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STOCHASTIC ORDERING RESULTS ON THE DURATION OF THE GAMBLER'S RUIN GAME

SHOOU-REN HSIAU,* *National Changhua University of Education*
YI-CHING YAO,** *Academia Sinica*

Abstract

In the classical gambler's ruin problem, the gambler plays an adversary with initial capitals z and $a - z$, respectively, where $a > 0$ and $0 < z < a$ are integers. At each round, the gambler wins or loses a dollar with probabilities p and $1 - p$. The game continues until one of the two players is ruined. For even a and $0 < z \leq a/2$, the family of distributions of the duration (total number of rounds) of the game indexed by $p \in [0, \frac{1}{2}]$ is shown to have monotone (increasing) likelihood ratio, while for $a/2 \leq z < a$, the family of distributions of the duration indexed by $p \in [\frac{1}{2}, 1]$ has monotone (decreasing) likelihood ratio. In particular, for $z = a/2$, in terms of the likelihood ratio order, the distribution of the duration is maximized over $p \in [0, 1]$ by $p = \frac{1}{2}$. The case of odd a is also considered in terms of the usual stochastic order. Furthermore, as a limit, the first exit time of Brownian motion is briefly discussed.

Keywords: Brownian motion; first exit time; likelihood ratio order; monotone likelihood ratio; usual stochastic order

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1. Introduction and main results

In [4, Chapter XIV], the classical gambler's ruin problem is studied in detail, in which the gambler plays an adversary with initial capitals z and $a - z$, respectively, where $a > 0$ and $0 \leq z \leq a$ are integers. At each round, the gambler wins or loses a dollar with probabilities p and q $(= 1 - p)$. The game continues until one of the two players is ruined (and the other player's capital reaches the maximum value a). We use the symbol $\mathbb{P}_{p,z,a}$ to denote the probability measure with parameters p, z, and a. The duration (total number of rounds) of the game is denoted by N, whose distribution depends on p, z, and a and is denoted by $\mathcal{L}_{p,z,a}(N)$. We are concerned with stochastic ordering relations for the family of distributions $\{\mathcal{L}_{p,z,a}(N)\colon 0 \leq p \leq 1,\ 0 \leq z \leq a\}$.

Letting $I = 0$ if the gambler is ruined and $I = 1$ otherwise, the generating function for N admits the following explicit expression [4, (4.11) and (4.12), p. 351]:

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* Postal address: Department of Mathematics, National Changhua University of Education, No. 1, Jin-De Road, Changhua 500, Taiwan, R.O.C. Email: srhsiau@cc.ncue.edu.tw
** Postal address: Institute of Statistical Science, Academia Sinica, No. 128 Academia Road, Section 2, Nankang, Taipei 11529, Taiwan, R.O.C. Email: yao@stat.sinica.edu.tw

$$\sum_{n=0}^{\infty} \mathbb{P}_{p,z,a}(N=n)s^n = \sum_{n=0}^{\infty} \mathbb{P}_{p,z,a}(N=n, I=0)s^n + \sum_{n=0}^{\infty} \mathbb{P}_{p,z,a}(N=n, I=1)s^n,$$

$$\sum_{n=0}^{\infty} \mathbb{P}_{p,z,a}(N=n, I=0)s^n = \left(\frac{q}{p}\right)^z \frac{\lambda_1^{a-z}(s) - \lambda_2^{a-z}(s)}{\lambda_1^a(s) - \lambda_2^a(s)},$$

$$\sum_{n=0}^{\infty} \mathbb{P}_{p,z,a}(N=n, I=1)s^n = \frac{\lambda_1^z(s) - \lambda_2^z(s)}{\lambda_1^a(s) - \lambda_2^a(s)},$$

where, for $0 < s < 1$,

$$\lambda_1(s) = \frac{1 + \sqrt{1 - 4pqs^2}}{2ps}, \qquad \lambda_2(s) = \frac{1 - \sqrt{1 - 4pqs^2}}{2ps}.$$

Furthermore, for even $n - z$ and $n > 1$ [4, (5.7) and (5.8), pp. 353–354],

$$\mathbb{P}_{p,z,a}(N=n, I=0) = a^{-1}2^{n+1}p^{(n-z)/2}q^{(n+z)/2} \sum_{1 \le \nu < a/2} \cos^{n-1}\frac{\pi\nu}{a} \sin\frac{\pi\nu}{a} \sin\frac{\pi z\nu}{a}, \tag{1}$$

and $\mathbb{P}_{p,z,a}(N=n, I=0) = 0$ for odd $n - z$. By symmetry, for even $n - a + z$ and $n > 1$,

$$\mathbb{P}_{p,z,a}(N=n, I=1) = a^{-1}2^{n+1}p^{(n+a-z)/2}q^{(n-a+z)/2} \sum_{1 \le \nu < a/2} \cos^{n-1}\frac{\pi\nu}{a} \sin\frac{\pi\nu}{a} \sin\frac{\pi(a-z)\nu}{a}, \tag{2}$$

and $\mathbb{P}_{p,z,a}(N=n, I=1) = 0$ for odd $n - a + z$.

When a is an even integer, it is shown in [8] that $\mathcal{L}_{1/2,a/2,a}(N)$ is stochastically larger than $\mathcal{L}_{p,a/2,a}(N)$ for $p \neq \frac{1}{2}$. In terms of the likelihood ratio order, which is stronger than the usual stochastic order (see, e.g., [10]), a stronger version of their result may be derived as follows. Let $X_{n,p}$, $n = 1, 2, \ldots$, be independent and identically distributed (i.i.d.) with

$$\mathbb{P}(X_{n,p} = 1) = p = 1 - \mathbb{P}(X_{n,p} = -1). \tag{3}$$

For $0 < z < a$ and $n \geq 1$, let

$$\mathcal{S}^+_{z,a}(n) = \{(\omega_1, \ldots, \omega_n) \in \{-1, 1\}^n : 0 < z + \omega_1 + \cdots + \omega_i < a,\ i = 1, \ldots, n-1,\ z + \omega_1 + \cdots + \omega_n = a\},$$

$$\mathcal{S}^-_{z,a}(n) = \{(\omega_1, \ldots, \omega_n) \in \{-1, 1\}^n : 0 < z + \omega_1 + \cdots + \omega_i < a,\ i = 1, \ldots, n-1,\ z + \omega_1 + \cdots + \omega_n = 0\}.$$

For $z \in \{0, a\}$ and $n \geq 1$, let $\mathcal{S}^+_{z,a}(n) = \mathcal{S}^-_{z,a}(n) = \emptyset$. Note that $\mathcal{S}^-_{z,a}(n) = \emptyset$ if n and z have opposite parity, while $\mathcal{S}^+_{z,a}(n) = \emptyset$ if n and $a - z$ have opposite parity. Assume a is even and $0 < z < a$, so that z and $a - z$ are of the same parity. Then we have $\mathbb{P}_{p,z,a}(N = n) = 0$ if n and z have opposite parity; and for $n = \min\{z, a - z\}, \min\{z, a - z\} + 2, \ldots,$

$$\begin{aligned}
\mathbb{P}_{p,z,a}(N=n) &= \mathbb{P}((X_{1,p}, \ldots, X_{n,p}) \in \mathcal{S}^+_{z,a}(n)) + \mathbb{P}((X_{1,p}, \ldots, X_{n,p}) \in \mathcal{S}^-_{z,a}(n)) \\
&= p^{(n+a-z)/2}q^{(n-a+z)/2}|\mathcal{S}^+_{z,a}(n)| + p^{(n-z)/2}q^{(n+z)/2}|\mathcal{S}^-_{z,a}(n)| \\
&= (pq)^{n/2}\left[\left(\frac{p}{q}\right)^{(a-z)/2}|\mathcal{S}^+_{z,a}(n)| + \left(\frac{q}{p}\right)^{z/2}|\mathcal{S}^-_{z,a}(n)|\right],
\end{aligned} \tag{4}$$

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where $|S|$ denotes the cardinality of the set S. For $z = a/2$, we have $|\mathcal{S}^+_{a/2,a}(n)| = |\mathcal{S}^-_{a/2,a}(n)|$ by symmetry, which together with (4) implies that

$$\mathbb{P}_{p,a/2,a}(N = n) = (pq)^{n/2}\left[\left(\frac{p}{q}\right)^{a/4} + \left(\frac{q}{p}\right)^{a/4}\right]|\mathcal{S}^+_{a/2,a}(n)|.$$

So, for $p, p' \in (0, 1)$ (with $q' = 1 - p'$) and $n = a/2, a/2 + 2, \ldots,$

$$\frac{\mathbb{P}_{p',a/2,a}(N = n)}{\mathbb{P}_{p,a/2,a}(N = n)} = \left(\frac{p'q'}{pq}\right)^{n/2}\left\{\left[\left(\frac{p'}{q'}\right)^{a/4} + \left(\frac{q'}{p'}\right)^{a/4}\right]\Big/\left[\left(\frac{p}{q}\right)^{a/4} + \left(\frac{q}{p}\right)^{a/4}\right]\right\},$$

which is increasing in n if $\left|p - \frac{1}{2}\right| > \left|p' - \frac{1}{2}\right|$. Consequently, for even a and $z = a/2$, if p and $p' \in [0, 1]$ satisfy $\left|p - \frac{1}{2}\right| > \left|p' - \frac{1}{2}\right|$, then $\mathcal{L}_{p',a/2,a}(N)$ is larger than $\mathcal{L}_{p,a/2,a}(N)$ in the likelihood ratio order. For a family of distributions indexed by $\theta \in \mathcal{I}$ (an interval) with probability mass/density functions $f_\theta(\cdot)$ on $\mathcal{X}$ (a subset of the real line), it is said to have monotone (increasing) likelihood ratio if

$$f_\theta(x)f_{\theta'}(x') \geq f_{\theta'}(x)f_\theta(x') \tag{5}$$

whenever $x, x' \in \mathcal{X}$ and $\theta, \theta' \in \mathcal{I}$ satisfy $x < x'$ and $\theta < \theta'$, and is said to have monotone (decreasing) likelihood ratio if the inequality (5) is reversed; see [6]. Indeed, we have shown the following result.

