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DIRECT FINITENESS OF CERTAIN MONOID ALGEBRAS

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A semigroup is said to be completely regular if and only if each of its elements lies in a subgroup. It is shown that the algebra of a completely regular monoid (semigroup with identity) over a field of characteristic zero is directly finite.

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A ring R with unity 1 is termed *directly finite* (or *von Neumann finite*) if and only if, for all $a,b\in R, ab=1$ implies $ba=1$. Kaplansky [3] has shown that the group algebra of an arbitrary group over a field of characteristic zero is directly finite. The purpose of this note is to generalise Kaplansky's result from group algebras to a wider class of monoid algebras, namely those in which the monoids are completely regular.

To facilitate the discussion, it is convenient to introduce a further concept. A ring R is said to be *quasidirectly finite* if and only if, for all $a,b\in R, ab=a+b$ implies $ab=ba$. It is easily seen that, for the case in which R has a unity, direct finiteness and quasidirect finiteness are equivalent properties. However, the second property can be useful in the study of a monoid algebra—for, in general, there are important auxiliary semigroup algebras that need not have unity elements. In fact we shall show that the algebra of a completely regular semigroup over a field of characteristic zero is quasidirectly finite. The result stated in the summary above is an immediate consequence.

We begin with an elementary lemma that provides a basis for induction.

Lemma 1. *Let R be a ring, let S be a subring of R and let T be an ideal of R such that $R=S\oplus T$. Then R is quasidirectly finite if and only if S and T are quasidirectly finite.*

Proof. It is clear that if R is quasidirectly finite then so also are S and T. Assume, conversely, that S and T are quasidirectly finite and suppose that $a,b\in R$ are such that $ab=a+b$. Write $a=a_1+a_2$ and $b=b_1+b_2$, where $a_1,b_1\in S$ and $a_2,b_2\in T$. Then

$$a_1b_1=a_1+b_1, \tag{1}$$

$$a_2b_2+a_1b_2+a_2b_1=a_2+b_2. \tag{2}$$

Since S is quasidirectly finite, (1) implies that

$$b_1a_1 = b_1 + a_1. \tag{3}$$

Now let R^1 be an overring of R containing a unity 1. We operate in R^1 and deduce results on R itself. From (3), the equation

$$(1-b_1)(1-a_1) = 1 \tag{4}$$

holds in R^1. Thus, from (4) and (2),

$$a_2(1-b_1)(1-a_1)b_2 = a_2(1-b_1) + (1-a_1)b_2.$$

Hence, since $a_2(1-b_1)$ and $(1-a_1)b_2$ lie in T and this ring is quasidirectly finite,

$$(1-a_1)b_2a_2(1-b_1) = (1-a_1)b_2 + a_2(1-b_1). \tag{5}$$

Pre- and post-multiplying both sides of (5) by $1-b_1$ and $1-a_1$ respectively, and using (4), we see that

$$b_2a_2 = b_2(1-a_1) + (1-b_1)a_2;$$

that is,

$$b_2a_2 + b_1a_2 + b_2a_1 = b_2 + a_2. \tag{6}$$

Finally, (3) and (6) combine to give $ba = b + a$. Thus R is quasidirectly finite. □

It is easy to deduce from Lemma 1 (or indeed to show directly) that if R is a quasidirectly finite ring and the ring R^1 is formed by adjoining a unity to R in the usual way then R^1 is directly finite.

By a *semilattice* we mean a commutative semigroup consisting of idempotents. A ring R is said to be *graded* by a semilattice Y if and only if R has a family of subrings $R_\alpha(\alpha \in Y)$ (called the *homogeneous components* of R) such that $R = \bigoplus_{\alpha \in Y} R_\alpha$ and, for all $\alpha, \beta \in Y, R_\alpha R_\beta \subseteq R_{\alpha\beta}$. The main result below relies on the fact that the semigroup algebras that we consider have a natural semilattice-grading.

