TOLERANCE RELATIONS ON LATTICES

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The lattice of all tolerance relations (that is, reflexive,
symmetric compatible relations) on a lattice is investigated.
For modular lattices some examples are given which show that
such relations do naturally occur.

A tolerance relation on a lattice is a reflexive and symmetric binary
relation satisfying the Substitution Property (with respect to the lattice
operations). Tolerance relations have been studied by Chajda, Zelinka,
and others during the past decade. Although there are now numerous
articles about tolerance relations on lattices, it has hardly ever been
made clear how tolerance relations are related to the structure theory of
lattices, nor why they deserve any interest. Now, tolerance relations
"generalize" congruence relations (drop transitivity), but this would
hardly serve to motivate them. For, there is no analogue amongst tolerance
relations to the relationship of congruences and homomorphisms. As we see
it there are two places where tolerance relations do arise naturally: the
first is the study of (local) polynomial functions on lattices, and the
second is the study of convex sublattices maximal with respect to certain
properties. Let us commence with the first aspect. A polynomial function
is any n-ary operation which is built up by the lattice operations and
constants. Kindermann proved the following result: an order-preserving
n-ary operation on a finite lattice is a polynomial function if and only if
it preserves all tolerance relations; in particular, every order-
preserving function is a polynomial function if and only if the finite

lattice is "tolerance simple", that is, there are at most two (trivial)
tolerance relations. It is quite clear that it is much easier to
determine whether a lattice has a proper tolerance relation than to test
all order-preserving functions. It is the second aspect which serves as
the main motivation for our present work. For a given lattice, we are
interested in convex sublattices endowed with certain properties, either in
their own right or since they form (in some sense) the building blocks of
the lattice. If the given property is hereditary to subintervals, one may
consider maximal convex sublattices with the property. It rarely happens
that they actually form the blocks (alias classes) of some congruence
relation; in general, they overlap, like maximal relatively complemented
convex sublattices of a modular lattice or maximal α-complete convex
sublattices of a distributive lattice. Nevertheless, these "blocks" share
some properties with congruence blocks, for instance, intervals which are
projective into some block must also belong to some block. In either case,
this is governed by a tolerance relation, where two elements are related
if and only if they belong to a common block.

    The organisation of the material is as follows. The first section
provides the basic tools to decide whether a given binary relation on a
lattice is a tolerance relation. In Section 2 the lattice of all tolerance
relations is investigated. In particular, this lattice turns out to be
pseudocomplemented, and (at least in the modular case) there is a
convenient description of pseudocomplements. This is applied in the final
section to some "canonical" tolerance relations arising in modular and
distributive lattices. Some results of Herrmann, Kurinnoĭ, and Jakubík can
be seen in this light.

1. Basic facts

    A reflexive and symmetric binary relation ξ on a lattice L is
called a *tolerance relation* if ξ is compatible, that is ξ is a sub-
lattice of L^2 . $\Xi(L)$ denotes the lattice of all tolerance relations on
L and $\Theta(L)$ denotes the congruence lattice of L .

    In order to prove that a given binary relation is a tolerance
relation, Lemma 1.1 often facilitates the computations involved. This is,
of course, analogous to the corresponding fact for congruence relations

(see [9], Lemma I.3.8).

LEMMA 1.1. *A reflexive binary relation* ξ *on* L *is a tolerance relation if and only if the following five properties are satisfied for* $t, x, y \in L$:

(1) $(x, y) \in \xi$ *if and only if* $(x \wedge y, x \vee y) \in \xi$,

(2) $(t, x), (t, y) \in \xi$, *and* $t \leq x, y$ *imply that* $(t, x \vee y) \in \xi$,

(3) $(t, x), (t, y) \in \xi$, *and* $x, y \leq t$ *imply that* $(t, x \wedge y) \in \xi$,

(4) $(t, x) \in \xi$ *and* $x \leq t, y$ *imply that* $(t \vee y, y) \in \xi$,

(5) $(t, y) \in \xi$ *and* $t, x \leq y$ *imply that* $(t \wedge x, x) \in \xi$.