Theorem 1. *For even $a \geq 4$ and $z = a/2$, the family of distributions $\left\{\mathcal{L}_{p,a/2,a}(N) \colon 0 \leq p \leq \frac{1}{2}\right\}$ has monotone (increasing) likelihood ratio, and the family of distributions $\left\{\mathcal{L}_{p,a/2,a}(N) \colon \frac{1}{2} \leq p \leq 1\right\}$ has monotone (decreasing) likelihood ratio.*

By Theorem 1, in terms of the likelihood ratio order, the distribution $\mathcal{L}_{p,a/2,a}(N)$ is maximized over $p \in [0, 1]$ by $p = \frac{1}{2}$, implying the result of [8].

We next consider the more general case with a even and $z \neq a/2$. We need to establish a crucial monotonicity result for $p = \frac{1}{2}$, which is of independent interest. For $p = \frac{1}{2}$, note that

$$\mathbb{P}_{p,z,a}(N = n, I = 1) = 2^{-n}|\mathcal{S}^+_{z,a}(n)|, \qquad \mathbb{P}_{p,z,a}(N = n, I = 0) = 2^{-n}|\mathcal{S}^-_{z,a}(n)|.$$

Theorem 2. *For $p = \frac{1}{2}$, even $a \geq 4$, and $0 < z < a/2$, as $n \in \{z, z + 2, \ldots\}$ increases to ∞,*

$$\frac{\mathbb{P}_{p,z,a}(N = n, I = 1)}{\mathbb{P}_{p,z,a}(N = n)} = \frac{|\mathcal{S}^+_{z,a}(n)|}{|\mathcal{S}^+_{z,a}(n)| + |\mathcal{S}^-_{z,a}(n)|}$$

monotonically increases to $\frac{1}{2}$. Equivalently, for $p = \frac{1}{2}$ and $0 < z < a/2$,

$$\frac{\mathbb{P}_{p,z,a}(N = n, I = 1)}{\mathbb{P}_{p,z,a}(N = n, I = 0)} = \frac{|\mathcal{S}^+_{z,a}(n)|}{|\mathcal{S}^-_{z,a}(n)|}$$

monotonically increases to 1 as $n \in \{z, z + 2, \ldots\}$ increases to ∞.

With the help of Theorem 2, the next theorem can be readily shown, which is an extension of Theorem 1 from $z = a/2$ to $z \neq a/2$.

Theorem 3. *For even $a \geq 4$ and $0 < z < a/2$, the family of distributions $\left\{\mathcal{L}_{p,z,a}(N) \colon 0 \leq p \leq \frac{1}{2}\right\}$ has monotone (increasing) likelihood ratio, and for $a/2 < z < a$, the family of distributions $\left\{\mathcal{L}_{p,z,a}(N) \colon \frac{1}{2} \leq p \leq 1\right\}$ has monotone (decreasing) likelihood ratio.*

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For the case of odd a, analogous results do not hold unless the likelihood ratio order is replaced by a weaker stochastic order. To see this, consider odd $a \geq 3$ and $0 < z < a$. Then z and $a - z$ have opposite parity. So, for all n, either $\mathcal{S}^-_{z,a}(n) = \emptyset$ or $\mathcal{S}^+_{z,a}(n) = \emptyset$. For even $n - z \geq 0$,

$$\mathbb{P}_{p,z,a}(N = n) = p^{(n-z)/2} q^{(n+z)/2} |\mathcal{S}^-_{z,a}(n)| = (pq)^{n/2} \left(\frac{q}{p}\right)^{z/2} |\mathcal{S}^-_{z,a}(n)|,$$

while for even $n - a + z \geq 0$,

$$\mathbb{P}_{p,z,a}(N = n) = p^{(n+a-z)/2} q^{(n-a+z)/2} |\mathcal{S}^+_{z,a}(n)| = (pq)^{n/2} \left(\frac{p}{q}\right)^{(a-z)/2} |\mathcal{S}^+_{z,a}(n)|.$$

We have

$$\frac{\mathbb{P}_{p',z,a}(N = n)}{\mathbb{P}_{p,z,a}(N = n)} = \begin{cases} \left(\frac{p'q'}{pq}\right)^{n/2} \left(\frac{pq'}{p'q}\right)^{z/2} & \text{for } n = z, z + 2, \ldots, \\ \left(\frac{p'q'}{pq}\right)^{n/2} \left(\frac{p'q}{pq'}\right)^{(a-z)/2} & \text{for } n = a - z, a - z + 2, \ldots \end{cases} \tag{6}$$

Consider $0 < p < p' \leq \frac{1}{2}$ and $0 < z < \frac{a}{2}$. By (6), for all $n \geq a - z\ (> z)$,

$$\left[\frac{\mathbb{P}_{p',z,a}(N = n + 2)}{\mathbb{P}_{p,z,a}(N = n + 2)}\right] \Big/ \left[\frac{\mathbb{P}_{p',z,a}(N = n)}{\mathbb{P}_{p,z,a}(N = n)}\right] = \frac{p'q'}{pq} > 1,$$

i.e. the likelihood ratio $\mathbb{P}_{p',z,a}(N = n)/\mathbb{P}_{p,z,a}(N = n)$ increases by a factor of $p'q'/pq$ when n increases by 2. On the other hand, by (6) again, we have, for $n = a - z, a - z + 2, \ldots,$

$$\begin{aligned} &\left[\frac{\mathbb{P}_{p',z,a}(N = n + 1)}{\mathbb{P}_{p,z,a}(N = n + 1)}\right] \Big/ \left[\frac{\mathbb{P}_{p',z,a}(N = n)}{\mathbb{P}_{p,z,a}(N = n)}\right] \\ &= \left[\left(\frac{p'q'}{pq}\right)^{(n+1)/2} \left(\frac{pq'}{p'q}\right)^{z/2}\right] \Big/ \left[\left(\frac{p'q'}{pq}\right)^{n/2} \left(\frac{p'q}{pq'}\right)^{(a-z)/2}\right] \\ &= \left(\frac{p}{p'}\right)^{(a-1)/2} \left(\frac{q'}{q}\right)^{(a+1)/2} < 1, \end{aligned}$$

showing that the likelihood ratio $\mathbb{P}_{p',z,a}(N = n)/\mathbb{P}_{p,z,a}(N = n)$ is not monotonically increasing in n. The next theorem gives stochastic ordering results for the odd a case in terms of the usual stochastic order.

Theorem 4. *Let $a \geq 3$ be an odd integer. For $0 < z < a/2$ and $0 \leq p < p' \leq \frac{1}{2}$, $\mathcal{L}_{p,z,a}(N)$ is stochastically smaller than $\mathcal{L}_{p',z,a}(N)$, and for $a/2 < z < a$ and $\frac{1}{2} \leq p < p' \leq 1$, $\mathcal{L}_{p,z,a}(N)$ is stochastically larger than $\mathcal{L}_{p',z,a}(N)$.*

The long and technical proof of Theorem 2 is given in Section 2, while Sections 3 and 4 present the proofs of Theorems 3 and 4, respectively. We close this section with a number of remarks on some implications of the above results and a conjecture for odd a related to the work of [8].

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Remark 1. For $p = \frac{1}{2}$, the conditional distributions of N given $I = 0$ and $I = 1$, denoted by $\mathcal{L}_{1/2,z,a}(N \mid I = 0)$ and $\mathcal{L}_{1/2,z,a}(N \mid I = 1)$, have respective probability mass functions

$$\mathbb{P}_{1/2,z,a}(N = n \mid I = 0) = \left(\frac{a}{a-z}\right)\mathbb{P}_{1/2,z,a}(N = n, I = 0), \quad n = z, z+2, \ldots,$$

$$\mathbb{P}_{1/2,z,a}(N = n \mid I = 1) = \left(\frac{a}{z}\right)\mathbb{P}_{1/2,z,a}(N = n, I = 1), \qquad n = a-z, a-z+2, \ldots$$

So, for even a and $0 < z < a/2$, Theorem 2 is equivalent to saying that $\mathcal{L}_{1/2,z,a}(N \mid I = 1)$ is larger than $\mathcal{L}_{1/2,z,a}(N \mid I = 0)$ in the likelihood ratio order.

Remark 2. For even a, Theorem 3 does not cover the case $a/2 < z < a$ and $p \le \frac{1}{2}$. In fact, for $a/2 < z < a$ and $0 < p < p' \le \frac{1}{2}$, $\mathcal{L}_{p,z,a}(N)$ is neither stochastically smaller nor stochastically larger than $\mathbb{L}_{p',z,a}(N)$. To see this, note that

$$\mathbb{P}_{p,z,a}(N \le a-z) = \mathbb{P}_{p,z,a}(N = a-z) = p^{a-z} < (p')^{a-z} = \mathbb{P}_{p',z,a}(N = a-z)$$
$$= \mathbb{P}_{p',z,a}(N \le a-z).$$

On the other hand, $\mathbb{P}_{p,z,a}(N \le n) > \mathbb{P}_{p',z,a}(N \le n)$ for large n, as shown below. Since $|\mathcal{S}^{+}_{z,a}(n)| = |\mathcal{S}^{-}_{a-z,a}(n)|$ and $|\mathcal{S}^{-}_{z,a}(n)| = |\mathcal{S}^{+}_{a-z,a}(n)|$, it follows from Theorem 2 that $|\mathcal{S}^{-}_{z,a}(n)|/|\mathcal{S}^{+}_{z,a}(n)| = |\mathcal{S}^{+}_{a-z,a}(n)|/|\mathcal{S}^{-}_{a-z,a}(n)|$ increases to 1 as $n \in \{z, z+2, \ldots\}$ increases to ∞. By (4), as $n \in \{z, z+2, \ldots\}$ increases to ∞,

$$\left(\frac{pq}{p'q'}\right)^{n/2}\left(\frac{\mathbb{P}_{p',z,a}(N = n)}{\mathbb{P}_{p,z,a}(N = n)}\right) \longrightarrow \frac{(p'/q')^{(a-z)/2} + (q'/p')^{z/2}}{(p/q)^{(a-z)/2} + (q/p)^{z/2}}. \tag{7}$$

Since $pq < p'q'$, we have $\lim_{n\to\infty} \mathbb{P}_{p,z,a}(N \ge n)/\mathbb{P}_{p',z,a}(N \ge n) = 0$ by (7). So $\mathbb{P}_{p,z,a}(N \le n) > \mathbb{P}_{p',z,a}(N \le n)$ for large n.