First, we establish

Lemma 2. *Let R be a semilattice-graded ring. Then R is quasidirectly finite if and only if each of its homogeneous components is quasidirectly finite.*

Proof. Let $R = \bigoplus_{\alpha \in Y} R_\alpha$, where Y is a semilattice, each $R_\alpha(\alpha \in Y)$ is a subring of R and, for all $\alpha, \beta \in Y, R_\alpha R_\beta \subseteq R_{\alpha\beta}$. Clearly, if R is quasidirectly finite then so is each R_α. Now suppose that, for all $\alpha \in Y$, R_α is quasidirectly finite. For $x \in R$, we denote the R_α-component of x by $x_\alpha(\alpha \in Y)$.

Let $a, b \in R$ be such that $ab = a + b$. We may assume that a and b are both nonzero. Let Z denote the subsemigroup of Y generated by the (finite) subset $\{\alpha \in Y : a_\alpha \neq 0$ or $b_\alpha \neq 0\}$. Since Y is locally finite, Z is a finite semilattice. Write $S := \bigoplus_{\alpha \in Z} R_\alpha$. Then S is a subring of R containing a and b and it suffices to show that S is quasidirectly finite. Note first that Z is partially ordered by the rule that

$$\alpha \leqq \beta \Leftrightarrow \alpha\beta (= \beta\alpha) = \alpha \quad (\alpha, \beta \in Z).$$

Let $n = |Z|$ and construct subsets $Z_1, Z_2, \ldots, Z_n$ of Z successively by taking $Z_1 = \{\omega\}$, where ω is the least element of Z, and $Z_{i+1} = Z_i \cup \{\alpha\}$ if $1 \leqq i < n$, where α is minimal in $Z \backslash Z_i$ under the partial ordering. Write $T_i := \bigoplus_{\alpha \in Z_i} R_\alpha$ $(i = 1, 2, \ldots, n)$. It is clear that each T_i is an ideal of S and that

$$T_1 \subset T_2 \subset \ldots \subset T_n = S.$$

Now T_1 is quasidirectly finite, since $T_1 = R_\omega$. Assume that $n > 1$ and that T_i is quasidirectly finite for $i < n$. By definition, $T_{i+1} = R_\alpha \oplus T_i$ for some $\alpha \in Z$. Hence, by Lemma 1, T_{i+1} is quasidirectly finite. Thus, by induction, we see that S is quasidirectly finite, as required. □

We adopt the basic terminology and notation for semigroups established in [**2**] (with the exception of the now-standard phrase 'completely regular' (see below)). Throughout, the symbol F denotes a field. The semigroup algebra [**2**, §5.2] of a semigroup S over F is denoted by $F[S]$ and, for a positive integer n, the F-algebra consisting of all $n \times n$ matrices over an F-algebra R (under the usual operations) is denoted by $M_n(R)$.

The following result was obtained by Kaplansky [**3**, p. 122]. (See also [**4**] and [**5**, Corollary 2.1.9 and Example 9, p. 65].)

Lemma 3. *Let F have characteristic zero and let G be a group. Then, for all positive integers n, $M_n(F[G])$ is directly finite.*

From Lemma 3 we derive

Lemma 4. *Let F have characteristic zero and let S be a completely simple semigroup. Then $F[S]$ is quasidirectly finite.*

Proof. By Rees's theorem [**2**, Theorem 3.5], $S \cong \mathscr{M}(G; I, \Lambda; P)$ for some group G, some nonempty sets I and Λ and some $\Lambda \times I$ matrix P over G. Then, as in [**2**, Lemma 5.17], without loss of generality we can assume that $F[S] = \mathscr{M}(F[G]; I, \Lambda; P)$, the algebra of all $I \times \Lambda$ matrices over $F[G]$ having at most finitely many nonzero entries, with the usual addition and scalar multiplication, and with multiplication $\circ$ defined in terms of ordinary matrix multiplication by