Proof. It is quite easy to see that every tolerance relation ξ has properties (1) through (5). Assume now that ξ is a reflexive binary relation satisfying (1)-(5). Symmetry is trivial. Suppose x, y belong to an interval $[u, v]$ where $(u, v) \in \xi$. Then by (4), $(x \wedge y, v) \in \xi$, and hence by (5), $(x \wedge y, x \vee y) \in \xi$. Thus, (1) implies

(6) $(u, v) \in \xi$ and $x, y \in [u, v]$ imply that $(x, y) \in \xi$.

Let $(u, v), (x, y) \in \xi$; we claim that $(u \vee x, v \vee y) \in \xi$. Indeed, we have $(u \wedge v, u \vee v), (x \wedge y, x \vee y) \in \xi$ by (1), and thus by (4),

$$\big((u \wedge v) \vee (x \wedge y), u \vee v \vee (x \wedge y)\big) ,$$

$$\big((u \wedge v) \vee (x \wedge y), (u \wedge v) \vee x \vee y\big) \in \xi . \quad .$$

Applying (2), we get

$$\big((u \wedge v) \vee (x \wedge y), u \vee v \vee x \vee y\big) \in \xi .$$

Since $u \vee x, v \vee y \in [(u \wedge v) \vee (x \wedge y), u \vee v \vee x \vee y]$, we conclude using (6) that $(u \vee x, v \vee y) \in \xi$. Similarly, using (3), (5), (6), one proves that $(u \wedge x, v \wedge y) \in \xi$.

The lemma is best explained by Figure 1, that is, a reflexive (and symmetric) relation ξ is seen to be a tolerance relation if and only if it passes the following test: whenever there are pairs in ξ labelled by "$\sim\!\!\sim$" in the six Hasse diagrams, then in either case the pair (a, b) must belong to ξ . Note that it is necessary to make the checks corresponding to each of the diagrams.
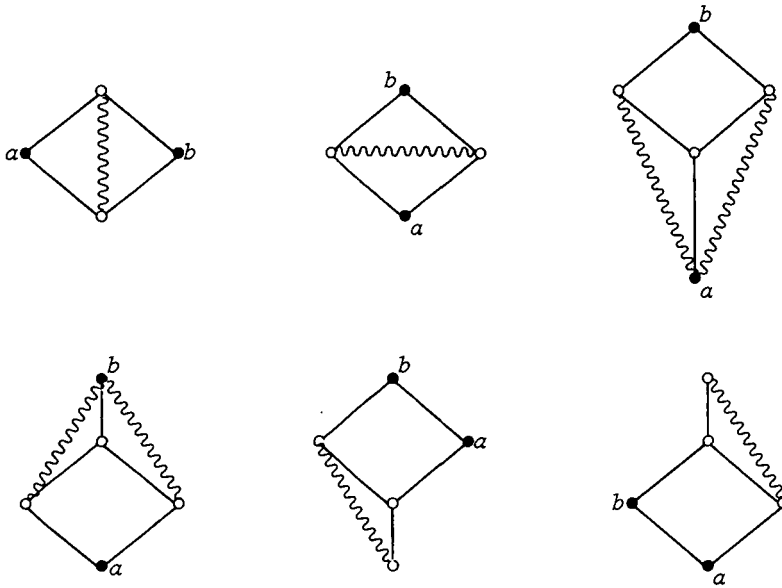
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FIGURE 1

For a subset A of L and a tolerance relation ξ , let $A\xi$ denote
the set of all $b \in L$ such that $(a, b) \in \xi$ for some $a \in A$. Whenever
A is a convex sublattice, so is $A\xi$. For, if $a, b \in A$ and
$(a, x), (b, y) \in \xi$, then for $t \in [x \wedge y, x \vee y]$, we have

$$s = \big(t \vee (a \wedge b)\big) \wedge (a \vee b) \in A \quad \text{and} \quad (s, t) \in \xi .$$