Remark 3. In view of (6) and (7), it can be shown that for $0 < z < a$ and $p, p' \in (0, 1)$,

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{\mathbb{P}_{p',z,a}(N \ge n)}{\mathbb{P}_{p,z,a}(N \ge n)}\right) = \frac{1}{2}\log\left(\frac{p'q'}{pq}\right).$$

Thus, $\mathcal{L}_{p,z,a}(N)$ has much lighter tails than $\mathcal{L}_{p',z,a}(N)$ for $\left|p - \frac{1}{2}\right| > \left|p' - \frac{1}{2}\right|$.

Remark 4. For even a and $z = a/2$, it is shown in [8] that the distribution $\mathcal{L}_{p,z,a}(N)$ is stochastically maximized over $p \in [0, 1]$ by $p = \frac{1}{2}$. Can an analogous result hold for odd a? For odd a, it seems natural to consider a random initial state z that takes on the two middle values $(a+1)/2$ and $(a-1)/2$ with equal probabilities. Then the distribution of the duration N is a mixture of the two distributions $\mathcal{L}_{p,(a+1)/2,a}(N)$ and $\mathcal{L}_{p,(a-1)/2,a}(N)$ with equal weights, denoted by $\frac{1}{2}\mathcal{L}_{p,(a+1)/2,a}(N) + \frac{1}{2}\mathcal{L}_{p,(a-1)/2,a}(N)$. Note that for $p = \frac{1}{2}$, $\mathcal{L}_{1/2,(a+1)/2,a}(N) = \mathcal{L}_{1/2,(a-1)/2,a}(N)$, and for $p + p' = 1$,

$$\frac{1}{2}\mathcal{L}_{p,(a+1)/2,a}(N) + \frac{1}{2}\mathcal{L}_{p,(a-1)/2,a}(N) = \frac{1}{2}\mathcal{L}_{p',(a+1)/2,a}(N) + \frac{1}{2}\mathcal{L}_{p',(a-1)/2,a}(N).$$

We conjecture that $\frac{1}{2}\mathcal{L}_{p,(a+1)/2,a}(N) + \frac{1}{2}\mathcal{L}_{p,(a-1)/2,a}(N)$ is stochastically maximized over $p \in [0, 1]$ by $p = \frac{1}{2}$, which can be shown to hold for small odd a. Moreover, this conjecture is equivalent to saying that $\frac{1}{2}\mathbb{P}_{p,(a+1)/2,a}(N \geq n) + \frac{1}{2}\mathbb{P}_{p,(a-1)/2,a}(N \geq n)$ is maximized over $p \in [0, 1]$ by $p = \frac{1}{2}$ for all n. It may be verified directly for small n.

Remark 5. [5, II.7] introduces a model of randomized random walks where instead of successive jumps occurring at epochs 1, 2, ..., the time intervals between successive jumps are assumed to be i.i.d. exponential random variables with mean 1. This model is a compound Poisson process $Z_t = \sum_{n=1}^{\Pi_t} X_{n,p}$, $t \geq 0$, where $X_{n,p}$ $(n = 1, 2, \dots)$ are i.i.d. as defined in (3), and Π_t is a Poisson process of constant rate 1 (independent of the $X_{n,p}$). (For a discussion of compound Poisson processes, see, e.g., [7, 16.9].) For $0 < z < a$, let $\mathcal{T} := \inf\{t > 0 \colon z + Z_t \notin (0, a)\}$, the first exit time of the process $z + Z_t$ from the interval $(0,a)$. We denote the distribution of $\mathcal{T}$ by $\mathcal{L}_{p,z,a}(\mathcal{T})$, which is the same as the distribution of $\sum_{i=1}^{N} \mathcal{E}_i$ where N is the duration of the gambler's ruin game and $\mathcal{E}_i$ $(i = 1, 2, \dots)$ are i.i.d. exponential with mean 1 (independent of N). We write $\mathcal{L}_{p,z,a}(\mathcal{T}) = \mathcal{L}_{p,z,a}\big(\sum_{i=1}^{N} \mathcal{E}_i\big)$. In view of this, if $\mathcal{L}_{p,z,a}(N)$ is stochastically smaller than $\mathcal{L}_{p',z,a}(N)$, then $\mathcal{L}_{p,z,a}(\mathcal{T})$ is stochastically smaller than $\mathcal{L}_{p',z,a}(\mathcal{T})$. In other words, the usual stochastic order relation concerning the distribution of N carries over to $\mathcal{T}$. However, the likelihood ratio order relation does not carry over. It follows from Theorems 3 and 4 that, for $0 < p < p' \leq \frac{1}{2}$ and $0 < z \leq a/2$, $\mathcal{L}_{p,z,a}(\mathcal{T})$ is stochastically smaller than $\mathcal{L}_{p',z,a}(\mathcal{T})$.

Remark 6. [4, XIV.6] discusses the connection of the gambler's ruin game with Brownian motion as a limit, which is briefly described below. To have Brownian motion as a limit, the time and step size for the random walk in the gambler's ruin problem may be rescaled such that there are r steps per unit time and each step causes a displacement equal to $\pm\delta$. Given real values c and $0 < \xi < \alpha$, let

$$\delta \to 0, \quad r \to \infty, \quad p \to \frac{1}{2}, \quad z \to \infty, \quad a \to \infty \tag{8}$$

in such a way that

$$(p - q)\delta r \to c, \quad 4pq\delta^2 r \to 1, \quad z\delta \to \xi, \quad a\delta \to \alpha. \tag{9}$$

Then the duration N of the gambler's ruin game with initial state z becomes, in the limit, the first exit time $\tau := \inf\{t > 0 \colon \xi + ct + B_t \notin (0, \alpha)\}$ from the interval $(0, \alpha)$, where B_t is standard Brownian motion and $\xi + ct + B_t$ is Brownian motion with drift parameter c and initial state ξ. Denote the density function of τ by $u_{c,\xi,\alpha}(t)$, which may be decomposed as $u_{c,\xi,\alpha}(t) = u^-_{c,\xi,\alpha}(t) + u^+_{c,\xi,\alpha}(t)$. Here, $u^-_{c,\xi,\alpha}(t)$ and $u^+_{c,\xi,\alpha}(t)$ denote, respectively, the density functions of τ when the Brownian motion process exits through the lower and upper boundaries, i.e., for $t > 0$,

$$\mathbb{P}(\tau \leq t,\ \xi + c\tau + B_\tau = 0) = \int_0^t u^-_{c,\xi,\alpha}(s)\,\mathrm{d}s,$$

$$\mathbb{P}(\tau \leq t,\ \xi + c\tau + B_\tau = \alpha) = \int_0^t u^+_{c,\xi,\alpha}(s)\,\mathrm{d}s.$$

Since $u^-_{c,\xi,\alpha}(t)$ and $u^+_{c,\xi,\alpha}(t)$ are the continuous-time counterparts of $\mathbb{P}_{p,z,a}(N = n, I = 0)$ and $\mathbb{P}_{p,z,a}(N = n, I = 1)$ as given in (1) and (2), applying a standard limiting argument to (1) and (2) yields

$$u^-_{c,\xi,\alpha}(t) = \pi\alpha^{-2}\mathrm{e}^{-c(ct+2\xi)/2} \sum_{\nu=1}^{\infty} \nu \mathrm{e}^{-\nu^2\pi^2 t/2\alpha^2} \sin\frac{\pi\xi\nu}{\alpha}, \tag{10}$$

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$$u^{+}_{c,\xi,\alpha}(t) = \pi\alpha^{-2}\mathrm{e}^{c(-ct+2\alpha-2\xi)/2}\sum_{\nu=1}^{\infty}\nu\mathrm{e}^{-\nu^2\pi^2 t/2\alpha^2}\sin\frac{\pi(\alpha-\xi)\nu}{\alpha}, \tag{11}$$

where (10) is [4, (6.15) (with $D=1$), p. 359] and (11) is due to $u^{+}_{c,\xi,\alpha}(t) = u^{-}_{-c,\alpha-\xi,\alpha}(t)$ by symmetry. In [4, Problem 22, p. 370], (10) and (11) are given in the following alternative form:

$$u^{-}_{c,\xi,\alpha}(t) = \frac{1}{\sqrt{2\pi t^3}}\mathrm{e}^{-c(ct+2\xi)/2}\sum_{k=-\infty}^{\infty}(\xi+2k\alpha)\mathrm{e}^{-(\xi+2k\alpha)^2/2t}, \tag{12}$$

$$u^{+}_{c,\xi,\alpha}(t) = \frac{1}{\sqrt{2\pi t^3}}\mathrm{e}^{c(-ct+2\alpha-2\xi)/2}\sum_{k=-\infty}^{\infty}(\alpha-\xi+2k\alpha)\mathrm{e}^{-(\alpha-\xi+2k\alpha)^2/2t}. \tag{13}$$

