Bull. Aust. Math. Soc. **100** (2019), 304–316
doi:10.1017/S000497271900025X

A SYSTEM OF FUNCTIONAL EQUATIONS SATISFIED BY COMPONENTS OF A QUADRATIC FUNCTION AND ITS STABILITY

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(Received 5 October 2018; accepted 11 January 2019; first published online 27 February 2019)

Abstract

A system of functional equations satisfied by the components of a quadratic function is derived via their corresponding circulant matrix. Given such a system of functional equations, general solutions are determined and a stability result for such a system is established.

2010 *Mathematics subject classification*: primary 39B72; secondary 39B82.

Keywords and phrases: quadratic functional equation, circulant matrix, component function, stability.

1. Introduction

Let $n \in \mathbb{N}$, $n \geq 2$, and let $\omega_n := \exp(2\pi i/n)$ be a primitive nth root of unity. A type-j function, first introduced by Schwaiger in [8], is a function $f : \mathbb{C} \to \mathbb{C}$ satisfying

$$f(\omega_n x) = \omega_n^j f(x).$$

They are referred to as the components of f because

$$f = \sum_{j=0}^{n-1} f_j \quad \text{where } f_j(x) = \frac{1}{n} \sum_{k=0}^{n-1} \omega_n^{-kj} f(\omega_n^k x).$$

Applying this concept, Schwaiger [8] derived and solved the following system of functional equations satisfied by the components of an exponential function.

$$f_j(x + \omega_n^m y) = \sum_{\ell=0}^{j} \omega_n^{(j-\ell)m} f_\ell(x) f_{j-\ell}(y) + \sum_{\ell=j+1}^{n-1} \omega_n^{(n+j-\ell)m} f_\ell(x) f_{n+j-\ell}(y), \tag{1.1}$$

for $j = 0, 1, \ldots, n-1$, where $m \in \{0, 1, \ldots, n-1\}$ is fixed. The stability of the system (1.1) was established one year later by Förg-Rob and Schwaiger in [2]. In 2005, Muldoon [7] simplified and systematised the results in [8] and [2] through the use of a circulant matrix.

A quadratic function is a function $q : \mathbb{C} \to \mathbb{C}$ satisfying

$$q(x+y) + q(x-y) = 2q(x) + 2q(y) \quad (x, y \in \mathbb{C}). \tag{Q}$$

Note that quadratic functions are even functions, that is, $q(-x) = q(x)$ for $x \in \mathbb{C}$.

Using Muldoon's approach, we find here a system of functional equations satisfied by the components of a quadratic function via their corresponding circulant matrix. Given such a system of functional equations, their solutions are determined and the stability of such a system is investigated.

2. Preliminary results

Throughout, let n be a fixed integer ≥ 2 and let $\omega_n = \exp(2\pi i/n)$ be a primitive nth root of unity. As in Muldoon [7], the following notation is adopted.

The $n \times n$ (symmetric) Fourier matrix and its complex-conjugate matrix are defined, respectively, by

$$\mathscr{F}_n = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega_n^{-1} & \cdots & \omega_n^{-(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{-(n-1)} & \cdots & \omega_n^{-(n-1)^2} \end{bmatrix}, \quad \mathscr{F}_n^* = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \cdots & \omega_n^{(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{(n-1)} & \cdots & \omega_n^{(n-1)^2} \end{bmatrix}.$$

Note that $\mathscr{F}_n$ is unitary, that is, $\mathscr{F}_n \mathscr{F}_n^* = I_n = \mathscr{F}_n^* \mathscr{F}_n$, where I_n denotes the $n \times n$ identity matrix.

The diagonal matrix Ω_n is defined by

$$\Omega_n = \operatorname{diag}(1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}) := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_n^{n-1} \end{bmatrix}.$$

Given a sequence $\{a_0, \ldots, a_{n-1}\} \subset \mathbb{C}$, its circulant matrix is defined by

$$\operatorname{circ}(a_0, a_1, \ldots, a_{n-1}) := \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{bmatrix}$$

and its diagonal matrix is defined by

$$\operatorname{diag}(a_0, a_1, \ldots, a_{n-1}) := \begin{bmatrix} a_0 & 0 & \cdots & 0 \\ 0 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n-1} \end{bmatrix}.$$

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The circulant matrix corresponding to the sequence $\{0, 1, 0, \ldots, 0\}$ is

$$\pi_n := \mathrm{circ}(0, 1, 0, \ldots, 0) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Observe that

(1) $\pi_n^{-1} = \pi_n^T$ (T denoting transpose), that is, π_n is orthogonal.

(2) The circulant matrix $\mathrm{circ}(a_0, a_1, \ldots, a_{n-1})$ can be written as

$$\mathrm{circ}(a_0, a_1, \ldots, a_{n-1}) = a_0 I_n + a_1 \pi_n + \cdots + a_{n-1}\pi_n^{n-1}.$$

(3) $\mathscr{F}_n^* \Omega_n \mathscr{F}_n = \pi_n$ and, equivalently, $\Omega_n = \mathscr{F}_n \pi_n \mathscr{F}_n^*$.