Every subset A of L which is maximal with respect to $A^2 \subseteq \xi$ is called
a *block* of ξ . Obviously, A is a block of ξ if and only if A is the
intersection of all $\{a\}\xi$ where $a \in A$. Hence every block of ξ is a
convex sublattice of L (*cf.* [7], Theorem 2). On the other hand, for a
standard sublattice S of L (in the sense of [8]), the relation

$$\Theta[S] = \big\{(x, y) \in L^2 \mid x \wedge y = \big((x \wedge y) \vee t\big) \wedge (x \vee y) \text{ and}$$
$$x \vee y = \big((x \vee y) \wedge s\big) \vee (x \wedge y) \text{ for some } s, t \in S\big\}$$

is a congruence relation and S is a block of $\Theta[S]$ (see [8]). We assert
that $\Theta[S]$ is the least tolerance relation ξ on L such that $S^2 \subseteq \xi$.
Indeed, if $x \leq y$ in L and $s, t \in S$ such that $x = (x \vee t) \wedge y$ and
$y = (y \wedge s) \vee x$, then $(s, t) \in \xi$ and $x = (y \wedge t) \vee x$, whence

$(x, y) \in \xi$. Note that this observation applies to [3], Theorem 3, and to
most results in [6].

To every tolerance relation ξ on L there corresponds the set Q
of all intervals $[a, b]$ of L which are contained in some block of ξ ,
that is $(a, b) \in \xi$. The next lemma characterizes the sets Q of
intervals which give rise to tolerance relations. Recall that an interval
$[x, y]$ is called *weakly perspective* into $[u, v]$ if either $u \leq x$ and
$v \vee x = y$ or $y \leq v$ and $u \wedge y = x$ (*cf.* [9]).

LEMMA 1.2. *Let Q be a set of intervals such that $L = \cup Q$. The
relation*

$$\xi_Q = \{(x, y) \in L^2 \mid [x \wedge y, x \vee y] \in Q\}$$

*is a tolerance relation if and only if the following conditions are
satisfied:*

(7) *if $[x, y]$ is weakly perspective into some $[u, v] \in Q$,
then $[x, y] \in Q$,*

(8) *if $[t, x], [t, y] \in Q$, then $[t, x \vee y] \in Q$,*

(9) *if $[x, z], [y, z] \in Q$, then $[x \wedge y, z] \in Q$.*

Proof. If ξ_Q is a tolerance relation, then (7), (8), (9) are
trivially satisfied. Assume now that the set Q satisfies (7), (8), (9).
The relation ξ_Q is symmetric by definition and is reflexive since Q
covers L and (7) holds. Now, (1) is trivially true, (7) implies (2) and
(3), (8) implies (4), (9) implies (5). Hence Lemma 1.1 completes the
proof.

The relation ξ_Q defined in Lemma 1.2 is a congruence relation if and
only if the conditions

(10) $[x, y], [y, z] \in Q$ imply that $[x, z] \in Q$
and (7) are satisfied.

Lemmas 1.1 and 1.2 provide us with the necessary information required
in Section 3 below. There are, however, many more ways to describe
tolerance relations. For instance, every compatible relation containing
the partial order $\leq$ corresponds to a tolerance relation, and *vice versa*

(see [1]). Proposition 1.3 offers a characterization of tolerance
relations in terms of certain mappings on the ideal lattice $J(L)$.
(Principal ideals are written as $(a]$ $(a \in L)$.)

   PROPOSITION 1.3. *For a lattice L , there is a* 1-1 *correspondence
between tolerance relations ξ and those mappings $\tau : J(L) \to J(L)$ which
satisfy $A \subseteq A\tau$ for all ideals A of L and preserve finite inter-
sections and (nonvoid) directed unions.*

   Proof. Given ξ , the mapping $\tau : A \longmapsto A\xi$ $\bigl(A \in J(L)\bigr)$ has the
desired properties. Moreover, τ satisfies

   (11) $(a, b) \in \xi$ if and only if $(a] \subseteq (b]\tau$ and $(b] \subseteq (a]\tau$.