(See also [3] for the Laplace transforms of $u^{-}_{c,\xi,\alpha}(t)$ and $u^{+}_{c,\xi,\alpha}(t)$.) Since a can be taken to be an even number as a increases to ∞ in (8) and (9), the monotonicity property of $\mathbb{P}_{p,z,a}(N=n, I=1)/\mathbb{P}_{p,z,a}(N=n, I=0)$ with $p=\frac{1}{2}$ and $0<z<a/2$ in Theorem 2 carries over to the continuous-time counterpart $u^{+}_{c,\xi,\alpha}(t)/u^{-}_{c,\xi,\alpha}(t)$ with $c=0$ and $0<\xi<\alpha/2$. Moreover, in (10) and (11), the term with $\nu=1$ is dominant for large t, so that $\lim_{t\to\infty} u^{+}_{0,\xi,\alpha}(t)/u^{-}_{0,\xi,\alpha}(t)=1$. On the other hand, in (12) and (13), the term with $k=0$ is dominant for small t, so that $\lim_{t\to 0+} u^{+}_{0,\xi,\alpha}(t)/u^{-}_{0,\xi,\alpha}(t)=0$ for $0<\xi<\alpha/2$. Hence, for $0<\xi<\alpha/2$, as t increases from 0 to ∞, $u^{+}_{0,\xi,\alpha}(t)/u^{-}_{0,\xi,\alpha}(t)$ monotonically increases from 0 to 1. (Equivalently, for $c=0$ and $0<\xi<\alpha/2$, the conditional distribution of τ given $\xi+B_\tau=0$ (which has probability density $(\alpha/(\alpha-\xi))u^{-}_{0,\xi,\alpha}(t)$) is smaller, in the likelihood ratio order, than the conditional distribution of τ given $\xi+B_\tau=\alpha$ (which has probability density $(\alpha/\xi)u^{+}_{0,\xi,\alpha}(t)$).) Furthermore, the monotone likelihood ratio property for N in Theorem 3 also carries over to τ. Specifically, for $0<\xi\le\alpha/2$, the family of distributions $\{\mathcal{L}_{c,\xi,\alpha}(\tau)\colon c\in(-\infty, 0]\}$ has monotone (increasing) likelihood ratio, while for $\alpha/2\le\xi<\alpha$, the family of distributions $\{\mathcal{L}_{c,\xi,\alpha}(\tau)\colon c\in[0,\infty)\}$ has monotone (decreasing) likelihood ratio. In particular, in terms of the likelihood ratio order, the distribution $\mathcal{L}_{c,\alpha/2,\alpha}(\tau)$ is maximized over $c\in(-\infty,\infty)$ by $c=0$. The first exit time τ of Brownian motion is a special case of the two-sided barrier problem in the subject of level-crossing problems for random processes. It is one of a limited number of cases where an explicit solution is available. See the survey articles [1, 2] for discussion of the related literature.

2. Proof of Theorem 2

To prove Theorem 2, we need to introduce some notation and establish a few lemmas. Let $a\ge 4$ be an even integer. For $0\le z\le a$ and $n\ge 1$, let $T^{+}_{z,a}(n) := |\mathcal{S}^{+}_{z,a}(n)|$ and $T^{-}_{z,a}(n) := |\mathcal{S}^{-}_{z,a}(n)|$. For $n\ge 2$ and $0<z<a$, since

$$\mathcal{S}^{+}_{z,a}(n) = [\mathcal{S}^{+}_{z,a}(n)\cap(\{-1\}\times\{-1,1\}^{n-1})]\cup[\mathcal{S}^{+}_{z,a}(n)\cap(\{1\}\times\{-1,1\}^{n-1})],$$

we have

$$T^{+}_{z,a}(n) = |\mathcal{S}^{+}_{z,a}(n)| = |\mathcal{S}^{+}_{z-1,a}(n-1)| + |\mathcal{S}^{+}_{z+1,a}(n-1)| = T^{+}_{z-1,a}(n-1) + T^{+}_{z+1,a}(n-1). \tag{14}$$

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Let $T^+_{z,a}(0) = 1$ or 0 according as $z = a$ or $0 \le z < a$. Then (14) also holds for $n = 1$ and $0 < z < a$. That is,

$$T^+_{z,a}(n) = T^+_{z-1,a}(n-1) + T^+_{z+1,a}(n-1) \qquad \text{for } n \ge 1 \text{ and } 0 < z < a. \tag{15}$$

Similarly,

$$T^-_{z,a}(n) = T^-_{z-1,a}(n-1) + T^-_{z+1,a}(n-1) \qquad \text{for } n \ge 1 \text{ and } 0 < z < a, \tag{16}$$

where $T^-_{z,a}(0) = 1$ or 0 according as $z = 0$ or $0 < z \le a$. Let $T_{z,a}(n) = T^+_{z,a}(n) + T^-_{z,a}(n)$ for $n \ge 0$ and $0 \le z \le a$. By (15) and (16), we have

$$T_{z,a}(n) = T_{z-1,a}(n-1) + T_{z+1,a}(n-1) \qquad \text{for } n \ge 1 \text{ and } 0 < z < a. \tag{17}$$

Applying (15) twice yields

$$\begin{aligned} T^+_{z,a}(n) &= T^+_{z-1,a}(n-1) + T^+_{z+1,a}(n-1) \\ &= T^+_{z-2,a}(n-2) + 2T^+_{z,a}(n-2) + T^+_{z+2,a}(n-2) \quad \text{for } n \ge 2 \text{ and } 2 \le z \le a-2. \end{aligned} \tag{18}$$

Similarly,

$$T_{z,a}(n) = T_{z-2,a}(n-2) + 2T_{z,a}(n-2) + T_{z+2,a}(n-2) \quad \text{for } n \ge 2 \text{ and } 2 \le z \le a-2. \tag{19}$$

We have, by symmetry,

$$T_{z,a}(n) = T_{z',a}(n) \qquad \text{for } z + z' = a. \tag{20}$$

Below we adopt the convention that $0/0 := 0$ and $c/0 := \infty$ for $c > 0$.

We are now ready to state and prove four lemmas. In particular, the inequality (21) given in Lemma 1 is a key observation for the proof of Theorem 2.

Lemma 1. *For even* $a \ge 6$ *and* $2 \le z \le a/2 - 1$,

$$\frac{T_{z,a}(n)}{T_{z-2,a}(n) + T_{z+2,a}(n)} \ge \frac{T_{z+2,a}(n)}{T_{z,a}(n) + T_{z+4,a}(n)} \qquad \text{for } n \ge 0. \tag{21}$$

Proof. By (20),

$$\frac{T_{a/2-1,a}(n)}{T_{a/2-3,a}(n) + T_{a/2+1,a}(n)} = \frac{T_{a/2+1,a}(n)}{T_{a/2-1,a}(n) + T_{a/2+3,a}(n)} \qquad \text{for } n \ge 0, \tag{22}$$

from which it follows that (21) holds for $z = a/2 - 1$.

We now prove (21) by induction on n. Since $T_{z+2,a}(0) = T_{z+2,a}(1) = 0$ for $2 \le z \le a/2 - 1$, (21) holds for $n = 0$ and $n = 1$. Suppose (21) holds for $2 \le z \le a/2 - 1$ and for $n \le m$ with some $m \ge 1$. We need to show that

$$\frac{T_{z,a}(m+1)}{T_{z-2,a}(m+1) + T_{z+2,a}(m+1)} \ge \frac{T_{z+2,a}(m+1)}{T_{z,a}(m+1) + T_{z+4,a}(m+1)} \tag{23}$$

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for $2 \le z \le a/2-1$. By (22), (23) holds for $z = a/2-1$. By (17), for $3 \le z \le a/2-2$, (23) is equivalent to

$$\frac{T_{z-1,a}(m)+T_{z+1,a}(m)}{T_{z-3,a}(m)+T_{z-1,a}(m)+T_{z+1,a}(m)+T_{z+3,a}(m)} \ge \frac{T_{z+1,a}(m)+T_{z+3,a}(m)}{T_{z-1,a}(m)+T_{z+1,a}(m)+T_{z+3,a}(m)+T_{z+5,a}(m)}. \quad (24)$$

For $3 \le z \le a/2-2$, we have, by the induction hypothesis,

$$\frac{T_{z-1,a}(m)}{T_{z-3,a}(m)+T_{z+1,a}(m)} \ge \frac{T_{z+1,a}(m)}{T_{z-1,a}(m)+T_{z+3,a}(m)}, \quad (25)$$

$$\frac{T_{z+1,a}(m)}{T_{z-1,a}(m)+T_{z+3,a}(m)} \ge \frac{T_{z+3,a}(m)}{T_{z+1,a}(m)+T_{z+5,a}(m)}. \quad (26)$$

Note that the right-hand side of (25) and the left-hand side of (26) are the same. Since for $c_i, c_i{}' \ge 0$ $(i = 1, 2)$, $c_1/c_2 \ge c_1'/c_2'$ implies

$$\frac{c_1}{c_2} \ge \frac{c_1+c_1'}{c_2+c_2'} \ge \frac{c_1'}{c_2'},$$

it follows from (25) and (26) that the left-hand side of (24) is greater than or equal to the right-hand side of (25) while the right-hand side of (24) is less than or equal to the left-hand side of (26). This establishes (24) (and hence (23)) for $3 \le z \le a/2-2$.

