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BOUNDS OF MODES AND UNIMODAL PROCESSES WITH INDEPENDENT INCREMENTS

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§ 1. Introduction

A probability measure μ is called unimodal if there is a point a such that the distribution function of μ is convex on $(-\infty, a)$ and concave on (a, ∞). The point a is called a mode of μ. When μ is unimodal, the mode of μ is not always unique; the set of modes is a one point set or a closed interval. If μ is a unimodal distribution with finite variance, Johnson and Rogers [6] give a bound

$$|a - m| \leq \sqrt{3v}\,, \tag{1.1}$$

where m and v are mean and variance of μ (see also [11]). Here $\sqrt{3}$ is the best constant. Let β_p be the absolute moment (possibly infinite) of μ of order p. We will extend the method of [6] and give a bound

$$|a| \leq \text{const}\ \beta_p^{1/p} \tag{1.2}$$

for any (not necessarily integer) $p > 0$. The constant depends only on p. We can give it explicitly, although it is not the best constant. Inequalities of the type (1.2) are proved in Section 2. We emphasize that they apply to distributions for which β_p is finite only for small p, such as non-Gaussian stable distributions.

In Section 3 we consider a stochastic process X_t with homogeneous independent increments. Some behaviors of its absolute moments as $t \to \infty$ are given. We use them to show some limit theorems of modes when X_t is unimodal. The inequalities in Section 2 can be used to give explicit bounds in the behaviors of modes. In Section 4 the modes of stable processes with index 1 are examined.

Bounds of modes for special classes of unimodal distributions are treated in some papers. Wolfe [12] and Sato-Yamazato [9] consider distri-

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butions of class L. Hall [4] studies unimodal sums of i.i.d. random variables in the domain of attraction of the Gaussian distribution.

§ 2. Bound of modes

The following lemma is basic to our discussion. It is suggested by a discussion in Johnson and Rogers [6].

LEMMA 2.1. *If μ is unimodal and the origin is a mode, then*

$$(q+1)^{1/q}\beta_q^{1/q} \le (p+1)^{1/p}\beta_p^{1/p} \tag{2.1}$$

for any $p > q > 0$.

Proof. By a result of Khintchine and Shepp ([1] V.9), the distribution of X is unimodal with a mode 0 if and only if there are independent random variables U and Y such that U is uniformly distributed on $[0, 1]$ and UY has the identical distribution with X. Hence $\beta_p = E|X|^p = EU^p E|Y|^p = (p+1)^{-1}E|Y|^p$, if X has distribution μ. Now (2.1) is a consequence of the moment inequality ([1] V.8) applied to the absolute moments of Y.

THEOREM 2.1. *Let μ be a unimodal probability measure. Let a be a mode of μ. Then,*

$$(q+1-(p+1)^{q/p})|a|^q \le (q+1)\beta_q + (p+1)^{q/p}\beta_p^{q/p} \tag{2.2}$$

for any p and q satisfying $0 < q \le 1$ and $q < p$.

Note that $(x+1)^{1/x}$ is decreasing in $x > 0$, so the coefficient of $|a|^q$ is positive.

Proof. Let X be a random variable with distribution μ. Then $X - a$ is unimodal with a mode 0. Applying Lemma 2.1, we have

$$(q+1)^{1/q}(E|X-a|^q)^{1/q} \le (p+1)^{1/p}(E|X-a|^p)^{1/p} \tag{2.3}$$

for $0 < q < p$. We use $||x|^\alpha - |y|^\alpha| \le |x-y|^\alpha \le |x|^\alpha + |y|^\alpha$ for $0 < \alpha \le 1$. If $0 < q < p \le 1$, then it follows that

$$\begin{aligned}(q+1)||a|^q - \beta_q| &\le (p+1)^{q/p}(|a|^p + \beta_p)^{q/p} \\ &\le (p+1)^{q/p}(|a|^q + \beta_p^{q/p}).\end{aligned}$$

If $0 < q \le 1 < p$, then we use Minkowski's inequality in the right-hand side of (2.3) to obtain

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$$(q+1)||a|^q - \beta_q| \leq (p+1)^{q/p}(|a| + \beta_p^{1/p})^q$$
$$\leq (p+1)^{q/p}(|a|^q + \beta_p^{q/p}).$$

Thus we get (2.2).