The following basic results are taken from [7].

Lemma 2.1 [7, Lemmas 2.1 and 2.2].

(I) *If* $A = \mathrm{circ}(a_0, a_1, \ldots, a_{n-1})$, *then*

$$\mathscr{F}_n A \mathscr{F}_n^* = \sqrt{n}\,\mathrm{diag}(\mathscr{F}_n^* \bar{a})^T, \quad \bar{a} = \begin{bmatrix} a_0 \\ \vdots \\ a_{n-1} \end{bmatrix}.$$

(II) *Let m be a nonnegative integer. If A is a circulant matrix, then*

$$\mathscr{F}_n(\Omega_n^{-m} A \Omega_n^m)\mathscr{F}_n^* = \pi_n^m(\mathscr{F}_n A \mathscr{F}_n^*)\pi_n^{-m}.$$

Lemma 2.2 [7, Lemma 2.4]. *Any $f : \mathbb{C} \to \mathbb{C}$ can be written uniquely as a sum of functions f_j ($j \in \{0, 1, \ldots, n-1\}$) of type-$j$ (called its j-components):*

$$f(x) = f_0(x) + f_1(x) + \cdots + f_{n-1}(x),$$

where

$$\begin{bmatrix} f_0(x) \\ f_1(x) \\ \vdots \\ f_{n-1}(x) \end{bmatrix} = \frac{1}{\sqrt{n}}\mathscr{F}_n \begin{bmatrix} f(x) \\ f(\omega_n x) \\ \vdots \\ f(\omega_n^{n-1} x) \end{bmatrix} = \frac{1}{\sqrt{n}}\mathscr{F}_n^* \begin{bmatrix} f(x) \\ f(\omega_n^{-1} x) \\ \vdots \\ f(\omega_n^{-(n-1)} x) \end{bmatrix}.$$

The circulant matrix corresponding to a function f, whose j-components are f_j, is

$$\mathbf{F}(x) := \mathrm{circ}(f_0(x), f_1(x), \ldots, f_{n-1}(x)) = \begin{bmatrix} f_0(x) & f_1(x) & \cdots & f_{n-1}(x) \\ f_{n-1}(x) & f_0(x) & \cdots & f_{n-2}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(x) & f_2(x) & \cdots & f_0(x) \end{bmatrix}.$$

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LEMMA 2.3. *The circulant matrix function* $\mathbf{F}(x)$ *corresponding to* $f : \mathbb{C} \to \mathbb{C}$ *satisfies*

(I) $\mathbf{F}(x) = \mathscr{F}_n^* \operatorname{diag}(f(x), f(\omega_n x), \ldots, f(\omega_n^{n-1} x))\mathscr{F}_n$ *and, equivalently,*

$$\mathscr{F}_n \mathbf{F}(x) \mathscr{F}_n^* = \operatorname{diag}(f(x), f(\omega_n x), \ldots, f(\omega_n^{n-1} x));$$

(II) $\mathbf{F}(\omega_n^m x) = \Omega_n^{-m} \mathbf{F}(x) \Omega_n^m$ *for each* $m \in \mathbb{N}$.

PROOF. Part I is Lemma 2.6 in [7]. The case $m = 1$ in Part II is Lemma 2.8 in [7]. We proceed now to prove the general case of $m \in \mathbb{N}$. By multiplying the three matrices,

$$\Omega_n^{-m} \mathbf{F}(x) \Omega_n^m = \begin{bmatrix} f_0(x) & \omega_n^m f_1(x) & \cdots & \omega_n^{(n-1)m} f_{n-1}(x) \\ \omega_n^{(n-1)m} f_{n-1}(x) & f_0(x) & \cdots & \omega_n^{(n-2)m} f_{n-2}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_n^m f_1(x) & \omega_n^{2m} f_2(x) & \cdots & f_0(x) \end{bmatrix}$$

$$= \begin{bmatrix} f_0(\omega_n^m x) & f_1(\omega_n^m x) & \cdots & f_{n-1}(\omega_n^m x) \\ f_{n-1}(\omega_n^m x) & f_0(\omega_n^m x) & \cdots & f_{n-2}(\omega_n^m x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(\omega_n^m x) & f_2(\omega_n^m x) & \cdots & f_0(\omega_n^m x) \end{bmatrix} = \mathbf{F}(\omega_n^m x).$$

This completes the proof of Lemma 2.3. □

LEMMA 2.4.

(I) *Let m be a nonnegative integer. If* $B = \operatorname{diag}(b_0, b_1, \ldots, b_{n-1})$, *then*

$$\pi_n^m B \pi_n^{-m} = \operatorname{diag}(b_m, b_{m+1}, \ldots, b_{m+n-1}),$$

where suffixes are taken modulo n.