It remains to prove (23) for $z = 2$; i.e.,

$$\frac{T_{2,a}(m+1)}{T_{0,a}(m+1)+T_{4,a}(m+1)} \ge \frac{T_{4,a}(m+1)}{T_{2,a}(m+1)+T_{6,a}(m+1)}. \quad (27)$$

Note that $T_{0,a}(m+1) = 0$ and that, for $a = 6$, (27) is an equality (since $T_{2,6}(m+1) = T_{4,6}(m+1)$ and $T_{6,6}(m+1) = 0$). By (19), for (even) $a \ge 8$, (27) is equivalent to

$$\frac{T_{0,a}(m-1)+2T_{2,a}(m-1)+T_{4,a}(m-1)}{T_{2,a}(m-1)+2T_{4,a}(m-1)+T_{6,a}(m-1)} \ge \frac{T_{2,a}(m-1)+2T_{4,a}(m-1)+T_{6,a}(m-1)}{T_{0,a}(m-1)+2T_{2,a}(m-1)+2T_{4,a}(m-1)+2T_{6,a}(m-1)+T_{8,a}(m-1)}. \quad (28)$$

Since $T_{2,a}(m-1) = T_{4,a}(m-1) = T_{6,a}(m-1) = 0$ for $m = 1$ and $a \ge 8$, (28) holds for $m = 1$. We now assume $m > 1$ (implying that $T_{0,a}(m-1) = T_{a,a}(m-1) = 0$). By (20), for $a = 8$, (28) reduces to

$$\frac{2T_{2,8}(m-1)+T_{4,8}(m-1)}{2T_{2,8}(m-1)+2T_{4,8}(m-1)} \ge \frac{T_{2,8}(m-1)+T_{4,8}(m-1)}{2T_{2,8}(m-1)+T_{4,8}(m-1)}. \quad (29)$$

The induction hypothesis applied to $a = 8$ and $z = 2$ yields

$$\frac{T_{2,8}(m-1)}{T_{4,8}(m-1)} = \frac{T_{2,8}(m-1)}{T_{0,8}(m-1)+T_{4,8}(m-1)} \ge \frac{T_{4,8}(m-1)}{T_{2,8}(m-1)+T_{6,8}(m-1)} = \frac{T_{4,8}(m-1)}{2T_{2,8}(m-1)},$$

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which implies (or more precisely, is equivalent to) (29). To show (28) for (even) $a \geq 10$, by the induction hypothesis applied to $a \geq 10$ and $z = 2, 4$ ($\leq a/2 - 1$), we have $A_1 \geq A_2 \geq A_3$, where

$$A_k = \frac{T_{2k,a}(m-1)}{T_{2k-2,a}(m-1) + T_{2k+2,a}(m-1)} \qquad \text{for } k = 1, 2, 3.$$

Note that

$$A_1 = \frac{T_{2,a}(m-1)}{T_{0,a}(m-1) + T_{4,a}(m-1)} = \frac{T_{2,a}(m-1)}{T_{4,a}(m-1)}.$$

If $T_{2,a}(m-1) = 0$, then necessarily $m-1$ (≥ 1) is odd and $T_{4,a}(m-1) = T_{6,a}(m-1) = T_{8,a}(m-1) = 0$, so that (28) holds trivially. Suppose $T_{2,a}(m-1) > 0$. Then each of the two sides of (28) is a weighted average of A_1, A_2, and A_3. Indeed, the left-hand side of (28) equals $c_1A_1 + c_2A_2$ with weights

$$c_1 = \frac{2T_{4,a}(m-1)}{T_{2,a}(m-1) + 2T_{4,a}(m-1) + T_{6,a}(m-1)},$$

$$c_2 = \frac{T_{2,a}(m-1) + T_{6,a}(m-1)}{T_{2,a}(m-1) + 2T_{4,a}(m-1) + T_{6,a}(m-1)},$$

while the right-hand side of (28) equals $c'_1A_1 + c'_2A_2 + c'_3A_3$ with weights

$$c'_1 = \frac{T_{4,a}(m-1)}{2T_{2,a}(m-1) + 2T_{4,a}(m-1) + 2T_{6,a}(m-1) + T_{8,a}(m-1)},$$

$$c'_2 = \frac{2T_{2,a}(m-1) + 2T_{6,a}(m-1)}{2T_{2,a}(m-1) + 2T_{4,a}(m-1) + 2T_{6,a}(m-1) + T_{8,a}(m-1)},$$

$$c'_3 = \frac{T_{4,a}(m-1) + T_{8,a}(m-1)}{2T_{2,a}(m-1) + 2T_{4,a}(m-1) + 2T_{6,a}(m-1) + T_{8,a}(m-1)}.$$

Since $c_1 \geq c'_1$, it follows from $A_1 \geq A_2 \geq A_3$ that $c_1A_1 + c_2A_2 \geq c'_1A_1 + c'_2A_2 + c'_3A_3$. This shows (28) for $a \geq 10$ and completes the induction proof.

Remark 7. The proof of Lemma 1 makes use of the induction method on n with the help of the recursion (17), which is related to first-step analysis in Markov chains (see, e.g., [9]). After the first step, the initial state z moves either down to $z-1$ or up to $z+1$. To apply the induction hypothesis, it is necessary to consider the boundary cases $z = 2$ and $z = a/2 - 1$ separately from $3 \leq z \leq a/2 - 2$. (The induction hypothesis is applicable neither in the case that $z = 2$ moves down to 1 nor in the case that $z = a/2 - 1$ moves up to $a/2$.) The proofs of Lemmas 2 and 3 and Theorem 2 also make use of the recursions (15), (18), and (19). Again, the boundary cases need to be treated separately.

Lemma 2. *For even $n \geq 2$, $T^+_{z,a}(n)/T_{z,a}(n)$ is increasing in $z \in \{2, 4, \ldots, a-2\}$. For odd $n \geq 1$, $T^+_{z,a}(n)/T_{z,a}(n)$ is increasing in $z \in \{1, 3, \ldots, a-1\}$.*

Proof. Let $\rho(n) = 1$ or 2 according as n is odd or even. We show that, for $n \geq 1$, $T^+_{z,a}(n)/T_{z,a}(n)$ is increasing in $z \in \{\rho(n), \rho(n)+2, \ldots, a-\rho(n)\}$; i.e., for $n \geq 1$,

$$\frac{T^+_{z+2,a}(n)}{T_{z+2,a}(n)} \geq \frac{T^+_{z,a}(n)}{T_{z,a}(n)} \qquad \text{for } z = \rho(n), \rho(n)+2, \ldots, a-\rho(n)-2. \tag{30}$$

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We proceed by induction on n. Since $T^+_{z,a}(1)/T_{z,a}(1) = 0$ for $z < a-1$, and $T^+_{a-1,a}(1)/T_{a-1,a}(1) = 1$, (30) holds for $n = 1$. Suppose (30) holds for $n \le m$ with some $m \ge 1$. We need to show that

$$\frac{T^+_{z+2,a}(m+1)}{T_{z+2,a}(m+1)} \ge \frac{T^+_{z,a}(m+1)}{T_{z,a}(m+1)} \qquad \text{for } z = \rho(m+1), \rho(m+1)+2, \ldots, a-\rho(m+1)-2. \tag{31}$$

Consider the case that m is even. Then $m \ge 2$ and $\rho(m+1) = 1$. For $z = 1, 3, \ldots, a-3$, we have, by (15) and (17),

$$\frac{T^+_{z,a}(m+1)}{T_{z,a}(m+1)} = \frac{T^+_{z-1,a}(m) + T^+_{z+1,a}(m)}{T_{z-1,a}(m) + T_{z+1,a}(m)}, \tag{32}$$

$$\frac{T^+_{z+2,a}(m+1)}{T_{z+2,a}(m+1)} = \frac{T^+_{z+1,a}(m) + T^+_{z+3,a}(m)}{T_{z+1,a}(m) + T_{z+3,a}(m)}. \tag{33}$$

The right-hand side of (32) equals $T^+_{z+1,a}(m)/T_{z+1,a}(m)$ for $z = 1$ and is less than or equal to $T^+_{z+1,a}(m)/T_{z+1,a}(m)$ for $z > 1$ since, by the induction hypothesis, for $z = 3, \ldots, a-3$,

$$\frac{T^+_{z+1,a}(m)}{T_{z+1,a}(m)} \ge \frac{T^+_{z-1,a}(m)}{T_{z-1,a}(m)}.$$

The right-hand side of (33) equals $T^+_{z+1,a}(m)/T_{z+1,a}(m)$ for $z = a-3$ and is greater than or equal to $T^+_{z+1,a}(m)/T_{z+1,a}(m)$ for $z < a-3$ since, by the induction hypothesis, for $z = 1, \ldots, a-5$,

$$\frac{T^+_{z+3,a}(m)}{T_{z+3,a}(m)} \ge \frac{T^+_{z+1,a}(m)}{T_{z+1,a}(m)}.$$

This proves (31) for the case of even m. The case of odd m can be treated similarly.

Lemma 3. *For even $a \ge 4$ and $n \ge 0$,*

$$\frac{T^+_{1,a}(n+2)}{T_{1,a}(n+2)} \ge \frac{T^+_{1,a}(n)}{T_{1,a}(n)}, \tag{34}$$

$$\frac{T^+_{2,a}(n+2)}{T_{2,a}(n+2)} \ge \frac{T^+_{2,a}(n)}{T_{2,a}(n)}. \tag{35}$$

Proof. Note that (34) holds trivially for even n since both sides of (34) are $0/0$ for even n. We now prove (34) for odd $n \ge 1$. By (15),

$$T^+_{1,a}(n+2) = T^+_{0,a}(n+1) + T^+_{2,a}(n+1) = T^+_{2,a}(n+1) = T^+_{1,a}(n) + T^+_{3,a}(n). \tag{36}$$

Similarly, by (17), $T_{1,a}(n+2) = T_{1,a}(n) + T_{3,a}(n)$, which together with (36) implies that

$$\frac{T^+_{1,a}(n+2)}{T_{1,a}(n+2)} = \frac{T^+_{1,a}(n) + T^+_{3,a}(n)}{T_{1,a}(n) + T_{3,a}(n)} \ge \frac{T^+_{1,a}(n)}{T_{1,a}(n)},$$

where the inequality follows from $T^+_{1,a}(n)/T_{1,a}(n) \le T^+_{3,a}(n)/T_{3,a}(n)$ (by Lemma 2).

