3

The quark model

3.1 Introduction

The quark model arose from the analysis of symmetry patterns observed when particles were grouped together according to their spin and parity. When the eight mesons with $J^p = 0^-$ are displayed in a strangeness (S) versus isospin (I_3) plane, they form the octet of Fig. 3.1. An identical pattern emerges for the eight vector mesons with $J^p = 1^-$ also shown in Fig. 3.1. The vector mesons are excited states of the particles in the $J^p = 0^-$ octet. The symmetry pattern was interpreted as a generalization of the isospin group SU(2) to the group SU(3) which incorporates both isospin and strangeness. Gell-Mann and Neeman (1964) proposed that the eight baryons with $J^p = \frac{1}{2}^+$ also belong to an octet of SU(3), thus establishing a parallelism between meson and baryon states. Finally, many static properties of the particles exhibit the SU(3) symmetry.

Since the fundamental representation of the group SU(3) is a triplet, it is natural to try to interpret the hadronic states in the octets as bound states of triplets or of triplets with antitriplets. If the fundamental fields also carry baryon number, the product of triplet $\otimes$ antitriplet would be mesons with zero baryon number. The product of three triplets carries baryon number and contain octets and a decuplet as was required by the observed states of baryons. This is the quark model of Gell-Mann (1964) and Zweig (1964).

The spectroscopy of particles and their SU(3) properties are covered in many books, for instance in the references at the end of this chapter, and we shall concentrate on symmetries of the currents, which are more relevant for the electroweak theory.

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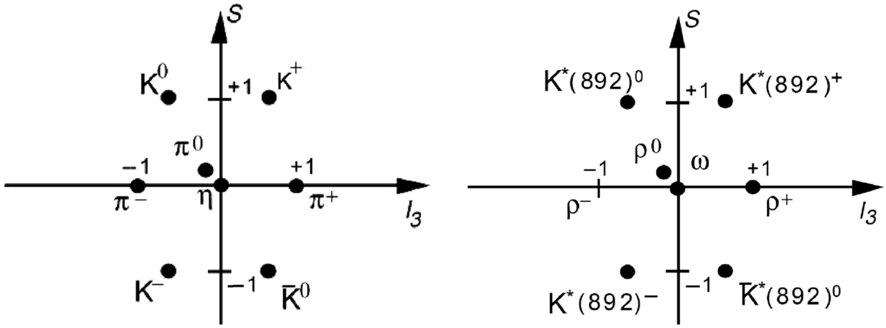

Figure 3.1. Meson octets with $J^P = 0^-$ and $J^P = 1^-$.

3.2 Current algebra

The original model contained a triplet of quarks

$$q = \begin{pmatrix} \mathrm{u} \\ \mathrm{d} \\ \mathrm{s} \end{pmatrix} \tag{3.1}$$

with the quantum numbers

Quark	Q/e	I	I_3	B	Y
u	2/3	1/2	1/2	1/3	1/3
d	−1/3	1/2	−1/2	1/3	1/3
s	−1/3	0	0	1/3	−2/3

where Q, I, I_3, B, and Y are the charge, isospin, third component of isospin, baryon number, and hypercharge, respectively. The quantum numbers of the quarks satisfy the Gell-Mann–Nishijima relation,

$$Q = T_3 + \frac{Y}{2}, \tag{3.2}$$

a rule that was established originally for hadronic states.

Next we shall rewrite the currents in terms of quark fields and formulate several of their properties. This approach is motivated by the fact that several properties of the currents and their couplings to hadrons are explained as symmetry properties of SU(3) and in many cases they are identical with predictions of the simple quark model. In fact, for a long time the quark model was used as a tool for abstracting properties and relations, whose validity is more general in field theories. In the early days the quark model was supplemented with strong interactions mediated by vector mesons in order to verify the validity of the results in theories with

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interactions. Among the regularities are relations between the masses of particles within a multiplet and regularities of the currents. We describe below conservation laws of the currents and outline the algebra of currents.