(II) *If B is a diagonal matrix, then $\mathscr{F}_n^* B \mathscr{F}_n$ is a circulant matrix.*

PROOF. (I) When $m = 1$, the result follows by multiplying the matrices:

$$\pi_n B \pi_n^{-1} = \pi_n \operatorname{diag}(b_0, b_1, \ldots, b_{n-1}) \pi_n^{-1}$$

$$= \begin{bmatrix} 0 & b_1 & 0 & \cdots & 0 \\ 0 & 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{n-1} \\ b_0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^T = \operatorname{diag}(b_1, b_2, \ldots, b_0).$$

Assume the result holds up to m, that is, $\pi_n^m B \pi_n^{-m} = \operatorname{diag}(b_m, b_{m+1}, \ldots, b_{m+n-1})$, where suffixes are taken modulo n. Since $\pi_n^{m+1} B \pi_n^{-m-1} = \pi_n(\pi_n^m B \pi_n^{-m})\pi_n^{-1}$, using the induction hypothesis and the result of the case $m = 1$,

$$\pi_n^{m+1} B \pi_n^{-m-1} = \pi_n \operatorname{diag}(b_m, b_{m+1}, \ldots, b_{m+n-1}) \pi_n^{-1} = \operatorname{diag}(b_{m+1}, b_{m+2}, \ldots, b_{m+n}),$$

as desired.

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(II) The result follows from another matrix calculation:

$$\mathscr{F}_n^* B \mathscr{F}_n = \begin{bmatrix} d_0 & d_1 & \cdots & d_{n-1} \\ d_{n-1} & d_0 & \cdots & d_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ d_1 & d_2 & \cdots & d_0 \end{bmatrix} = \operatorname{circ}(d_0, d_1, \ldots, d_{n-1}),$$

where $d_j = (1/n) \sum_{k=0}^{n-1} \omega_n^{n-kj} b_k \ \ (j = 0, 1, \ldots, n-1)$. □

3. A system of functional equations

In this section, we first find a system of functional equations satisfied by the components of a quadratic function (see (Q) in Section 1) via the corresponding circulant matrix, and then consider the problem of solving such a system.

Theorem 3.1. *If $f : \mathbb{C} \to \mathbb{C}$ satisfies (Q), then its corresponding circulant matrix function $\mathbf{F}(x)$ satisfies*

$$\mathbf{F}(\omega_n^m x + y) + \mathbf{F}(\omega_n^m x - y) = 2\Omega_n^{-m} \mathbf{F}(x) \Omega_n^m + 2\mathbf{F}(y)$$

for any $m \in \{0, 1, \ldots, n-1\}$.

Proof. From Lemma 2.3(I) and Lemma 2.3(II),

$$\begin{aligned} &\mathbf{F}(\omega_n^m x + y) + \mathbf{F}(\omega_n^m x - y) \\ &\quad = \mathscr{F}_n^* \operatorname{diag}(f(\omega_n^m x + y) + f(\omega_n^m x - y), f(\omega_n(\omega_n^m x + y)) + f(\omega_n(\omega_n^m x - y)), \\ &\qquad \ldots, f(\omega_n^{n-1}(\omega_n^m x + y)) + f(\omega_n^{n-1}(\omega_n^m x - y))) \mathscr{F}_n \\ &\quad = \mathscr{F}_n^* \operatorname{diag}(2f(\omega_n^m x) + 2f(y), 2f(\omega_n(\omega_n^m x)) + 2f(\omega_n y), \\ &\qquad \ldots, 2f(\omega_n^{n-1}(\omega_n^m x)) + 2f(\omega_n^{n-1} y)) \mathscr{F}_n \\ &\quad = 2\mathbf{F}(\omega_n^m x) + 2\mathbf{F}(y) = 2\Omega_n^{-m} \mathbf{F}(x) \Omega_n^m + 2\mathbf{F}(y). \end{aligned}$$

□

The following lemma, whose easy proof is omitted, is needed in the proof of Theorem 3.3.

Lemma 3.2. *Let $m \in \{0, 1, \ldots, n-1\}$ be fixed and let $d = \gcd(n, m)$. Then for every $s, u \in \{0, 1, \ldots, d-1\}$ and $t, v \in \{0, 1, \ldots, n/d - 1\}$,*

$$s + tm \not\equiv u + vm \pmod{n},$$

except when $s = u$ and $t = v$.