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Next, to prove (35), it suffices to consider the case of even n. For $n=0$, the right-hand side of (35) is 0, so (35) holds. For even $n \geq 2$ and $a=4$, both sides of (35) equal $\frac{1}{2}$ by symmetry. For even $n \geq 2$ and $a \geq 6$, we have, by (18) and (19),

$$\frac{T^+_{2,a}(n+2)}{T_{2,a}(n+2)} = \frac{T^+_{0,a}(n) + 2T^+_{2,a}(n) + T^+_{4,a}(n)}{T_{0,a}(n) + 2T_{2,a}(n) + T_{4,a}(n)}$$

$$= \frac{2T^+_{2,a}(n) + T^+_{4,a}(n)}{2T_{2,a}(n) + T_{4,a}(n)} \geq \frac{T^+_{2,a}(n)}{T_{2,a}(n)},$$

where the inequality follows from $T^+_{2,a}(n)/T_{2,a}(n) \leq T^+_{4,a}(n)/T_{4,a}(n)$ (by Lemma 2). The proof is complete.

Lemma 4. *Let* $\alpha_1 > 0$ *and* $\alpha_i \geq \beta_i \geq 0$, $i=1, 2, 3, 4$. *Suppose*

$$\frac{\beta_4}{\alpha_4} \geq \frac{\beta_1}{\alpha_1}, \qquad \frac{\beta_1+\beta_3}{\alpha_1+\alpha_3} \geq \frac{\beta_2}{\alpha_2}, \qquad \frac{\beta_2+\beta_4}{\alpha_2+\alpha_4} \geq \frac{\beta_3}{\alpha_3}, \qquad \frac{\alpha_2}{\alpha_1+\alpha_3} \geq \frac{\alpha_3}{\alpha_2+\alpha_4}.$$

Then

$$\frac{\beta_1+\beta_4}{\alpha_1+\alpha_4} \geq \frac{\beta_2+\beta_3}{\alpha_2+\alpha_3}.$$

Proof. If $\alpha_2 = 0$ then $\alpha_3 = 0$ since $\alpha_2/(\alpha_1+\alpha_3) \geq \alpha_3/(\alpha_2+\alpha_4)$. So

$$\frac{\beta_1+\beta_4}{\alpha_1+\alpha_4} \geq 0 = \frac{\beta_2+\beta_3}{\alpha_2+\alpha_3}.$$

If $\alpha_3 = 0$, then

$$\frac{\beta_4}{\alpha_4} \geq \frac{\beta_1}{\alpha_1} = \frac{\beta_1+\beta_3}{\alpha_1+\alpha_3} \geq \frac{\beta_2}{\alpha_2},$$

implying that

$$\frac{\beta_1+\beta_4}{\alpha_1+\alpha_4} \geq \frac{\beta_2}{\alpha_2} = \frac{\beta_2+\beta_3}{\alpha_2+\alpha_3}.$$

Now suppose $\alpha_2 > 0$ and $\alpha_3 > 0$. We have

$$0 \leq \frac{\alpha_2\alpha_3}{\alpha_1+\alpha_3}\left[\frac{\beta_2+\beta_4}{\alpha_2+\alpha_4} - \frac{\beta_3}{\alpha_3}\right] + \alpha_2\left[\frac{\beta_1+\beta_3}{\alpha_1+\alpha_3} - \frac{\beta_2}{\alpha_2}\right]$$

$$= \frac{\alpha_2(\alpha_2+\alpha_4)\beta_1 + \alpha_2\alpha_3\beta_4 - (\alpha_1\alpha_2+\alpha_1\alpha_4+\alpha_3\alpha_4)\beta_2}{(\alpha_1+\alpha_3)(\alpha_2+\alpha_4)},$$

$$0 \leq \frac{\alpha_2\alpha_3}{\alpha_2+\alpha_4}\left[\frac{\beta_1+\beta_3}{\alpha_1+\alpha_3} - \frac{\beta_2}{\alpha_2}\right] + \alpha_3\left[\frac{\beta_2+\beta_4}{\alpha_2+\alpha_4} - \frac{\beta_3}{\alpha_3}\right]$$

$$= \frac{\alpha_2\alpha_3\beta_1 + \alpha_3(\alpha_1+\alpha_3)\beta_4 - (\alpha_1\alpha_2+\alpha_1\alpha_4+\alpha_3\alpha_4)\beta_3}{(\alpha_1+\alpha_3)(\alpha_2+\alpha_4)},$$

implying that

$$\frac{\alpha_2(\alpha_2+\alpha_4)\beta_1 + \alpha_2\alpha_3\beta_4}{\alpha_1\alpha_2+\alpha_1\alpha_4+\alpha_3\alpha_4} \geq \beta_2, \qquad \frac{\alpha_2\alpha_3\beta_1 + \alpha_3(\alpha_1+\alpha_3)\beta_4}{\alpha_1\alpha_2+\alpha_1\alpha_4+\alpha_3\alpha_4} \geq \beta_3.$$

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So $C \geq \beta_2 + \beta_3$, where

$$C := \frac{\alpha_2(\alpha_2+\alpha_3+\alpha_4)\beta_1 + \alpha_3(\alpha_1+\alpha_2+\alpha_3)\beta_4}{\alpha_1\alpha_2+\alpha_1\alpha_4+\alpha_3\alpha_4}.$$

To show $(\beta_1+\beta_4)/(\alpha_1+\alpha_4) \geq (\beta_2+\beta_3)/(\alpha_2+\alpha_3)$, since $C \geq \beta_2+\beta_3$ it suffices to verify that

$$C_1 := (\alpha_2+\alpha_3)(\beta_1+\beta_4) - (\alpha_1+\alpha_4)C \geq 0. \tag{37}$$

We have

$$\begin{aligned} C_1(\alpha_1\alpha_2+\alpha_1\alpha_4+\alpha_3\alpha_4) &= \alpha_4[\alpha_3(\alpha_1+\alpha_3) - \alpha_2(\alpha_2+\alpha_4)]\beta_1 \\ &\quad + \alpha_1[\alpha_2(\alpha_2+\alpha_4) - \alpha_1(\alpha_1+\alpha_3)]\beta_4 \\ &= [\alpha_2(\alpha_2+\alpha_4) - \alpha_3(\alpha_1+\alpha_3)](\alpha_1\beta_4 - \alpha_4\beta_1) \geq 0, \end{aligned}$$

since $\alpha_2/(\alpha_1+\alpha_3) \geq \alpha_3/(\alpha_2+\alpha_4)$ and $\beta_4/\alpha_4 \geq \beta_1/\alpha_1$. This proves (37), and completes the proof.

Proof of Theorem 2. We claim that, for even $a \geq 4$ and $0 < z < a/2$,

$$\frac{T^+_{z,a}(n+2)}{T_{z,a}(n+2)} \geq \frac{T^+_{z,a}(n)}{T_{z,a}(n)}, \qquad n \geq 0. \tag{38}$$

By Lemma 3, (38) holds for $z = 1$ and $z = 2$. Consequently, (38) holds for $a = 4$ and $a = 6$. Note also that

$$\frac{T^+_{a/2,a}(n+2)}{T_{a/2,a}(n+2)} \geq \frac{T^+_{a/2,a}(n)}{T_{a/2,a}(n)}, \qquad n \geq 0. \tag{39}$$

(If n and $a/2$ have opposite parity, both sides of (39) are $0/0 = 0$. For n and $a/2$ of the same parity, $T^+_{a/2,a}(n)/T_{a/2,a}(n) = 0$ or $\frac{1}{2}$ according as $n < a/2$ or $\geq a/2$. This shows (39).)

We now prove (38) for $a \geq 8$ and $3 \leq z < a/2$ by induction on n. For $n = 0$ and $n = 1$, the right-hand side of (38) equals 0 since $T^+_{z,a}(0) = T^+_{z,a}(1) = 0$ for $0 < z < a/2$. So (38) holds for $n \leq 1$. Suppose (38) holds for $n \leq m$ with some $m \geq 1$. We need to show that

$$\frac{T^+_{z,a}(m+3)}{T_{z,a}(m+3)} \geq \frac{T^+_{z,a}(m+1)}{T_{z,a}(m+1)} \qquad \text{for even } a \geq 8 \text{ and } 3 \leq z < \frac{a}{2}.$$

By (15) and (17), this is equivalent to

$$\frac{T^+_{z-1,a}(m+2) + T^+_{z+1,a}(m+2)}{T_{z-1,a}(m+2) + T_{z+1,a}(m+2)} \geq \frac{T^+_{z-1,a}(m) + T^+_{z+1,a}(m)}{T_{z-1,a}(m) + T_{z+1,a}(m)} \tag{40}$$

for $a \geq 8$ and $3 \leq z < a/2$. By (18) and (19) applied to the left-hand side of (40), (40) is equivalent to

$$\frac{T^+_{z-3,a}(m) + 3T^+_{z-1,a}(m) + 3T^+_{z+1,a}(m) + T^+_{z+3,a}(m)}{T_{z-3,a}(m) + 3T_{z-1,a}(m) + 3T_{z+1,a}(m) + T_{z+3,a}(m)} \geq \frac{T^+_{z-1,a}(m) + T^+_{z+1,a}(m)}{T_{z-1,a}(m) + T_{z+1,a}(m)} \tag{41}$$

for $a \geq 8$ and $3 \leq z < a/2$. Note that (41) holds trivially if z and m are of the same parity. Suppose z and m have opposite parity. If $T_{z-1,a}(m+2) = 0$, then $m+2 < z-1$ $(< a/2 - 1)$, so that both sides of (40) (and hence (41)) are 0.