The electromagnetic interaction of leptons is

$$\mathcal{L}_{\text{int}}^{\text{em}} = e j_\mu^{\text{em}}(x) A^\mu(x), \tag{3.3}$$

with

$$j_\mu(x) = \bar{e}(x)\gamma_\mu e(x) + \bar{\mu}(x)\gamma_\mu \mu(x) + \cdots. \tag{3.4}$$

Similarly, we can construct the electromagnetic current of quarks,

$$\begin{aligned} j_\mu(x) &= \sum_i e_{q_i} \bar{q}_i \gamma_\mu q_i \\ &= \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s \\ &= \frac{1}{2}(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d) + \frac{1}{6}(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d - 2\bar{s}\gamma_\mu s). \end{aligned} \tag{3.5}$$

In the last equation we separated the current into two parts, in order to show explicitly its SU(3) content. Let λ^a be the Gell-Mann matrices for SU(3), then we define vector and axial currents

$$j_\mu^a(x) = \bar{q}(x)\gamma_\mu \frac{\lambda^a}{2} q(x) \tag{3.6}$$

$$j_{\mu 5}^a(x) = \bar{q}(x)\gamma_\mu \gamma_5 \frac{\lambda^a}{2} q(x) \tag{3.7}$$

with q given in Eq. (3.1). Then the electromagnetic current in (3.5) is

$$j_\mu^{\text{em}}(x) = j_\mu^3 + \frac{1}{\sqrt{3}} j_\mu^8, \tag{3.8}$$

which reproduces Eq. (2.11).

With the quark currents it is also convenient to study the symmetries of the Lagrangian

$$\mathcal{L}_{\text{quark}} = \mathcal{L}_0 + \mathcal{L}_{\text{mass}}, \tag{3.9}$$

with

$$\mathcal{L}_0 = \mathrm{i}\bar{q}\,\partial\!\!\!/ q \quad \text{and} \quad \mathcal{L}_{\text{mass}} = m_{\text{u}}\bar{u}u + m_{\text{d}}\bar{d}d + m_{\text{s}}\bar{s}s. \tag{3.10}$$

We now state the invariance properties under global transformations. $\mathcal{L}_0$ is invariant under the transformation

$$q \longrightarrow q' = Uq \quad \text{with} \quad U = \mathrm{e}^{\mathrm{i}\theta_\alpha \lambda^a/2}, \tag{3.11}$$

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where θ_α are constants, i.e. independent of space and time. Such a transformation is called global. In proving the invariance under unitary transformations recall that $e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$, provided that A, B commute with $[A, B]$. The term $\mathcal{L}_{\text{mass}}$ is not, in general, invariant under the global transformation. It becomes invariant only when all quark masses are equal:

$$m = m_u = m_d = m_s. \tag{3.12}$$

A consequence of the symmetry is the conservation of all vector currents. Consider

$$\frac{\partial}{\partial x^\mu}\left\{\bar{u}(x, p')\gamma_\mu d(x, p)\right\} = \frac{\partial}{\partial x^\mu} j_\mu^\dagger(x), \tag{3.13}$$

which in momentum space becomes

$$\bar{u}(x, p')\left[\not{p}' - \not{p}\right]d(x, p) = (m_u - m_d)\bar{u}(x, p')d(x, p). \tag{3.14}$$

This current is conserved when the two masses become equal.

Let us try to repeat this argument for axial transformations:

$$q \longrightarrow q' = Vq = e^{i\Phi_\alpha \lambda^a/2\cdot\gamma_5} q. \tag{3.15}$$

Now the kinetic term $\mathcal{L}_0$ is again invariant, but the mass term is not invariant even when the masses are equal. The Lagrangian is invariant under global γ_5 transformations when all quark masses are zero. In fact the axial current is not conserved and its divergence is

$$\frac{\partial}{\partial x^\mu} j_{\mu 5}^\dagger(x) = \bar{u}(x, p')\left[\not{p}' - \not{p}\right]\gamma_5 d(x, p) = (m_u + m_d)\bar{u}(x, p')\gamma_5 d(x, p). \tag{3.16}$$

When the Lagrangian is invariant under the axial transformations (3.15), all quark masses must vanish and the axial current is conserved. The two cases are examples of Noether's theorem, which states that, for every continuous global transformation that leaves the Lagrangian invariant, there is a current that is conserved.