If $\beta_1 < \infty$ for μ, denote the mean by m and the central absolute moment of order p by γ_p,

$$\gamma_p = \int |x - m|^p \mu(dx).$$

THEOREM 2.2. *Let μ be unimodal with a mode a. Let $p > 1$. Then*

(2.4) $$((q+1)^{1/q} - (p+1)^{1/p})|a| \leq (q+1)^{1/q}\beta_q^{1/q} + (p+1)^{1/p}\beta_p^{1/p}$$

for $1 \leq q < p$, and

(2.5) $$(2 - (p+1)^{1/p})|a - m| \leq (p+1)^{1/p}\gamma_p^{1/p},$$

(2.6) $$(2 - (p+1)^{1/p})|a| \leq 2|m| + (p+1)^{1/p}\beta_p^{1/p}.$$

Proof. For $1 \leq q < p$, we apply Minkowski's inequality to the both sides of (2.4). Then we have

$$(q+1)^{1/q}(|a| - \beta_q^{1/q}) \leq (p+1)^{1/p}(|a| + \beta_p^{1/p}),$$

which is identical with (2.4). In order to get (2.5), we let $q = 1$ in (2.3), and then

$$2E|X - a| \leq (p+1)^{1/p}(|m - a| + \gamma_p^{1/p}).$$

Since $|m - a| \leq E|X - a|$, we have (2.5). The bound (2.6) follows from (2.5), because $\gamma_p^{1/p} \leq |m| + \beta_p^{1/p}$. The proof is complete.

THEOREM 2.3. *For each $p > 0$, there is a constant A_p such that, if μ is unimodal with a mode a, then*

(2.7) $$|a| \leq A_p\beta_p^{1/p}.$$

If A_p is the best such constant, then A_p is non-increasing in p and

(2.8) $$A_p \leq \frac{2 + (p+1)^{1/p}}{2 - (p+1)^{1/p}} \qquad \text{for } p > 1,$$

(2.9) $$A_p \leq \inf_{0<q\leq 1} A_{p,q} \qquad \text{for } 1 < p < 2,$$

(2.10) $$A_p \leq \inf_{0<q<p} A_{p,q} \qquad \text{for } 0 < p \leq 1,$$

where

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$$(2.11)\qquad A_{p,q} = \left(\frac{q+1+(p+1)^{q/p}}{q+1-(p+1)^{q/p}}\right)^{1/q} \qquad \textit{for } 0 < q \leq 1,\ q < p\,.$$

Proof. Use the moment inequality $\beta_q^{1/q} \leq \beta_p^{1/p}$ for $0 < q < p$ in (2.2) or (2.4). Then we get

$$(2.12)\qquad |a| \leq A_{p,q}\beta_p^{1/p} \qquad \text{for } 0 < q < p\,.$$

where $A_{p,q}$ is (2.11) or

$$(2.13)\qquad A_{p,q} = \frac{(q+1)^{1/q} + (p+1)^{1/p}}{(q+1)^{1/q} - (p+1)^{1/p}} \qquad \text{for } 1 \leq q < p\,.$$

This shows existence of A_p satisfying (2.7). Let A_p be the best such constant. The bounds (2.8)–(2.10) are immediate since $A_p \leq A_{p,q}$. Obviously $A_p \geq A_{p'}$, for $p < p'$.

Remark. For $p > 1$, the $A_{p,q}$ of (2.13) is increasing in $q \in [1, p)$, since $(q+1)^{1/q}$ is decreasing in q. Hence $\min_{1\leq q<p} A_{p,q} = A_{p,1}$. For $p \geq 2$, we have $\min_{0<q\leq 1} A_{p,q} = A_{p,1}$ because $A_{p,q}$ is decreasing in $q \in (0, 1]$. In fact, fix $p \geq 2$ and let $f(q) = q + 1 + (p+1)^{q/p}$ and $g(q) = q + 1 - (p+1)^{q/p}$ for $0 < q \leq 1$. Then

$$(d/dq)\log A_{p,q} = -q^{-2}\log(f(q)/g(q)) + q^{-1}(f'(q)/f(q) - g'(q)/g(q))$$

and we have

$$g'(q) = 1 - (p+1)^{q/p}p^{-1}\log(p+1) \geq g'(1) \geq 1 - 3^{1/2}2^{-1}\log 3 > 0\,,$$
$$f(q)/g(q) \geq 1 + 2/g(q) \geq 1 + 2/g(1) \geq 3\,,$$
$$f'(q)/f(q) \leq 2^{-1}f'(1) \leq 2^{-1}(1 + 3^{1/2}2^{-1}\log 3) < 1\,.$$