Theorem 3.3. *Let $\mathbf{F}(x)$ be a circulant matrix with first row $(f_0(x), f_1(x), \ldots, f_{n-1}(x))$, where $f_i : \mathbb{C} \to \mathbb{C}$ are arbitrary functions which need not be components of the same function. If $\mathbf{F}$ satisfies*

$$\mathbf{F}(\omega_n^m x + y) + \mathbf{F}(\omega_n^m x - y) = 2\Omega_n^{-m} \mathbf{F}(x) \Omega_n^m + 2\mathbf{F}(y), \tag{3.1}$$

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for a fixed $m \in \{0, 1, \ldots, n-1\}$*, then putting* $d := \gcd(m, n)$*, when* $d = n$*,*

$$\begin{bmatrix} f_0(x) \\ f_1(x) \\ \vdots \\ f_{n-1}(x) \end{bmatrix} = \frac{1}{\sqrt{n}} \mathscr{F}_n \begin{bmatrix} B_0(x,x) \\ B_1(x,x) \\ \vdots \\ B_{n-1}(x,x) \end{bmatrix},$$

and when $1 \le d < n$*,*

$$\begin{bmatrix} f_0(x) \\ f_1(x) \\ \vdots \\ f_{n-1}(x) \end{bmatrix} = \frac{1}{\sqrt{n}} \mathscr{F}_n \begin{bmatrix} \alpha(x) \\ \alpha(\omega_n^m x) \\ \vdots \\ \alpha(\omega_n^{m(n/d-1)} x) \end{bmatrix} \quad \text{with } \alpha(x) = \begin{bmatrix} B_0(x,x) \\ B_1(x,x) \\ \vdots \\ B_{d-1}(x,x) \end{bmatrix},$$

where the $B_i : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *are symmetric, bi-additive functions defined by*

$$B_i(x,y) = \frac{1}{4}(g_i(x+y) - g_i(x-y)), \quad \text{with } g_i(x) := \sum_{k=0}^{n-1} \omega^{ik} f_k(x) \quad (i = 0, \ldots, n-1).$$

Proof. Suppose that $\mathbf{F}(x)$ satisfies (3.1). Then

$$\mathscr{F}_n \mathbf{F}(\omega_n^m x + y) \mathscr{F}_n^* + \mathscr{F}_n \mathbf{F}(\omega_n^m x - y) \mathscr{F}_n^* = 2\mathscr{F}_n \Omega_n^{-m} \mathbf{F}(x) \Omega_n^m \mathscr{F}_n^* + 2\mathscr{F}_n \mathbf{F}(y) \mathscr{F}_n^*.$$

Using Lemma 2.1(I) and (II), this equation becomes

$$\mathbf{G}(\omega_n^m x + y) + \mathbf{G}(\omega_n^m x - y) = 2\mathbf{G}_m(x) + 2\mathbf{G}(y), \tag{3.2}$$

where we write

$$\operatorname{diag}(g_0(x), g_1(x), \ldots, g_{n-1}(x)) = \mathbf{G}(x) = \mathscr{F}_n \mathbf{F}(x) \mathscr{F}_n^* = \sqrt{n} \operatorname{diag}(\mathscr{F}_n^* \bar{f}(x))^T, \tag{3.3}$$
$$\mathbf{G}_m(x) = \pi_n^m \mathbf{G}(x) \pi_n^{-m}.$$

Equation (3.2) and Lemma 2.4(I) yield a system of n equations

$$\begin{aligned} g_0(\omega_n^m x + y) + g_0(\omega_n^m x - y) &= 2g_m(x) + 2g_0(y) \\ g_1(\omega_n^m x + y) + g_1(\omega_n^m x - y) &= 2g_{m+1}(x) + 2g_1(y) \\ &\vdots \\ g_{n-1}(\omega_n^m x + y) + g_{n-1}(\omega_n^m x - y) &= 2g_{m+n-1}(x) + 2g_{n-1}(y). \end{aligned}$$

Using Lemma 3.2, we subdivide these n equations into d different classes each with n/d equations:

$$g_{k+jm}(\omega_n^m x + y) + g_{k+jm}(\omega_n^m x - y) = 2g_{k+(j+1)m}(x) + 2g_{k+jm}(y), \tag{3.4}$$

where $j = 0, 1, \ldots, n/d - 1$ and $k = 0, 1, \ldots, d-1$. Substituting $x = y = 0$ in (3.4),

$$g_k(0) = g_{k+m}(0) = \cdots = g_{k+(n/d)m}(0) = 0. \tag{3.5}$$

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Substituting $y = 0$ in (3.4) and using (3.5),

$$g_{k+(j+1)m}(x) = g_{k+jm}(\omega_n^m x) \quad (j = 0, 1, \ldots, n/d - 1;\ k = 0, 1, \ldots, d - 1). \tag{3.6}$$

Substituting (3.6) into (3.4),

$$g_{k+jm}(\omega_n^m x + y) + g_{k+jm}(\omega_n^m x - y) = 2g_{k+jm}(\omega_n^m x) + 2g_{k+jm}(y). \tag{3.7}$$

