma_{\mathfrak{n}}(\pi_*(\mathcal{F})) \longrightarrow \mathrm{Ext}^{\infty/2+i}_{u_q}(\mathbf{k}, \Psi(\mathcal{F})).$$

---

[1] In fact, $D^b(\mathrm{Coh}^{\mathbb{G}_m}(\tilde{\mathcal{N}}))$ can be identified with the derived category of a block in the category of graded modules over u_q compatible with a certain grading on u_q, as defined in [AJS94]. However, unlike the natural grading by weights and its modifications, this grading is neither explicit nor elementary; it is similar to a grading on the category O of $\mathfrak{g}$-modules with highest weight arising from Hodge weights on the Hom space between Hodge D-modules or from Frobenius weights.

https://doi.org/10.1112/S0010437X09004382 Published online by Cambridge University Press

In view of Proposition 1, to check that this map is an isomorphism it suffices to show that the pro-object $\widehat{\Psi(\mathcal{O})}$ in $D_{\infty/2}$ defined by $\widehat{\Psi(\mathcal{O})} = \varprojlim \Psi(\mathcal{O}_{\tilde{\mathcal{N}}}/\pi^*(I))$ represents the same functor on $D_{\infty/2}$ as the object $\mathbf{k} = \Psi(\mathcal{O}) \in D$.

Let $X \in D_{\infty/2}$, and let $f_1, \ldots, f_n$ be a regular sequence in $\mathcal{O}(\mathcal{N})$ whose common set of zeros equals $\mathfrak{n}$. We can and will assume that f_i is an eigenfunction for the action of $C \times \mathbb{G}_m$. There exists N such that f_i^N maps to $0 \in \mathrm{End}^\bullet(X)$. Then any morphism $\mathbf{k} \to X$ factors through $\mathbf{k}_N = \Psi(\mathcal{O}/(f_i^N))$. This shows that the map $\varinjlim \mathrm{Hom}(\mathbf{k}_N, X) \to \mathrm{Hom}(\mathbf{k}, X)$ is surjective. Similarly, for large N, the map $\mathrm{Hom}(\mathbf{k}_N, X) \to \mathrm{Hom}(\mathbf{k}_{2N}, X)$ kills the kernel of the map $\mathrm{Hom}(\mathbf{k}_N, X) \to \mathrm{Hom}(\mathbf{k}, X)$. Thus the map $\varinjlim \mathrm{Hom}(\mathbf{k}_N, X) \to \mathrm{Hom}(\mathbf{k}, X)$ is injective. □

Corollary 2. *Let T be a tilting module over Lusztig's 'big' quantum group U_q. The semi-infinite cohomology $\mathrm{Ext}_{u_q}^{\infty/2+i}(\mathbf{k}, T)$ either vanishes or is canonically isomorphic to $R\Gamma_{\mathfrak{n}}(\mathcal{F})$, where $\mathcal{F} \in D^b(\mathrm{Coh}^G(\mathcal{N}))$ is a certain (explicit) irreducible object in the heart of the perverse t-structure corresponding to the middle perversity (see [AB10, Bez00a]).*

Proof. By the result of [Bez06], we have $T = \Psi(\tilde{F})$ for some $\tilde{F} \in D^b(\mathrm{Coh}^{G\times\mathbb{G}_m}(\tilde{\mathcal{N}}))$ such that $\pi_*(\tilde{F})$ either vanishes or is an (explicit) irreducible perverse equivariant coherent sheaf as above. The statement now follows from Theorem 2. □

Example 3. If $T = \mathbf{k}$ is the trivial module, then it is clear from the construction of [Bez06] that we can set $\tilde{F} = \mathcal{O}_{\tilde{\mathcal{N}}}$. Thus $\mathcal{F} \cong \mathcal{O}_{\mathcal{N}}$, and so Corollary 2 yields the main result of [Ark97a, Ark98b].

Acknowledgements

We are grateful to S. Arkhipov for helpful discussions. This work owes its existence to W. Soergel: when refereeing the (presently unpublished) preprint [Bez00b] for the *Journal of Algebra*, he suggested extending the results to greater generality. This is accomplished in the present paper, and we thank Wolfgang for his stimulating suggestion.