Hence $(d/dq)\log A_{p,q} < -q^{-2}\log 3 + q^{-1} < 0$.

§ 3. Processes with homogeneous independent increments

Let X_t, $t \geq 0$, be a real-valued process with homogeneous independent increments with $X_0 = 0$. We give estimates of its absolute moments. Let $\beta_p = E|X_1|^p$.

THEOREM 3.1. *Let* $0 < p \leq 1$. *Suppose that* $E|X_t|^p < \infty$. *Then, for* $0 < q < p$,

$$(3.1)\qquad E|X_t|^q \leq B_q(2q^{-1} + e(p-q)^{-1})(2^{2-p}\beta_p t)^{q/p} \qquad \textit{for } t \geq 1\,,$$

where

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(3.2) $$B_q = 2\pi^{-1}\Gamma(q+1)\sin(2^{-1}q\pi)\,.$$

Moreover, if $p \neq 1$, *then, for* $0 < q \leq p$,

(3.3) $$E|X_t|^q = o(t^{q/p}) \qquad \text{as } t \to \infty\,.$$

Remark. For any $p > 0$, the condition $E|X_t|^p < \infty$ for some $t > 0$ implies $E|X_t|^p < \infty$ for all $t > 0$, because this is equivalent to the condition $\int_{|x|>1} |x|^p \nu(dx) < \infty$ for the Lévy measure ν of X_t ([7], [8]).

Proof of Theorem 3.1. Let $E\exp(izX_t) = \varphi_t(z) = e^{-t\psi(z)}$, the characteristic function of X_t. Define $h_{t,p}(z)$ by

(3.4) $$\varphi_t(z) = 1 - |z|^p h_{t,p}(z)\,.$$

Then, $h_{t,p}(z)$ is bounded in z. By Theorems 2.1 and 4.1 of Hsu [5], we have

(3.5) $$\int_{-1}^{1} |z|^{-1}|h_{t,p}(z)|\,dz < \infty \qquad \text{if } p < 1\,,$$

(3.6) $$\int_{-1}^{1} |z|^{-1}|\mathrm{Re}\,h_{t,p}(z)|\,dz < \infty \qquad \text{if } p = 1\,,$$

and

(3.7) $$E|X_t|^p = B_p\int_0^\infty z^{-1}\,\mathrm{Re}\,h_{t,p}(z)dz\,.$$

An explicit bound of $h_{t,p}(z)$ is known:

(3.8) $$|h_{t,p}(z)| \leq 2^{1-p}E|X_t|^p\,.$$

Indeed,

$$|\varphi_t(z) - 1| \leq \int |e^{izx} - 1|\mu_t(dx) = 2\int |\sin 2^{-1}zx|\,\mu_t(dx)$$
$$\leq 2\int |\sin 2^{-1}zx|^p \mu_t(dx) \leq 2^{1-p}|z|^p \int |x|^p \mu_t(dx)\,,$$

where μ_t is the distribution of X_t. Since

$$|\log(1+w)| = \left|\int_1^{1+w} v^{-1}dv\right| \leq 2|w|$$

for any complex number w satisfying $|w| \leq 2^{-1}$, we have

(3.9) $$|\psi(z)| = |\log(1 - |z|^p h_{1,p}(z))| \leq 2|z|^p|h_{1,p}(z)|$$
$$\leq 2^{2-p}\beta_p|z|^p = c^{-p}|z|^p$$