Relations of the second class are abstracted from the quark model and establish equal-time commutation relations of currents.

In quantum field theory the quark fields satisfy the following equal-time canonical anticommutation relations:

$$\begin{aligned} \left\{q_\tau^\dagger(x), q_{\tau'}(x')\right\}_{x_0=x_0'} &= \delta_{\tau\tau'}\delta^{(3)}(\vec{x} - \vec{x}'), \\ \left\{q_\tau(x), q_{\tau'}(x')\right\}_{x_0=x_0'} &= \left\{q_\tau^\dagger(x), q_{\tau'}^\dagger(x')\right\}_{x_0=x_0'} = 0, \end{aligned} \tag{3.17}$$

where τ and τ' run from 1 to 12, i.e. there are three flavors and to each of them there correspond four spinor components. One can derive equal-time commutation

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relations for the SU(3) currents using the identity

$$[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB. \quad (3.18)$$

The final result, outlined in Problem 2, is

$$\left[j_\mu^a(x), j_0^b(x')\right]_{x_0=x_0'} = -f^{abc} j_\mu^c(x)\delta^{(3)}(\vec{x} - \vec{x}'). \quad (3.19)$$

On integrating this equation over three-dimensional space we arrive at

$$\left[j_\mu^a(x), Q^b(x_0)\right]_{x_0=x_0'} = \mathrm{i} f^{abc} j_\mu^c(x), \quad (3.20)$$

where $Q^b(x_0)$ is the charge corresponding to the vector current, defined by

$$Q^b(x_0) = \int \mathrm{d}^3x \; j_0^b(x). \quad (3.21)$$

It is now straightforward to derive from (3.20) the commutation relation for the charges:

$$\left[Q^a, Q^b\right] = \mathrm{i} f^{abc} Q^c. \quad (3.22)$$

Similarly, we can repeat the above steps for the axial current to obtain

$$\left[Q^a, Q_5^b\right] = \mathrm{i} f^{abc} Q_5^c, \quad (3.23)$$
$$\left[Q_5^a, Q_5^b\right] = \mathrm{i} f^{abc} Q^c. \quad (3.24)$$

We see that the vector and axial charges form an algebra that closes under commutation relations. If we define left- and right-handed charges

$$Q_{\mathrm{L,R}}^a = \frac{1}{2}\left(Q^a \mp Q_5^b\right), \quad (3.25)$$

they also satisfy the algebra

$$\begin{aligned} &\left[Q_\mathrm{L}^a, Q_\mathrm{R}^b\right] = 0, \\ &\left[Q_\mathrm{L}^a, Q_\mathrm{L}^b\right] = \mathrm{i} f^{abc} Q_\mathrm{L}^c, \quad \left[Q_\mathrm{R}^a, Q_\mathrm{R}^b\right] = \mathrm{i} f^{abc} Q_\mathrm{R}^c. \end{aligned} \quad (3.26)$$

It says that the left-handed sector does not communicate with the right-handed sector. Thus each sector by itself forms an SU(3) algebra. The group now is $\mathrm{SU(3)_L} \times \mathrm{SU(3)_R}$, known as the chiral group. The theory based on the chiral group and the approximation that the u and d quark masses are very small, relative to those of the other quarks, is known as chiral theory. The chiral theory can explain many of the regularities observed at small masses and momenta. We shall have the opportunity to remark on the implications of such a theory in Sections 5.2 and 15.6.