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Now suppose $T_{z-1,a}(m+2) > 0$. We first prove (41) for $z = 3$ (and even m), in which case we have $T_{z-3,a}(m) = T^+_{z-3,a}(m) = 0$, so that (41) becomes

$$\frac{3T^+_{2,a}(m) + 3T^+_{4,a}(m) + T^+_{6,a}(m)}{3T_{2,a}(m) + 3T_{4,a}(m) + T_{6,a}(m)} \geq \frac{T^+_{2,a}(m) + T^+_{4,a}(m)}{T_{2,a}(m) + T_{4,a}(m)}.$$

This inequality holds since, by Lemma 2,

$$\frac{T^+_{6,a}(m)}{T_{6,a}(m)} \geq \frac{T^+_{4,a}(m)}{T_{4,a}(m)} \geq \frac{T^+_{2,a}(m)}{T_{2,a}(m)}.$$

We now prove (41) for $4 \leq z < a/2$ (in which case necessarily $a \geq 10$). Note that $T_{z-1,a}(m+2) > 0$ implies $T_{z-3,a}(m) > 0$. By the induction hypothesis together with (39), we have

$$\frac{T^+_{z-1,a}(m+2)}{T_{z-1,a}(m+2)} \geq \frac{T^+_{z-1,a}(m)}{T_{z-1,a}(m)}, \qquad \frac{T^+_{z+1,a}(m+2)}{T_{z+1,a}(m+2)} \geq \frac{T^+_{z+1,a}(m)}{T_{z+1,a}(m)}.$$

By (18) and (19) applied to the left-hand sides of each of these inequalities, we have

$$\frac{T^+_{z-3,a}(m) + 2T^+_{z-1,a}(m) + T^+_{z+1,a}(m)}{T_{z-3,a}(m) + 2T_{z-1,a}(m) + T_{z+1,a}(m)} \geq \frac{T^+_{z-1,a}(m)}{T_{z-1,a}(m)}, \tag{42}$$

$$\frac{T^+_{z-1,a}(m) + 2T^+_{z+1,a}(m) + T^+_{z+3,a}(m)}{T_{z-1,a}(m) + 2T_{z+1,a}(m) + T_{z+3,a}(m)} \geq \frac{T^+_{z+1,a}(m)}{T_{z+1,a}(m)}. \tag{43}$$

Noting that the left-hand side of (42) equals

$$c\left(\frac{T^+_{z-3,a}(m) + T^+_{z+1,a}(m)}{T_{z-3,a}(m) + T_{z+1,a}(m)}\right) + (1-c)\left(\frac{T^+_{z-1,a}(m)}{T_{z-1,a}(m)}\right),$$

where

$$c = \frac{T_{z-3,a}(m) + T_{z+1,a}(m)}{T_{z-3,a}(m) + 2T_{z-1,a}(m) + T_{z+1,a}(m)} > 0,$$

the inequality in (42) implies that

$$\frac{T^+_{z-3,a}(m) + T^+_{z+1,a}(m)}{T_{z-3,a}(m) + T_{z+1,a}(m)} \geq \frac{T^+_{z-1,a}(m)}{T_{z-1,a}(m)}. \tag{44}$$

Similarly, if $T_{z-1,a}(m) > 0$, then the inequality in (43) implies that

$$\frac{T^+_{z-1,a}(m) + T^+_{z+3,a}(m)}{T_{z-1,a}(m) + T_{z+3,a}(m)} \geq \frac{T^+_{z+1,a}(m)}{T_{z+1,a}(m)}. \tag{45}$$

If $T_{z-1,a}(m) = 0$, then $T_{z+1,a}(m) = T_{z+3,a}(m) = 0$ (since $z-1 < z+3 \leq a-(z-1)$), so the inequality in (45) holds trivially.

Let

$$\alpha_1 = T_{z-3,a}(m), \qquad \alpha_2 = T_{z-1,a}(m), \qquad \alpha_3 = T_{z+1,a}(m), \qquad \alpha_4 = T_{z+3,a}(m),$$

$$\beta_1 = T^+_{z-3,a}(m), \qquad \beta_2 = T^+_{z-1,a}(m), \qquad \beta_3 = T^+_{z+1,a}(m), \qquad \beta_4 = T^+_{z+3,a}(m).$$

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Since $T_{z-1,a}(m+2) > 0$, we have $\alpha_1 = T_{z-3,a}(m) > 0$. By (44) and (45),

$$\frac{\beta_1 + \beta_3}{\alpha_1 + \alpha_3} \geq \frac{\beta_2}{\alpha_2}, \qquad \frac{\beta_2 + \beta_4}{\alpha_2 + \alpha_4} \geq \frac{\beta_3}{\alpha_3}.$$

Furthermore, by Lemma 1,

$$\frac{\alpha_2}{\alpha_1 + \alpha_3} = \frac{T_{z-1,a}(m)}{T_{z-3,a}(m) + T_{z+1,a}(m)} \geq \frac{T_{z+1,a}(m)}{T_{z-1,a}(m) + T_{z+3,a}(m)} = \frac{\alpha_3}{\alpha_2 + \alpha_4},$$

and by Lemma 2,

$$\frac{\beta_1}{\alpha_1} = \frac{T^+_{z-3,a}(m)}{T_{z-3,a}(m)} \leq \frac{T^+_{z+3,a}(m)}{T_{z+3,a}(m)} = \frac{\beta_4}{\alpha_4}.$$

It follows from Lemma 4 that

$$\frac{\beta_1 + \beta_4}{\alpha_1 + \alpha_4} \geq \frac{\beta_2 + \beta_3}{\alpha_2 + \alpha_3},$$

implying that

$$\begin{aligned}\frac{T^+_{z-1,a}(m) + T^+_{z+1,a}(m)}{T_{z-1,a}(m) + T_{z+1,a}(m)} &= \frac{\beta_2 + \beta_3}{\alpha_2 + \alpha_3} \\ &\leq \frac{\beta_1 + 3\beta_2 + 3\beta_3 + \beta_4}{\alpha_1 + 3\alpha_2 + 3\alpha_3 + \alpha_4} \\ &= \frac{T^+_{z-3,a}(m) + 3T^+_{z-1,a}(m) + 3T^+_{z+1,a}(m) + T^+_{z+3,a}(m)}{T_{z-3,a}(m) + 3T_{z-1,a}(m) + 3T_{z+1,a}(m) + T_{z+3,a}(m)},\end{aligned}$$

proving (41) for $4 \leq z < a/2$. This completes the proof of (38), which implies that

$$\frac{|\mathcal{S}^+_{z,a}(n)|}{|\mathcal{S}^+_{z,a}(n)| + |\mathcal{S}^-_{z,a}(n)|} = \frac{T^+_{z,a}(n)}{T_{z,a}(n)}$$

is increasing as $n \in \{z, z+2, \ldots\}$ increases. Finally, as $n \in \{z, z+2, \ldots\}$ tends to ∞, it follows from (1) and (2) (with $p = \frac{1}{2}$) that

$$\frac{|\mathcal{S}^+_{z,a}(n)|}{|\mathcal{S}^-_{z,a}(n)|} = \frac{\mathbb{P}_{p,z,a}(N=n, I=1)}{\mathbb{P}_{p,z,a}(N=n, I=0)} = \frac{\sum_{1 \leq \nu < a/2} \cos^{n-1}(\pi\nu/a) \sin(\pi\nu/a) \sin(\pi(a-z)\nu/a)}{\sum_{1 \leq \nu < a/2} \cos^{n-1}(\pi\nu/a) \sin(\pi\nu/a) \sin(\pi z\nu/a)}$$

approaches 1, implying that $|\mathcal{S}^+_{z,a}(n)|/(|\mathcal{S}^+_{z,a}(n)| + |\mathcal{S}^-_{z,a}(n)|)$ increases to $\frac{1}{2}$ in the limit. The proof of Theorem 2 is complete.

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3. Proof of Theorem 3

Theorem 2 plays a key role in the following proof of Theorem 3.

Proof of Theorem 3. By (4), for $0 < z < a/2$ and $n = z, z+2, \ldots,$

$$\begin{aligned}\mathbb{P}_{p,z,a}(N=n) &= (pq)^{n/2}\left[\left(\frac{p}{q}\right)^{(a-z)/2}|\mathcal{S}^{+}_{z,a}(n)| + \left(\frac{q}{p}\right)^{z/2}|\mathcal{S}^{-}_{z,a}(n)|\right]\\ &= (pq)^{n/2}\left[\left(\frac{p}{q}\right)^{(a-z)/2}\frac{T^{+}_{z,a}(n)}{T_{z,a}(n)} + \left(\frac{q}{p}\right)^{z/2}\left(1-\frac{T^{+}_{z,a}(n)}{T_{z,a}(n)}\right)\right]T_{z,a}(n)\\ &= (pq)^{n/2}\left\{\left[\left(\frac{p}{q}\right)^{(a-z)/2} - \left(\frac{q}{p}\right)^{z/2}\right]\frac{T^{+}_{z,a}(n)}{T_{z,a}(n)} + \left(\frac{q}{p}\right)^{z/2}\right\}T_{z,a}(n).\end{aligned}$$

So, for $0 < p < p' \leq \frac{1}{2}$, $0 < z < a/2$, and $n = z, z+2, \ldots,$

$$\frac{\mathbb{P}_{p',z,a}(N=n)}{\mathbb{P}_{p,z,a}(N=n)} = \left(\frac{p'q'}{pq}\right)^{n/2} H_{p,p',z,a}\left(\frac{T^{+}_{z,a}(n)}{T_{z,a}(n)}\right), \tag{46}$$

where

$$H_{p,p',z,a}(x) = \frac{[(p'/q')^{(a-z)/2} - (q'/p')^{z/2}]x + (q'/p')^{z/2}}{[(p/q)^{(a-z)/2} - (q/p)^{z/2}]x + (q/p)^{z/2}} \qquad \text{for } 0 \leq x \leq 1.$$