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if $|z| \leq c$, where $c = (2^{2-p}\beta_p)^{-1/p}$. Define $g_p(z)$ by $\psi(z) = |z|^p g_p(z)$. Now let $0 < q < p$. Formula (3.7) with p replaced by q yields

$$E|X_t|^q = B_q \int_0^\infty z^{-1} \operatorname{Re} h_{t,q}(z) dz \leq B_q \int_0^\infty z^{-q-1} |1 - e^{t\psi(z)}| dz$$
$$= B_q \int_0^\infty z^{-q-1} |1 - \exp(tz^p g_p(z))| dz = B_q t^{q/p}(I + J),$$

where I and J are the integrals of $z^{-q-1}|1 - \exp(z^p g_p(t^{-1/p}z))|$ over the intervals $(0, c)$ and (c, ∞), respectively. Since $|e^{t\psi(z)}| \leq 1$, we have

$$J \leq 2\int_c^\infty z^{-q-1} dz = 2q^{-1}c^{-q}.$$

Since

$$|1 - e^w| = \left|\int_0^w e^v dv\right| \leq |w| e^{|w|}$$

for any complex w, we have

$$I \leq \int_0^c z^{p-q-1} |g_p(t^{-1/p}z)| \exp|z^p g_p(t^{-1/p}z)| dz.$$

It follows from (3.9) that

$$I \leq ec^{-p}\int_0^c z^{p-q-1} dz = e(p-q)^{-1}c^{-q}.$$

Thus we obtain (3.1).

Next we show (3.3). Let $0 < q \leq p < 1$. It follows from (3.5) that

$$\int_{-1}^1 |z|^{-1}|g_p(z)| dz < \infty. \tag{3.10}$$

By the calculation above, we have

$$E|X_t|^q \leq B_q t^{q/p}(I(u) + J(u))$$

for any $u > 0$, where $I(u)$ and $J(u)$ are the integrals of $z^{-q-1}|1 - \exp(z^p g_p(t^{-1/p}z))|$ over intervals $(0, u)$ and (u, ∞), respectively. For any given $\varepsilon > 0$, we can find u such that

$$J(u) \leq 2\int_u^\infty z^{-q-1} dz < \varepsilon.$$

Since $g_p(z)$ is locally bounded, there is a constant K (depending on u) such that

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$$|1 - \exp(z^p g_p(t^{-1/p}z))| \leq K z^p |g_p(t^{-1/p}z)|$$

for $0 \leq z \leq u$ and $t \geq 1$. Hence, for $t \geq 1$,

$$I(u) \leq K \int_0^u z^{p-q-1} |g_p(t^{-1/p}z)| dz$$
$$\leq K c^{p-q} \int_0^u z^{-1} |g_p(t^{-1/p}z)| dz \leq K c^{p-q} \int_0^{ut^{-1/p}} z^{-1} |g_p(z)| dz ,$$

which tends to zero as $t \to \infty$ by virtue of (3.10). Thus (3.3) follows. The proof is complete.

THEOREM 3.2. *Let $1 < p \leq 2$. Assume that $E|X_t|^p < \infty$ and $EX_t = 0$. Then, for $1 \leq q < p$,*

(3.11) $$E|X_t|^q \leq B_q(2q^{-1} + e(p-q)^{-1})(2^{3-p}p^{-1}\beta_p t)^{q/p} \qquad \text{for } t \geq 1 ,$$

where B_q is given by (3.2). *Moreover, if $p \neq 2$, then, for $1 \leq q \leq p$, we have*

(3.12) $$E|X_t|^q = o(t^{q/p}) \qquad \text{as } t \to \infty .$$

Proof. Define $h_{t,p}(z)$ again by (3.4). By Theorem 2.1 of Hsu [5], (3.5) and (3.7) are true also for $1 < p < 2$. An explicit bound of $h_{t,p}(z)$ is

(3.13) $$|h_{t,p}(z)| \leq 2^{2-p} p^{-1} E|X_t|^p$$

in place of (3.8), because

$$\varphi_t(z) - 1 - \varphi_t'(0)z = \int_0^z (\varphi_t'(u) - \varphi_t'(0))du$$

and

$$|\varphi_t'(z) - \varphi_t'(0)| \leq \int |x||e^{izx} - 1|\mu_t(dx) = 2\int |x||\sin 2^{-1}zx|\mu_t(dx)$$
$$\leq 2\int |x||\sin 2^{-1}zx|^{p-1}\mu_t(dx) \leq 2^{2-p}|z|^{p-1}\int |x|^p \mu_t(dx) .$$

Letting $c = (2^{3-p}p^{-1}\beta_p)^{-1/p}$, we have

$$|\psi(z)| \leq c^{-p}|z|^p \qquad \text{if } |z| \leq c .$$

Now, if $1 \leq q < p$, then, starting with (3.7) for $E|X_t|^q$, we can proceed along the same line as the proof of (3.1) and obtain (3.11). If $1 \leq q \leq p$ and $1 < p < 2$, then the proof of (3.12) is wholly similar to that of (3.3).