Replacing x by $\omega_n^{-m} x$ in (3.7),

$$g_{k+jm}(x + y) + g_{k+jm}(x - y) = 2g_{k+jm}(x) + 2g_{k+jm}(y).$$

This last relation shows that each g_{k+jm} is a quadratic function. Invoking Theorem 4.1 of [6, page 222],

$$g_{k+jm}(x) = B_{k+jm}(x, x), \tag{3.8}$$

where $B_{k+jm} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ are symmetric bi-additive functions given by

$$B_{k+jm}(x, y) = \tfrac{1}{4}(g_{k+jm}(x + y) - g_{k+jm}(x - y)).$$

If $d = n$, then $m = 0$ and from (3.8),

$$g_k(x) = B_k(x, x) \quad (k = 1, \ldots, n - 1).$$

From (3.3) and the above relation,

$$\begin{bmatrix} f_0(x) \\ f_1(x) \\ \vdots \\ f_{n-1}(x) \end{bmatrix} = \frac{1}{\sqrt{n}} \mathscr{F}_n \begin{bmatrix} g_0(x) \\ g_1(x) \\ \vdots \\ g_{n-1}(x) \end{bmatrix} = \frac{1}{\sqrt{n}} \mathscr{F}_n \begin{bmatrix} B_0(x, x) \\ B_1(x, x) \\ \vdots \\ B_{n-1}(x, x) \end{bmatrix}.$$

If $1 \le d < n$, then the system (3.6) can be rewritten as

$$\begin{aligned} g_{k+m}(x) &= g_k(\omega_n^m x) \\ g_{k+2m}(x) &= g_{k+m}(\omega_n^m x) = g_k(\omega_n^{2m} x) \\ &\vdots \\ g_{k+(n/d)m}(x) &= g_{k+(n/d-1)m}(\omega_n^m x) = \cdots = g_k(\omega_n^{(n/d)m} x). \end{aligned}$$

From (3.8) and these relations,

$$g_{k+jm}(x) = g_k(\omega_n^{jm} x) = B_k(\omega_n^{jm} x, \omega_n^{jm} x) \quad (j = 0, 1, \ldots, n/d - 1;\ k = 0, 1, \ldots, d - 1).$$

From (3.3) and the last relation,

$$\begin{bmatrix} f_0(x) \\ f_1(x) \\ \vdots \\ f_{n-1}(x) \end{bmatrix} = \frac{1}{\sqrt{n}} \mathscr{F}_n \begin{bmatrix} g_0(x) \\ g_1(x) \\ \vdots \\ g_{n-1}(x) \end{bmatrix} = \frac{1}{\sqrt{n}} \mathscr{F}_n \begin{bmatrix} \alpha(x) \\ \alpha(\omega_n^m x) \\ \vdots \\ \alpha(\omega_n^{m(n/d-1)} x) \end{bmatrix}$$

with $\alpha(x) = [B_0(x, x)\ B_1(x, x)\ \cdots\ B_{d-1}(x, x)]^T$. □

We work out two examples for the results of Theorem 3.3 corresponding to the cases $n = 2$ and 3, respectively.

EXAMPLE 3.4. If $n = 2$, then (3.1) becomes

$$\mathbf{F}(\omega_2^m x + y) + \mathbf{F}(\omega_2^m x - y) = 2\Omega_2^{-m}\mathbf{F}(x)\Omega_2^m + 2\mathbf{F}(y) \quad (m = 0, 1) \tag{3.9}$$

where $\mathbf{F}(x) = \mathrm{circ}(f_0(x), f_1(x))$.

For $m = 0$, (3.9) reads

$$\mathbf{F}(x + y) + \mathbf{F}(x - y) = 2\mathbf{F}(x) + 2\mathbf{F}(y) \tag{3.10}$$

and Theorem 3.3 gives $f_0(x) = \frac{1}{2}(B_0(x, x) + B_1(x, x))$, $f_1(x) = \frac{1}{2}(B_0(x, x) - B_1(x, x))$, where B_0, B_1 are symmetric, bi-additive functions. Equating the elements in (3.10),

$$f_i(x + y) + f_i(x - y) = 2f_i(x) + 2f_i(y) \quad (i = 0, 1),$$

showing that f_0, f_1 are quadratic functions.

If we assume that f_0, f_1 are components of a function f, that is, $f(x) = f_0(x) + f_1(x)$, then f is a quadratic function, and so is an even function. Thus, its odd part $f_1(x) \equiv 0$ yielding $B_0(x, x) = B_1(x, x)$ and $f(x) = f_0(x) = B_0(x, x)$, that is, f has only trivial components.

For $m = 1$, (3.9) reads

$$\mathbf{F}(\omega_2 x + y) + \mathbf{F}(\omega_2 x - y) = 2\Omega_2^{-1}\mathbf{F}(x)\Omega_2 + 2\mathbf{F}(y) \tag{3.11}$$

and Theorem 3.3 gives

$$f_0(x) = \tfrac{1}{2}(B_0(x, x) + B_0(\omega_2 x, \omega_2 x)), f_1(x) = \tfrac{1}{2}(B_0(x, x) - B_0(\omega_2 x, \omega_2 x)),$$

where B_0 is a symmetric, bi-additive function. Equating the elements in (3.11),

$$f_i(\omega_2 x + y) + f_i(\omega_2 x - y) = 2\omega_2^i f_i(x) + 2f_i(y) \quad (i = 0, 1). \tag{3.12}$$

Substituting $x = y = 0$ in (3.12),

$$f_i(0) = 0. \tag{3.13}$$

Substituting $y = 0$ in (3.12) and using (3.13),

$$f_i(\omega_2 x) = \omega_2^i f_i(x). \tag{3.14}$$

Replacing y by $\omega_2 y$ in (3.12) and using (3.14),

$$f_i(x + y) + f_i(x - y) = 2f_i(x) + 2f_i(y) \quad (i = 0, 1),$$

showing again that f_0, f_1 are quadratic functions.