We can also express the weak hadronic current given by

$$j_\mu^{\mathrm{had}} = \bar{u}\gamma_\mu(1 - \gamma_5)d \cos\theta_\mathrm{c} + \bar{u}\gamma_\mu(1 - \gamma_5)s \sin\theta_\mathrm{c} \quad (3.27)$$

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in terms of the octet currents (3.6) and (3.7). Defining

$$V_\mu^a = j_\mu^a, \quad A_\mu^a = j_{\mu 5}^a, \tag{3.28}$$

one obtains

$$j_\mu^{\text{had}} = \left[\left(V_\mu^1 + \mathrm{i}V_\mu^2\right) - \left(A_\mu^1 + \mathrm{i}A_\mu^2\right)\right]\cos\theta_\mathrm{c} + \left[\left(V_\mu^4 + \mathrm{i}V_\mu^5\right) - \left(A_\mu^4 + \mathrm{i}A_\mu^5\right)\right]\sin\theta_\mathrm{c} \tag{3.29}$$

and we recover the $\Delta S = 0$ part of (2.17) and the $\Delta S = 1$ part of (2.18).

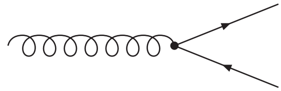
Expressing the hadronic currents in terms of quark fields sets them in one-to-one correspondence with the leptonic currents. The equal-time commutators give non-linear relations between observables, thus determining their relative strengths. Prominent among them are several sum rules that are valid at small and large momentum transfers.

3.3 Quantum chromodynamics

In spite of its successes, the quark model was received with a lot of skepticism because there was no experimental evidence for particles with fractional charges. To some authors this remained a mystery; to others the quarks remained a mnemonic for deriving useful rules. An additional objection concerned the fact that there was no theory describing the strong interactions among quarks. The attitude changed in the late sixties when inelastic electron–nucleon-scattering experiments provided evidence for point-like constituents, partons, within hadrons. Furthermore, correlations between electron- and neutrino-induced reactions provided evidence that the partons carried the quark quantum numbers. These topics will be studied in detail in Chapters 10 and 11. The final result was the formulation of a theory for the strong interactions whose fundamental fields are the quark and vector mesons – the gluons.

The theory of strong interactions is known as quantum chromodynamics or, in short, QCD. There are strong indications that each quark carries an additional quantum number called color; hence the name of the theory chromodynamics (color = *chroma*). The choice of names of the colors as red, white, and blue, or another triplet of names, is arbitrary but the fact that they are three in number is important. The quarks interact with each other by the exchange of vector bosons that change the colors of the quarks (Gross and Wilczek, 1973; Politzer, 1973).

We include in this section a few introductory remarks and discuss topics related to QCD in various sections of the book.



$$ig_s\gamma_\mu \frac{\lambda_a}{2}$$

Figure 3.2. A gluon–fermion vertex.

The theory of strong interactions is in many respects similar to QED. We write a quark of a specific flavor as a triplet of color SU(3):

$$q(x) = \begin{pmatrix} q_r \\ q_w \\ q_b \end{pmatrix}.$$

The theory also contains eight vector mesons – the gluons – coupled to quarks. There is again a vector vertex, with the new coupling constant g_s for the strong interactions, and a λ^α matrix acting on the quarks. The effective coupling constant for the strong interactions,

$$\alpha_s(p) = \frac{g_s^2(p)}{4\pi},$$

is now large and calculations with the exchange of a single gluon are neither accurate nor useful. One considers the cumulative effect from the exchange of many gluons, which modify the coupling constant, making it a function of momentum carried by the gluon.

The strong coupling constant has a remarkable property. At small momenta it is very large, binding the quarks into hadrons, so quarks cannot be separated as asymptotic particles. At large momenta of the gluons, the strong coupling constant becomes small, making perturbative calculations possible. As a consequence there are two types of calculations in QCD. One of them involves large momenta, for which perturbative summations of many gluons are possible. In a second class of calculations, numerical simulations of QCD replace continuous space-time by a finite but large four-dimensional lattice for space and time. Sophisticated computer programs have been written for handling gluon and quark fields on the lattice. These are non-perturbative calculations that should produce, among other results, confinement.