From now on we assume that the distribution of X_t is unimodal for every t. For example, if X_t has distribution of class L, then it is unimodal,

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which is proved by Yamazato [13]. Let $a(t)$ be a mode of X_t. Let $m = EX_1$ (if it exists) and $\gamma_p = E|X_1 - m|^p$. To see asymptotic behavior of $a(t)$ as $t \to \infty$, we use the following lemma.

LEMMA 3.1. *Let $\{\mu_n\}$ be a sequence of probability measures that converges weakly to μ. Suppose that, for each n, μ_n is unimodal with a mode a_n, μ is unimodal, and the mode a of μ is unique. Then, $a_n \to a$.*

This is obvious from the proof of Theorem 4 of Gnedendo-Kolmogorov [3], Section 32.

THEOREM 3.3. *Let $0 < p \leq 1$ and assume that $E|X_t|^p < \infty$. Then*

$$|a(t)| \leq A_q B_q^{1/q}(2q^{-1} + e(p-q)^{-1})^{1/q}(2^{2-p}\beta_p t)^{1/p} \qquad \text{for } t \geq 1\,, \tag{3.14}$$

where q is an arbitrary number satisfying $0 < q < p$. If $p \neq 1$, then

$$a(t) = o(t^{1/p}) \qquad \text{as } t \to \infty\,. \tag{3.15}$$

If $p = 1$, then

$$a(t) = mt + o(t) \qquad \text{as } t \to \infty\,. \tag{3.16}$$

Proof. The bound (3.14) is a conclusion of Theorems 2.3 and 3.1. In order to prove (3.15) for $p \neq 1$, choose $0 < q \leq p$. Then (3.3) of Theorem 3.1 says that $E|t^{-1/p}X_t|^q \to 0$ as $t \to \infty$. It follows that $t^{-1/p}X_t$ tends to 0 in distribution. Hence we get (3.15) from Lemma 3.1. If $p = 1$, then $t^{-1}(X_t - mt)$ tends to 0 in distribution by the law of large numbers. Thus we have (3.16) by Lemma 3.1. The proof is complete.

THEOREM 3.4. *Let $1 < p \leq 2$ and let $E|X_t|^p < \infty$. Then*

$$a(t) = mt + o(t^{1/p}) \qquad \text{as } t \to \infty \tag{3.17}$$

and

$$|a(t) - mt| \leq C_q B_q^{1/q}(2q^{-1} + e(p-q)^{-1})^{1/q}(2^{3-p}p^{-1}\gamma_p t)^{1/p} \qquad \text{for } t \geq 1\,, \tag{3.18}$$

where q is an arbitrary number satisfying $1 < q < p$ and

$$C_q = \frac{(q+1)^{1/q}}{2 - (q+1)^{1/q}}\,. \tag{3.19}$$

Proof. Let $Y_t = X_t - mt$. Then Y_t is a process with homogeneous independent increments with mean 0 and $E|Y_t|^p < \infty$. It is unimodal with a mode $a(t) - mt$. Hence we get (3.18), combining Theorems 2.2 and 3.2.

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Let $p \neq 2$. By Theorem 3.2, we have $E|t^{-1/p}Y_t|^q \to 0$ as $t \to \infty$ for $1 \leq q \leq p$. Thus $t^{-1/p}Y_t$ tends to 0 in distribution and (3.17) follows from Lemma 3.1. In case $p = 2$, the central limit theorem implies convergence of the distribution of $t^{-1/2}Y_t$ to a Gaussian distribution with mean 0 and, hence, $t^{-1/2}(a(t) - mt)$ tends to 0 by Lemma 3.1. This completes the proof.