If we assume that f_0, f_1 are components of a function f, then as in the previous case f is a quadratic function, $f_1(x) = 0$, and $f(x) = f_0(x) = B_0(x, x)$, that is, f has only trivial components.

EXAMPLE 3.5. If $n = 3$, then (3.1) becomes

$$\mathbf{F}(\omega_3^m x + y) + \mathbf{F}(\omega_3^m x - y) = 2\Omega_3^{-m}\mathbf{F}(x)\Omega_3^m + 2\mathbf{F}(y) \quad (m = 0, 1, 2) \tag{3.15}$$

where $\mathbf{F}(x) = \mathrm{circ}(f_0(x), f(1)(x), f_2(x))$.

For $m = 0$, (3.15) reads

$$\mathbf{F}(x + y) + \mathbf{F}(x - y) = 2\mathbf{F}(x) + 2\mathbf{F}(y) \tag{3.16}$$

and Theorem 3.3 gives

$$\begin{aligned} f_0(x) &= \tfrac{1}{3}(B_0(x, x) + B_1(x, x) + B_2(x, x)) \\ f_1(x) &= \tfrac{1}{3}(B_0(x, x) + \omega_3^2 B_1(x, x) + \omega_3 B_2(x, x)) \\ f_2(x) &= \tfrac{1}{3}(B_0(x, x) + \omega_3 B_1(x, x) + \omega_3^2 B_2(x, x)), \end{aligned} \tag{3.17}$$

where B_0, B_1, B_2 are symmetric, bi-additive functions. Equating the elements in (3.16),

$$f_i(x + y) + f_i(x - y) = 2f_i(x) + 2f_i(y) \quad (i = 0, 1, 2),$$

showing that f_0, f_1, f_2 are quadratic functions.

If we assume that f_0, f_1, f_2 are components of a function f, that is, f is given by $f(x) = f_0(x) + f_1(x) + f_2(x)$, then f is also a quadratic function. In contrast to the case $n = 2$, we now show that f can have nontrivial components. So, suppose that f has only trivial components, that is, the following three possibilities occur.

$$\begin{aligned} &\text{Either} \quad f(x) = f_0(x) \quad \text{and} \quad f_1(x) = f_2(x) = 0; \\ &\text{or} \quad f(x) = f_1(x) \quad \text{and} \quad f_0(x) = f_2(x) = 0; \\ &\text{or} \quad f(x) = f_2(x) \quad \text{and} \quad f_0(x) = f_1(x) = 0. \end{aligned}$$

If $f(x) = f_0(x)$ and $f_1(x) = f_2(x) = 0$, by solving the system (3.17),

$$B_0(x, x) = B_1(x, x) = B_2(x, x). \tag{3.18}$$

If $f(x) = f_1(x)$ and $f_0(x) = f_2(x) = 0$, by solving the system (3.17),

$$B_1(x, x) = \omega_3 B_0(x, x), \quad B_2(x, x) = \omega_3^2 B_0(x, x). \tag{3.19}$$

If $f(x) = f_2(x)$ and $f_0(x) = f_1(x) = 0$, by solving the system (3.17),

$$B_1(x, x) = \omega_3^2 B_0(x, x), \quad B_2(x, x) = \omega_3 B_0(x, x). \tag{3.20}$$

Since the three symmetric bi-additive functions B_0, B_1, B_2 are arbitrary, it is possible to choose these B_j in such a way that that the three requirements (3.18), (3.19) and (3.20) do not hold.

We leave the discussion of the remaining cases ($m = 1, 2$) to the reader.