Note that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}H_{p,p',z,a}(x)\\ &= \left\{\left[\left(\frac{p}{q}\right)^{(a-z)/2} - \left(\frac{q}{p}\right)^{z/2}\right]x + \left(\frac{q}{p}\right)^{z/2}\right\}^{-2}\left(\frac{qq'}{pp'}\right)^{z/2}\left[\left(p'/q'\right)^{a/2} - \left(\frac{p}{q}\right)^{a/2}\right] > 0.\end{aligned} \tag{47}$$

Since, by Theorem 2, $T^{+}_{z,a}(n)/T_{z,a}(n)$ is increasing in $n \in \{z, z+2, \ldots\}$, it follows from (46) and (47) that $\mathbb{P}_{p',z,a}(N=n)/\mathbb{P}_{p,z,a}(N=n)$ is increasing in $n \in \{z, z+2, \ldots\}$. This proves that $\{\mathcal{L}_{p,z,a}(N)\colon 0 \leq p \leq \frac{1}{2}\}$ has monotone (increasing) likelihood ratio. For $a/2 < z < a$, note that $\mathcal{L}_{p,z,a}(N) = \mathcal{L}_{p',z',a}(N)$ with $p' = 1-p$ and $z' = a-z$, implying that $\{\mathcal{L}_{p,z,a}(N)\colon \frac{1}{2} \leq p \leq 1\}$ has monotone (decreasing) likelihood ratio, completing the proof.

4. Proof of Theorem 4

Proof of Theorem 4. Fix $0 < z < a/2$. We claim that, for $n \geq 0$,

$$f(p, n) := \mathbb{P}_{p,z,a}(N > n) \text{ is increasing in } p \in \left[0, \tfrac{1}{2}\right]. \tag{48}$$

Let

$$S_z(n) := \{(\omega_1, \ldots, \omega_n) \in \{-1, 1\}^n \colon 0 < z + \omega_1 + \cdots + \omega_i < a,\, i = 1, \ldots, n\}, \tag{49}$$

and let $(\omega_1, \ldots, \omega_n)_z := \left(z, z+\omega_1, z+\omega_1+\omega_2, \ldots, z+\sum_{i=1}^{n}\omega_i\right)$, which is the sample path starting at z with successive increments $\omega_1, \ldots, \omega_n$. Since z is fixed, for convenience we may identify $(\omega_1, \ldots, \omega_n)$ with the corresponding sample path $(\omega_1, \ldots, \omega_n)_z$. In particular, we refer to $S_z(n)$ as the collection of all sample paths starting at z and strictly staying between 0 and a up to time n. (By abusing notation, for $(\omega_1, \ldots, \omega_n) \in S_z(n)$, we also write $(\omega_1, \ldots, \omega_n)_z \in S_z(n)$.)

We now prove (48) by induction on n. Plainly, (48) holds for $n = 0$. Suppose (48) holds for $n \le m$ with some $m \ge 0$. We need to show that

$$f(p, m+1) \text{ is increasing in } p \in \left[0, \tfrac{1}{2}\right]. \tag{50}$$

By (49), $f(p, m+1) := \mathbb{P}_{p,z,a}(N > m+1) = \mathbb{P}\{(X_{1,p}, \ldots, X_{m+1,p}) \in S_z(m+1)\}$, where the $X_{i,p}$ are defined as in (3). We partition the sample paths of $S_z(m+1)$ into subsets Δ_i, $i = 0, 1, \ldots, m+1$, where Δ_0 is the subset of those sample paths that always stay below $a-1$, and Δ_i $(i = 1, \ldots, m+1)$ is the subset of those sample paths that visit $a-1$ at time i for the first time. (Note that for $i \ge 1$, $\Delta_i = \emptyset$ if i and $a-z-1$ have opposite parity.) Then $f(p, m+1) = \sum_{i=0}^{m+1} g(p, m+1, i)$, where $g(p, m+1, i) := \mathbb{P}\{(X_{1,p}, \ldots, X_{m+1,p})_z \in \Delta_i\}$. To prove (50), it suffices to show that, for $i = 0, 1, \ldots, m+1$, $g(p, m+1, i)$ is increasing in $p \in \left[0, \frac{1}{2}\right]$. To show $g(p, m+1, 0)$ is increasing in $p \in \left[0, \frac{1}{2}\right]$, we have

$$\begin{aligned} g(p, m+1, 0) &= \mathbb{P}\{(X_{1,p}, \ldots, X_{m+1,p})_z \in \Delta_0\} \\ &= \mathbb{P}\{(X_{1,p}, \ldots, X_{m+1,p})_z \text{ stays strictly between 0 and } a-1\} \\ &= \mathbb{P}_{p,z,a-1}(N > m+1), \end{aligned}$$

which, by Theorems 1 and 3, is increasing in $p \in \left[0, \frac{1}{2}\right]$.

To show that $g(p, m+1, i)$ is increasing in $p \in \left[0, \frac{1}{2}\right]$ for $i = 1, \ldots, m+1$, we further partition the Δ_i into $\Delta_{i,j}$ $(i \le j \le m+1)$ where $\Delta_{i,i}$ is the subset of those sample paths that after time i never revisit z, and $\Delta_{i,j}$ $(j > i)$ is the subset of those sample paths that after time i revisit z at time j for the first time. Let $h(p, m+1, i, j) := \mathbb{P}\{(X_{1,p}, \ldots, X_{m+1,p})_z \in \Delta_{i,j}\}$. We have $g(p, m+1, i) = \sum_{j=i}^{m+1} h(p, m+1, i, j)$. It suffices to show that each $h(p, m+1, i, j)$ is increasing in $p \in \left[0, \frac{1}{2}\right]$.

For $i < j \le m+1$, let

$$\begin{aligned} A_{i,j} := \{(\omega_1, \ldots, \omega_j) \in \{-1, 1\}^j :\ & 0 < z + \omega_1 + \cdots + \omega_\ell < a-1,\ \ell = 1, \ldots, i-1; \\ & z + \omega_1 + \cdots + \omega_i = a-1; \\ & z < a-1 + \omega_{i+1} + \cdots + \omega_\ell < a,\ \ell = i+1, \ldots, j-1; \\ & \omega_{i+1} + \cdots + \omega_j = -(a-z-1)\}, \end{aligned}$$

and

$$\begin{aligned} S_z(m+1-j) := &\ \{(\omega_1, \ldots, \omega_{m+1-j}) \in \{-1, 1\}^{m+1-j} : \\ &\quad 0 < z + \omega_1 + \cdots + \omega_\ell < a,\ \ \ell = 1, \ldots, m+1-j\} \\ = &\ \{(\omega_{j+1}, \ldots, \omega_{m+1}) \in \{-1, 1\}^{m+1-j} : \\ &\quad 0 < z + \omega_{j+1} + \cdots + \omega_\ell < a,\ \ \ell = j+1, \ldots, m+1\}. \end{aligned}$$

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Since $(\omega_1, \ldots, \omega_{m+1})_z \in \Delta_{i,j}$ if and only if $(\omega_1, \ldots, \omega_{m+1}) \in A_{i,j} \times S_z(m+1-j)$,

$$\begin{aligned} h(p, m+1, i, j) &= \mathbb{P}\{(X_{1,p}, \ldots, X_{m+1,p})_z \in \Delta_{i,j}\} \\ &= \mathbb{P}\{(X_{1,p}, \ldots, X_{m+1,p}) \in A_{i,j} \times S_z(m+1-j)\} \\ &= \mathbb{P}\{(X_{1,p}, \ldots, X_{j,p}) \in A_{i,j}\}\mathbb{P}\{(X_{j+1,p}, \ldots, X_{m+1,p}) \in S_z(m+1-j)\} \\ &= \mathbb{P}\{(X_{1,p}, \ldots, X_{j,p}) \in A_{i,j}\}\mathbb{P}_{p,z,a}(N > m+1-j). \end{aligned} \tag{51}$$

By the induction hypothesis, $\mathbb{P}_{p,z,a}(N > m+1-j)$ is increasing in $p \in \left[0, \frac{1}{2}\right]$. Also,

$$\mathbb{P}\{(X_{1,p}, \ldots, X_{j,p}) \in A_{i,j}\} = \begin{cases} (pq)^{j/2}|A_{i,j}| & \text{if } j \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

which is increasing in $p \in \left[0, \frac{1}{2}\right]$. By (51), $h(p, m+1, i, j)$ is increasing in $p \in \left[0, \frac{1}{2}\right]$.

It remains to show that $h(p, m+1, i, i)$ is increasing in $p \in \left[0, \frac{1}{2}\right]$. Observe that each sample path $(\omega_1, \ldots, \omega_{m+1})_z \in \Delta_{i,i}$ ends above z at time $m+1$, so that there are more $+1$ increments than -1 increments. It follows that the probability of each sample path in $\Delta_{i,i}$ is increasing in $p \in \left[0, \frac{1}{2}\right]$. Consequently, $h(p, m+1, i, i) = \mathbb{P}\{(X_{1,p}, \ldots, X_{m+1,p})_z \in \Delta_{i,i}\}$ is increasing in $p \in \left[0, \frac{1}{2}\right]$. This completes the induction proof of (48).

It follows from (48) that, for $0 < z < a/2$, the distribution $\mathcal{L}_{p,z,a}(N)$ is stochastically increasing in $p \in \left[0, \frac{1}{2}\right]$. Since $\mathcal{L}_{p,z,a}(N) = \mathcal{L}_{p',z',a}(N)$ for $p' = 1-p$ and $z' = a-z$, it follows that $\mathcal{L}_{p,z,a}(N)$ is stochastically decreasing in $p \in \left[\frac{1}{2}, 1\right]$ for $a/2 < z < a$. The proof is complete.

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