Remark 1. In the proof we do not get any information on speed of convergence in the asymptotic behavior (3.15), (3.16), and (3.17) of the mode $a(t)$, because Lemma 3.1 does not tell anything about the speed. However, we can give an alternative proof to (3.15) for $0 < p < 1$ and to (3.17) for $1 < p < 2$ without resort to Lemma 3.1, combining the bounds (3.3) and (3.12) of absolute moments with the bounds (2.5) and (2.7) of the modes. So, in case $0 < p < 1$ or $1 < p < 2$, we can estimate speed of convergence in (3.15) or (3.17), if estimate of speed of convergence in (3.3) or (3.12) is given. In order to do this, estimate of the convergence

$$\int_{-u}^{u} |z|^{-1}|h_{1,p}(z)|\,dz \to 0 \qquad (u \to 0) \tag{3.20}$$

is essential, as is seen from examination of the proof of (3.3) and (3.12). Concerning (3.20) we note

$$\begin{aligned}\int_{-u}^{u} |z|^{-1}|h_{1,p}(z)|\,dz &= \int_{-u}^{u} |z|^{-p-1}|\varphi_1(z) - 1|\,dz \\ &\leq \int_{-u}^{u} |z|^{-p-1}dz \int_{-\infty}^{\infty} |e^{izx} - 1|\mu_1(dx) \\ &= 2\int_{-\infty}^{\infty} |x|^p \mu_1(dx) \int_0^{u|x|} |z|^{-p-1}|e^{iz} - 1|\,dz\end{aligned}$$

for $0 < p < 1$ and a similar relation

$$\begin{aligned}&\int_{-u}^{u} |z|^{-p-1}|\varphi_1(z) - 1 - \varphi_1'(0)z|\,dz \\ &\qquad \leq 2\int_{-\infty}^{\infty} |x|^p \mu_1(dx) \int_0^{u|x|} |z|^{-p-1}|e^{iz} - 1 - iz|\,dz\end{aligned}$$

for $1 < p < 2$.

Remark 2. It is known that, if $E|X_t|^p < \infty$ for some $0 < p < 1$, then $t^{-1/p}X_t$ tends to 0 almost surely as $t \to \infty$ ([2], [10]). This fact implies (3.15) by Lemma 3.1.

§ 4. Stable processes with index 1

Let X_t be a stable process with index $0 < \alpha < 2$. A general form of its characteristic function is as follows:

$$\varphi_t(z) = \exp\,[t\lambda(i\gamma z - |z|^\alpha + i\sigma(\tan 2^{-1}\pi\alpha)z|z|^{\alpha-1})] \qquad (\alpha \neq 1)\,, \tag{4.1}$$

$$\varphi_t(z) = \exp\,[t\lambda(i\gamma z - |z| - i\sigma 2\pi^{-1} z \log|z|)] \qquad (\alpha = 1)\,, \tag{4.2}$$

where λ, γ, and σ are real parameters, $\lambda > 0$, $-1 \leq \sigma \leq 1$. The Lévy measure is supported on the positive half line if and only if $\sigma = 1$. It is supported on the negative half line if and only if $\sigma = -1$. We assume that $\lambda = 1$ and $\gamma = 0$. It does not do harm to generality (consider $X_{t/\lambda} - t\gamma$ instead of X_t). It is a special case of Yamazato's result [13] that X_t is unimodal for each t. Furthermore the mode of X_t is unique for each t (Sato-Yamazato [9]). We denote it by $a(t)$. When the index α is not one, Zolotarev [14] gives some information on $a(t)$. Thus he proves that

$$\begin{aligned} \operatorname{sgn} a(t) &= \operatorname{sgn}\sigma && \text{if } 0 < \alpha < 1\,, \\ \operatorname{sgn} a(t) &= -\operatorname{sgn}\sigma && \text{if } 1 < \alpha < 2\,, \end{aligned}$$

where $\operatorname{sgn} x = 1, 0, -1$ according as $x > 0$, $x = 0$, $x < 0$, respectively. In case $\alpha = 1$, however, to get information on $a(t)$ is more difficult. By numerical calculation he finds that $a(1) < 0$ if $\alpha = 1$ and $\sigma = k/20$, $k = 1, 2, \cdots, 20$ ([14] p. 172). But no proof is given to the assertion that $\operatorname{sgn} a(1) = -\operatorname{sgn}\sigma$ for $\alpha = 1$.