4. Stability

The concept of the stability of functional equations arose in 1940 when Ulam in [9] asked: Under what conditions does there exist an additive mapping near an approximately additive mapping? This question was answered in 1941 by Hyers [3] with the result: If $f : E_1 \to E_2$ is a mapping satisfying

$$\|f(x + y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E_1$, where E_1 and E_2 are Banach spaces and δ is a given positive number, then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E_1$. If $f(x)$ is a real continuous function of x over $\mathbb{R}$, and

$$|f(x + y) - f(x) - f(y)| \leq \delta,$$

it was shown by Hyers and Ulam [5] that there exists a constant k such that

$$|f(x) - kx| \leq 2\delta.$$

For recent developments, see [1, 4]. In this section, we establish the stability of the circulant matrix functional equation

$$\mathbf{F}(\omega_n^m x + y) + \mathbf{F}(\omega_n^m x - y) = 2\Omega_n^{-m}\mathbf{F}(x)\Omega_n^m + 2\mathbf{F}(y).$$

As in [7], we use the usual 1-norm for a square matrix $A = (a_{ij})$ defined by

$$\|A\| = \max_{0\leq i\leq n-1} \sum_{j=0}^{n-1} |a_{ij}|.$$

Theorem 4.1. *Let* $\mathbf{F}(x)$ *be a circulant matrix whose first row is* $(f_0(x), f_1(x), \ldots, f_{n-1}(x))$, *where* $f_i : \mathbb{C} \to \mathbb{C}$ *are arbitrary functions which need not be components of the same function, and let* $\varepsilon > 0$. *If* $\mathbf{F}$ *satisfies*

$$\|\mathbf{F}(\omega_n^m x + y) + \mathbf{F}(\omega_n^m x - y) - 2\Omega_n^{-m}\mathbf{F}(x)\Omega_n^m - 2\mathbf{F}(y)\| \leq \varepsilon, \tag{4.1}$$

for a fixed $m \in \{0, 1, \ldots, n-1\}$, *then there exists a circulant matrix* $\mathbf{Q}(x)$ *satisfying the matrix functional equation*

$$\mathbf{Q}(x + y) + \mathbf{Q}(x - y) = 2\mathbf{Q}(x) + 2\mathbf{Q}(y) \tag{4.2}$$

such that

$$\|\mathbf{F}(x) - \mathbf{Q}(x)\| \leq \frac{5n^3}{2}\varepsilon.$$

https://doi.org/10.1017/S000497271900025X Published online by Cambridge University Press

Proof. Multiplying by $\|\mathscr{F}_n\|$ on the left-hand side and by $\|\mathscr{F}_n^*\|$ on the right-hand side of (4.1),

$$\|\mathscr{F}_n\mathbf{F}(\omega_n^m x+y)\mathscr{F}_n^* + \mathscr{F}_n\mathbf{F}(\omega_n^m x-y)\mathscr{F}_n^* - 2\mathscr{F}_n\Omega_n^{-m}\mathbf{F}(x)\Omega_n^m\mathscr{F}_n^* - 2\mathscr{F}_n\mathbf{F}(y)\mathscr{F}_n^*\| \\ \leq \|\mathscr{F}_n\|\varepsilon\|\mathscr{F}_n^*\|.$$

By Lemma 2.1(I) and (II), this last inequality becomes

$$\|\mathbf{G}(\omega_n^m x+y)+\mathbf{G}(\omega_n^m x-y)-2\mathbf{G}_m(x)-2\mathbf{G}(y)\| \leq n\varepsilon, \tag{4.3}$$

where

$$\operatorname{diag}(g_0(x), g_1(x), \ldots, g_{n-1}(x)) = \mathbf{G}(x) = \mathscr{F}_n\mathbf{F}(x)\mathscr{F}_n^* = \sqrt{n}\,\operatorname{diag}\,(\mathscr{F}_n^*\bar{f}(x))^T, \\ \mathbf{G}_m(x) = \pi_n^m\mathbf{G}(x)\pi_n^{-m}.$$

Putting $x=y=0$ in (4.3),

$$\|\mathbf{G}_m(0)\| \leq \frac{n\varepsilon}{2}. \tag{4.4}$$

Putting $x=0$ in (4.3) and using (4.4),

$$\|\mathbf{G}(y)-\mathbf{G}(-y)\| \leq 2n\varepsilon \quad (y\in\mathbb{C}). \tag{4.5}$$

Replacing x by $x+z$ and $x-z$, respectively, in (4.3),

$$\|\mathbf{G}(\omega_n^m x+\omega_n^m z+y)+\mathbf{G}(\omega_n^m x+\omega_n^m z-y)-2\mathbf{G}_m(x+z)-2\mathbf{G}(y)\| \leq n\varepsilon \tag{4.6}$$
$$\|\mathbf{G}(\omega_n^m x-\omega_n^m z+y)+\mathbf{G}(\omega_n^m x-\omega_n^m z-y)-2\mathbf{G}_m(x-z)-2\mathbf{G}(y)\| \leq n\varepsilon. \tag{4.7}$$

Replacing y by $y+\omega_n^m z$ and $y-\omega_n^m z$, respectively, in (4.3),

$$\|\mathbf{G}(\omega_n^m x+y+\omega_n^m z)+\mathbf{G}(\omega_n^m x-y-\omega_n^m z)-2\mathbf{G}_m(x)-2\mathbf{G}(y+\omega_n^m z)\| \leq n\varepsilon \tag{4.8}$$
$$\|\mathbf{G}(\omega_n^m x+y-\omega_n^m z)+\mathbf{G}(\omega_n^m x-y+\omega_n^m z)-2\mathbf{G}_m(x)-2\mathbf{G}(y-\omega_n^m z)\| \leq n\varepsilon. \tag{4.9}$$