References

AJS94 H. H. Andersen, J. C. Jantzen and W. Soergel, *Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p*, Astérisque **220** (1994).

AB10 D. Arinkin and R. Bezrukavnikov, *Perverse coherent sheaves*, Moscow Math. J. (2010), to appear.

Ark97a S. Arkhipov, *Semiinfinite cohomology of quantum groups*, Commun. Math. Phys. **188** (1997), 379–405.

Ark97b S. Arkhipov, *Semi-infinite cohomology of associative algebras and bar duality*, Int. Math. Res. Not. **17** (1997), 833–863.

Ark98a S. Arkhipov, *Semi-infinite cohomology of quantum groups. II*, in *Topics in quantum groups and finite-type invariants*, American Mathematical Society Translations, Series 2, vol. 185 (American Mathematical Society, Providence, RI, 1998), 3–42.

Ark98b S. Arkhipov, *A proof of Feigin's conjecture*, Math. Res. Lett. **5** (1998), 403–422.

ABG04 S. Arkhipov, R. Bezrukavnikov and V. Ginzburg, *Quantum groups, the loop Grassmannian, and the Springer resolution*, J. Amer. Math. Soc. **17** (2004), 595–678.

Bez00a R. Bezrukavnikov, *Perverse coherent sheaves (after Deligne)*, Preprint (2000), arXiv:math.AG/0005152.

https://doi.org/10.1112/S0010437X09004382 Published online by Cambridge University Press

Bez00b R. Bezrukavnikov, *On semi-infinite cohomology of finite dimensional algebras*, Preprint (2000), arXiv:math.RT/0005148.

Bez06 R. Bezrukavnikov, *Cohomology of tilting modules over quantum groups, and t-structures on derived categories of coherent sheaves*, Invent. Math. **166** (2006), 327–357.

BFS98 R. Bezrukavnikov, M. Finkelberg and V. Schechtman, *Factorizable sheaves and quantum groups*, Lecture Notes in Mathematics, vol. 1691 (Springer, Berlin, 1998).

BL07 R. Bezrukavnikov and A. Lachowska, *The small quantum group and the Springer resolution*, in *Quantum groups (Proceedings of the Haifa conference, 2004, in memory of J. Donin)*, Contemporary Mathematics, vol. 433 (American Mathematical Society, Providence, RI, 2007), 89–101.

Brz02 T. Brzezinski, *The structure of corings: induction functors, Maschke-type theorem, and Frobenius and Galois-type properties*, Algebr. Represent. Theory **5** (2002), 389–410.

BW03 T. Brzezinski and R. Wisbauer, *Corings and comodules*, London Mathematical Society Lecture Note Series, vol. 309 (Cambridge University Press, Cambridge, 2003).

GK93 V. Ginzburg and S. Kumar, *Cohomology of quantum groups at roots of unity*, Duke Math. J. **69** (1993), 179–198.

Har66 R. Hartshorne, *Residues and duality*, Lecture Notes in Mathematics, vol. 20 (Springer, Berlin, 1966).

Lus90 G. Lusztig, *Finite-dimensional Hopf algebras arising from quantized universal enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 257–296.

Pos07 L. Positselski, *Homological algebra of semimodules and semicontramodules*, in *Semi-infinite homological algebra of associative algebraic structures* (with appendices coauthored by S. Arkhipov and D. Rumynin), Preprint (2007), arXiv:0708.3398, to appear in Birkhäuser's series Monografie Matematyczne, vol. 70 (2010).

Sev01 A. Sevostyanov, *Semi-infinite cohomology and Hecke algebras*, Adv. Math. **159** (2001), 83–141.

Vor93 A. Voronov, *Semi-infinite homological algebra*, Invent. Math. **113** (1993), 103–146.

Roman Bezrukavnikov bezrukav@math.mit.edu
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Leonid Positselski posic@mccme.ru
Sector of Algebra and Number Theory, Institute for Information Transmission Problems,
Moscow 127994, Russia

https://doi.org/10.1112/S0010437X09004382 Published online by Cambridge University Press