Throughout this book we shall study decays and reactions that involve both strong and weak interactions. The weak interactions of hadrons will be expressed in terms of the quark substructure by writing the currents in terms of quark fields and estimating or calculating matrix elements of quark operators for transitions between hadronic states. The success of these methods varies from process to process. This is a still developing field of research, as will become evident in several sections of this book.

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Problems for Chapter 3

1. The lowest-lying baryon states are built with three quarks with $L = 0$. There are ten states with $J^p = \frac{3^+}{2}$. To this decuplet belongs Δ^{++}(uuu). Construct the wave function of Δ^{++} with space, spin, and color contributions so that it obeys Fermi statistics. Finally, argue that color is necessary in order for the Pauli principle to be preserved. See Kokkedee (1969).
2. The weak vector current builds, together with the electromagnetic current, an algebra. For a better understanding we consider the SU(2) algebra. The generators of the group SU(2) are the matrices τ^1, τ^2, and τ^3, with the following property:

$$\left[\tau^i, \tau^j\right] = 2\mathrm{i}\varepsilon^{ijk}\tau_k.$$

We now define the currents

$$j_\mu^a = \bar{q}(x)\frac{\tau^a}{2}\gamma_\mu q(x),$$

which satisfy an algebra.

The fermion fields obey the canonical quantization

$$\left\{q_\tau^\dagger(x), q_{\tau'}(y)\right\}_{x_0=y_0} = \delta_{\tau\tau'}\delta^{(3)}(\vec{x}-\vec{y})$$

and

$$\{q_\tau(x), q_{\tau'}(y)\}_{x_0=y_0} = \left\{q_\tau^\dagger(x), q_{\tau'}^\dagger(y)\right\}_{x_0=y_0} = 0.$$

(a) Show the following relation:

$$\left[\Gamma_\alpha\tau^a, \Gamma_\beta\tau^b\right] = \frac{1}{2}\left\{\Gamma_\alpha, \Gamma_\beta\right\}\left[\tau^a, \tau^b\right] + \frac{1}{2}\left[\Gamma_\alpha, \Gamma_\beta\right]\left\{\tau^a, \tau^b\right\},$$

where Γ_α and Γ_β are arbitrary Dirac matrices.

(b) Using the identity

$$[AB, CD] = -AC\{B, D\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB,$$

show that

$$\begin{aligned}&\left[q_\sigma^\dagger(x)q_\tau(x), q_{\sigma'}^\dagger(y)q_{\tau'}(y)\right]_{x_0=y_0}\\ &\quad = \left\{q_\sigma^\dagger(x)\delta_{\sigma'\tau}q_{\tau'}(y) - q_{\sigma'}^\dagger(y)\delta_{\sigma\tau'}q_\tau(x)\right\}\delta^{(3)}(\vec{x}-\vec{y})|_{x_0=y_0}.\end{aligned}$$

(c) It follows now that

$$\left[\mathrm{i}\bar{q}\gamma_\mu\frac{\tau^a}{2}q(x), \mathrm{i}\bar{q}\gamma_0\frac{\tau^b}{2}q(y)\right]_{x_0=y_0} = -\mathrm{i}\varepsilon^{abc}\bar{q}(x)\frac{\tau^c}{2}\gamma_\mu q(y)\delta^{(3)}(\vec{x}-\vec{y}).$$

References

Gell-Mann, M. (1964), *Phys. Lett.* **8**, 214
Gell-Mann, M., and Neeman, Y. (1964), *The Eightfold Way* (New York, Benjamin)
Gross, D. J., and Wilczek, F. (1973), *Phys. Rev. Lett.* **39**, 1343
Kokkedee, J. J. (1969), *The Quark Model* (New York, Benjamin)

Politzer, D. (1973), *Phys. Rev. Lett*. **30**, 1346
Zweig, G. (1964), CERN preprint 8182/TH-401

Select bibliography

Kokkedee, J. J. (1969), *The Quark Model* (New York, Benjamin)
Greenberg, O. W. (1964), *Phys. Rev. Lett*. **13**, 598
Han, M. Y., and Nambu, Y. (1965), *Phys. Rev*. **139B**, 1006