$$X \circ Y = XPY \quad (X, Y \in F[S]).$$

Let $A, B \in F[S]$ be such that $A \circ B = A + B$. We have to show that $A \circ B = B \circ A$. Choose nonempty finite subsets I_1 and Λ_1 of I and Λ, respectively, such that all nonzero entries of A and B in each case lie in the $I_1 \times \Lambda_1$ submatrix. Let T be the subalgebra of $F[S]$ generated by A and B. Then each element of T is such that all entries lying outside the $I_1 \times \Lambda_1$ submatrix are zero. Let P_1 denote the $\Lambda_1 \times I_1$ submatrix of P and let M denote the algebra $\mathscr{M}(F[G]; I_1, \Lambda_1; P_1)$. Clearly, the mapping from T into M defined by $X \mapsto X_1 (X \in T)$, where X_1 is the $I_1 \times \Lambda_1$ submatrix of X, is an injective algebra homomorphism. Thus

$$A_1 P_1 B_1 = A_1 + B_1 \tag{1}$$

and it suffices to show that $A_1 P_1 B_1 = B_1 P_1 A_1$.

Let N denote the F-algebra of all $I_1 \times I_1$ matrices over $F[G]$ under the usual operations. Now $A_1 P_1$ and $B_1 P_1$ lie in N; and, from (1), $(A_1 P_1)(B_1 P_1) = A_1 P_1 + B_1 P_1$. But, by Lemma 3 (since I_1 is finite), N is directly finite and so quasidirectly finite. Hence $A_1 P_1 + B_1 P_1 = (B_1 P_1)(A_1 P_1)$. Post-multiplying by B_1 and applying (1), we find that

$$A_1 P_1 B_1 + B_1 P_1 B_1 = B_1 P_1 A_1 P_1 B_1 = B_1 P_1 (A_1 + B_1) = B_1 P_1 A_1 + B_1 P_1 B_1.$$

Thus $A_1 P_1 B_1 = B_1 P_1 A_1$. □

Remark. The same argument yields the more general result that the contracted semigroup algebra $F_0[S]$ of a completely 0-simple semigroup S over a field F of characteristic zero is quasidirectly finite. (Contracted semigroup algebras are defined in [**2**, §5.2].)

A semigroup S is said to be *completely regular* if and only if each element of S lies in a subgroup of S; that is, if and only if S is a union of groups. Such semigroups (under the title 'semigroups admitting relative inverses') were first studied by Clifford [**1**], who characterised them as semilattices of completely simple semigroups (see [**2**, Chapter 4]).

We now have the

Theorem. *Let F be a field of characteristic zero and let S be a completely regular semigroup. Then $F[S]$ is quasidirectly finite.*

Proof. By [**2**, Theorem 4.6], there exists a semilattice Y and a family of pairwise-disjoint completely simple semigroups $S_\alpha (\alpha \in Y)$ such that $S = \bigcup_{\alpha \in Y} S_\alpha$ and, for all $\alpha, \beta \in Y$, $S_\alpha S_\beta \subseteq S_{\alpha\beta}$. (In fact, Y is isomorphic to the semilattice of principal ideals of S and the S_α are the $\mathscr{J}$-classes of S.) Then $F[S] = \bigoplus_{\alpha \in Y} F[S_\alpha]$ and, for all $\alpha, \beta \in Y$, $F[S_\alpha] F[S_\beta] \subseteq F[S_{\alpha\beta}]$; that is, $F[S]$ is graded by Y and has homogeneous components $F[S_\alpha]$ $(\alpha \in Y)$. By Lemma 4, each $F[S_\alpha]$ is quasidirectly finite. Hence, by Lemma 2, $F[S]$ is quasidirectly finite. □

Corollary. *Let F be a field of characteristic zero and let S be a completely regular monoid. Then $F[S]$ is directly finite.*

Remark. Since, for any positive integer n, $M_n(F[G])$ is directly finite when G is an abelian group and F is an arbitrary field, a theorem analogous to that above can be obtained by replacing the hypothesis that F has characteristic zero by the requirement that every subgroup of S be abelian. In particular, it follows that if F is an arbitrary field and S is a band (that is, a semigroup of idempotents) then $F[S]$ is quasidirectly finite.

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