Using (4.6) and (4.8),

$$\|\mathbf{G}(\omega_n^m x+\omega_n^m z-y)-\mathbf{G}(\omega_n^m x-y-\omega_n^m z)-2\mathbf{G}_m(x+z) \\ -2\mathbf{G}(y)+2\mathbf{G}_m(x)+2\mathbf{G}(y+\omega_n^m z)\| \leq 2n\varepsilon. \tag{4.10}$$

Using (4.7) and (4.9),

$$\|\mathbf{G}(\omega_n^m x-\omega_n^m z-y)-\mathbf{G}(\omega_n^m x-y+\omega_n^m z)-2\mathbf{G}_m(x-z) \\ -2\mathbf{G}(y)+2\mathbf{G}_m(x)+2\mathbf{G}(y-\omega_n^m z)\| \leq 2n\varepsilon. \tag{4.11}$$

Using (4.10) and (4.11),

$$\|\mathbf{G}_m(x+z)+\mathbf{G}_m(x-z)-2\mathbf{G}_m(x)+2\mathbf{G}(y)-\mathbf{G}(y+\omega_n^m z)-\mathbf{G}(y-\omega_n^m z)\| \leq 2n\varepsilon. \tag{4.12}$$

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Replacing x by z in (4.3),

$$\|\mathbf{G}(\omega_n^m z + y) + \mathbf{G}(\omega_n^m z - y) - 2\mathbf{G}_m(z) - 2\mathbf{G}(y)\| \le n\varepsilon. \tag{4.13}$$

Using (4.12) and (4.13),

$$\|\mathbf{G}_m(x+z) + \mathbf{G}_m(x-z) - 2\mathbf{G}_m(x) - 2\mathbf{G}_m(z) + \mathbf{G}(\omega_n^m z - y) - \mathbf{G}(y - \omega_n^m z)\| \le 3n\varepsilon. \tag{4.14}$$

Using (4.5), the inequality (4.14) becomes

$$\|\mathbf{G}_m(x+z) + \mathbf{G}_m(x-z) - 2\mathbf{G}_m(x) - 2\mathbf{G}_m(z)\| \le 5n\varepsilon.$$

By Lemma 2.4(I), the elements of $\mathbf{G}_m(x)$ and $\mathbf{G}(x)$ are the same (but possibly in a different order), and so

$$\|\mathbf{G}(x+z) + \mathbf{G}(x-z) - 2\mathbf{G}(x) - 2\mathbf{G}(z)\| \le 5n\varepsilon.$$

Since $\mathbf{G}(x) = \operatorname{diag}(g_0(x), g_1(x), \dots, g_{n-1}(x))$, by the definition of norm,

$$|g_i(x+z) + g_i(x-z) - 2g_i(x) - 2g_i(z)| \le 5n\varepsilon \quad (i = 0, 1, \dots, n-1).$$

By Theorem 6.24 of [6, page 323], there exist unique quadratic functions $h_i : \mathbb{C} \to \mathbb{C}$ satisfying (Q) such that

$$|g_i(x) - h_i(x)| \le \frac{5n}{2}\varepsilon \quad (i = 0, 1, \dots, n-1).$$

Let $\mathbf{H}(x) := \operatorname{diag}(h_0(x), h_1(x), \dots, h_{n-1}(x))$. Using the definition of norm,

$$\|\mathbf{G}(x) - \mathbf{H}(x)\| = \max_{0\le i\le n-1} \sum_{j=0}^{n-1} |g_{ij}(x) - h_{ij}(x)| \le \frac{5n^2}{2}\varepsilon.$$

Multiplying by $\|\mathscr{F}_n^*\|$ on the left-hand side and by $\|\mathscr{F}_n\|$ on the right-hand side of the last relation and noting that $\mathbf{F}(x) = \mathscr{F}_n^* \mathbf{G}(x) \mathscr{F}_n$,

$$\|\mathbf{F}(x) - \mathbf{Q}(x)\| \le \frac{5n^3}{2}\varepsilon,$$

where $\mathbf{Q}(x) = \mathscr{F}_n^* \mathbf{H}(x) \mathscr{F}_n$. Since $\mathbf{H}(x)$ is a diagonal matrix, Lemma 2.4(II) and its proof show that $\mathbf{Q}(x)$ is a circulant matrix whose first row is $(q_0(x), q_1(x), \dots, q_{n-1}(x))$, where

$$q_j(x) = \frac{1}{n} \sum_{k=0}^{n-1} \omega_n^{n-kj} h_k(x) \ \ (j = 0, 1, \dots, n-1).$$

Since each h_k satisfies (Q), the function elements q_j satisfy (Q), that is $\mathbf{Q}(x)$ satisfies (4.2). □

https://doi.org/10.1017/S000497271900025X Published online by Cambridge University Press

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