We restrict our consideration to the case of index $\alpha = 1$. Thus the characteristic function of X_t is

$$\varphi_t(z) = \exp\,[t(-|z| - i\sigma 2\pi^{-1} z \log|z|)]\,. \tag{4.3}$$

Denote the mode of X_t by $a_\sigma(t)$.

Proposition 4.1. (i) *$a_\sigma(t)$ is a continuous function of two variables (σ, t). $a_0(t) = 0$, $a_\sigma(0) = 0$, $a_{-\sigma}(t) = -a_\sigma(t)$.*

(ii) *For any fixed σ,*

$$a_\sigma(t) = t a_\sigma(1) + 2\pi^{-1}\sigma t \log t\,. \tag{4.4}$$

Let $0 < \sigma \leq 1$. The derivative $a'_\sigma(t)$ strictly increases from $-\infty$ to $+\infty$ as t moves from 0 to $+\infty$. The mode $a_\sigma(t)$ strictly decreases from 0 to a minimum negative value until an epoch s_σ and then strictly increases to $+\infty$. There is a unique epoch $t_\sigma > 0$ such that $a_\sigma(t_\sigma) = 0$.

(iii) *As* $\sigma \to 0$,

(4.5) $$a_\sigma(t) = \sigma(-Kt + 2\pi^{-1} t \log t) + tO(\sigma^3),$$

where $O(\sigma^3)$ *does not depend on* t *and*

(4.6) $$K = \pi^{-1}\Gamma'(3) > 0 .$$

(iv) *As* σ *decreases to* 0,

(4.7) $$s_\sigma = \exp(-1 + 2^{-1}\pi K) + O(\sigma^2),$$

(4.8) $$t_\sigma = \exp(2^{-1}\pi K) + O(\sigma^2).$$

(v) *There exists* $T_1 > 0$ *such that, if* $0 < t \leq T_1$ *and* $0 < \sigma \leq 1$, *then* $a_\sigma(t) < 0$.

(vi) *There exists* $T_2 > 0$ *such that, if* $t \geq T_2$ *and* $0 < \sigma \leq 1$, *then* $a_\sigma(t) > 0$.

(vii) *An explicit bound of* $a_\sigma(1)$ *is*

(4.9) $$|a_\sigma(1)| \leq A_p B_p^{1/p}(2p^{-1} + (1-p)^{-1} + 4\pi^{-2}(2-p)^{-3}\sigma^2)^{1/p}$$

for $0 < p < 1$, *where* A_p *and* B_p *are of* (2.7) *and* (3.2).

Proof. (i) Continuity of $a_\sigma(t)$ follows from Lemma 3.1 because $\varphi_t(z)$ is continuous in (σ, t) for each z. If $\sigma = 0$, then X_t is symmetric and $a_\sigma(t) = 0$. Since X_t starts at the origin, $a_\sigma(0) = 0$. The relation $a_{-\sigma}(t) = -a_\sigma(t)$ is seen from (4.3).

(ii) The equality (4.4) is already observed by Zolotarev [14]. It is an easy consequence of the space-time relation

$$\varphi_t(z) = \varphi_1(tz)\exp(i2\pi^{-1}\sigma zt \log t).$$

Since we have

(4.10) $$a'_\sigma(t) = a_\sigma(1) + 2\pi^{-1}\sigma(1 + \log t)$$

from (4.4), the rest of the assertion is obvious.

(iii) It is enough to prove (4.5) for $t = 1$. Let $f(x)$ be the density function of X. By the Fourier inversion we have

$$f(x) = (2\pi)^{-1}\int_{-\infty}^{\infty} \exp(-ixz - |z| - i2\pi^{-1}\sigma z \log|z|)dz .$$

Write $a_\sigma(1) = a$. Since a is the zero point of $f'(x)$, we get

$$\int_{-\infty}^{\infty} z\exp(-iaz - |z| - i2\pi^{-1}\sigma z \log|z|)dz = 0 .$$

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Taking the imaginary part,

$$\int_0^\infty ze^{-z}\sin(az+2\pi^{-1}\sigma z\log z)dz=0\,.$$

Using $|\sin x - x| \leq \text{const}\,|x|^3$, we have

$$\int_0^\infty z^2e^{-z}(a+2\pi^{-1}\sigma\log z)dz+R=0$$

where

$$|R|\leq \text{const}\int_0^\infty z^4e^{-z}|a+2\pi^{-1}\sigma\log z|^3dz\leq \text{const}\,(|a|^3+|\sigma|^3)\,.$$

Therefore

$$a+K\sigma=O(|a|^3)+O(|\sigma|^3)\,,$$

where

$$K=\pi^{-1}\int_0^\infty z^2e^{-z}\log z\,dz=\pi^{-1}\Gamma'(3)>0$$

(Γ' is the derivative of the gamma function). Hence

$$a(1+O(a^2))=-K\sigma(1+O(\sigma^2))\,.$$

Since $a\to 0$ as $\sigma\to 0$ by (i), we have $a=-K\sigma(1+o(1))$ and hence

$$a=-K\sigma(1+O(\sigma^2))\,.$$

(iv) is a consequence of (iii), as we have

$$s_\sigma=\exp(-1-\pi(2\sigma)^{-1}a_\sigma(1))\,,$$
$$t_\sigma=\exp(-\pi(2\sigma)^{-1}a_\sigma(1))$$

from (4.4) and (4.10).

(v) It follows from (iii) that there exists $\sigma_1>0$ such that $a_\sigma(1)<0$ for $0<\sigma\leq\sigma_1$. Hence $a_\sigma(t)<0$ for $0<\sigma\leq\sigma_1$ and $0<t\leq 1$. If $\sigma_1\leq\sigma\leq 1$ and $0<t\leq 1$, then

$$a_\sigma(t)\leq ta_\sigma(1)+2\pi^{-1}\sigma_1 t\log t$$

by (4.4). Since $a_\sigma(1)$ is bounded in σ by continuity, it follows that $a_\sigma(t)<0$ for $\sigma_1\leq\sigma\leq 1$ if t is small enough.

(vi) If $T>1$ is big enough, we see from (4.5) that $\sigma^{-1}a_\sigma(T)$ tends to a positive number as $\sigma\to 0$. Hence there is $\sigma_2>0$ such that $a_\sigma(T)>0$ for $0<\sigma\leq\sigma_2$. Hence $a_\sigma(t)>0$ for $0<\sigma\leq\sigma_2$ and $t\geq T$. Since

$$a_\sigma(t) \geq ta_\sigma(1) + 2\pi^{-1}\sigma_2 t \log t$$

for $\sigma_2 \leq \sigma \leq 1$ and $t \geq 1$, we see, using boundedness of $a_\sigma(1)$ again, that $a_\sigma(t) > 0$ for $\sigma_2 \leq \sigma \leq 1$ if t is sufficiently large.

(vii) Use (3.7). Then

$$E|X_1|^p = B_p \int_0^\infty z^{-p-1}[1 - e^{-z} \cos(2\pi^{-1}\sigma z \log z)]dz = B_p(I + J),$$

where I and J are the integrals over the intervals $(0, 1)$ and $(1, \infty)$, respectively. We have

$$I \leq \int_0^1 z^{-p} dz + 2\pi^{-2}\sigma^2 \int_0^1 z^{1-p}(\log z)^2 dz,$$

$$J \leq 2\int_1^\infty z^{-p-1} dz,$$

using $1 - e^{-z} \cos y \leq z + 2^{-1}y^2$. Thus we obtain (4.9) from Theorem 2.3. The proof is complete.

Added in proof. The author has found the best constants A_p and D_p in the inequalities $|a| \leq A_p \beta_p^{1/p}$ $(p > 0)$ and $|a - m| \leq D_p \gamma_p^{1/p}$ $(p \geq 1)$. The results are that A_p is the unique zero point of $x^{p+1} - (p + 1)x - p$ for $x > 1$, and that $D_p = (p + 1)^{1/p}$. Proof will be published in Ann. Statist. Math. A under the title "Modes and moments of unimodal distributions".

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