Conversely, given the mapping τ , define a relation ξ by (11). It is
readily verified that ξ is a tolerance relation. Further, if $b \in (a]\tau$,
then $(a \wedge b] \subseteq (b]\tau$ and $(b] \subseteq (a]\tau \cap (b]\tau = (a \wedge b]\tau$, whence
$(a \wedge b, b) \in \xi$ and thus

$$(a]\tau \subseteq (a]\xi = \bigcup_{x \leq a} \{x\}\xi \subseteq \bigcup_{x \leq a} (x]\tau = (a]\tau \ .$$

Consequently, for every ideal A , we have

$$A\tau = \bigcup_{a \in A} (a]\tau = \bigcup_{a \in A} (a]\xi = A\xi \ ,$$

completing the proof.

2. The lattice $\Xi(L)$

   Very little is known about the tolerance lattice $\Xi(L)$ of a lattice
L . Since $\Xi(L)$ can be regarded as a subalgebra lattice of a universal
algebra, $\Xi(L)$ is algebraic (*cf*. Section 3 in [5]). What about
identities? If L is distributive, so is $\Xi(L)$ (see [5], Theorem 16).
However, if L is modular, then $\Xi(L)$ is in general not modular:
consider the tolerance relations ξ_0, ξ_1, ξ_2 on the lattice L given by
Figure 2. The diagrams exhibit the blocks of ξ_0, ξ_1, ξ_2 , and
$\xi_0 \cap \xi_1 = \xi_0 \cap \xi_2$. Since $\xi_0 \vee \xi_1 = \xi_0 \vee \xi_2 = L^2$, we have an N_5
sitting inside $\Xi(L)$.

ξ_0

ξ_1

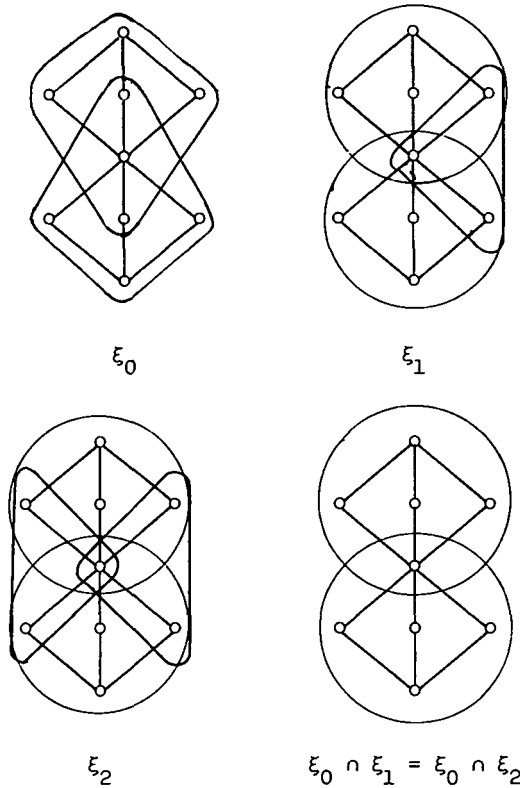ξ_2

$\xi_0 \cap \xi_1 = \xi_0 \cap \xi_2$

FIGURE 2

Therefore the variety of distributive lattices is the only nontrivial
lattice variety V for which all members have modular tolerance lattices.
It might be true that the tolerance lattices of members in any nontrivial
variety V other than the variety of distributive lattices generate the
variety of all lattices. At least, this "tolerance generated" variety must
contain V. Indeed, every lattice L is isomorphic to a sublattice of
$\Xi(M)$, where M is obtained by adjoining a new 0 to L; the desired
embedding of L in $\Xi(M)$ is given by $x \mapsto \xi_x = (x]^2 \cup L^2 \quad (x \in L)$.

We now proceed to prove the main result of this section. As we have
already seen $\Xi(L)$ is in general far from being distributive. However,
$\Xi(L)$ shares with $\Theta(L)$ a local distributivity at the bottom ω, that is
$\Xi(L)$ is 0-modular and 0-distributive (in the sense of [15]).

Recall that weak projectivity is the transitive extension of weak perspectivity: if for some natural number n and

$$[u, v] = [a_0, b_0], \ [a_1, b_1], \ \ldots, \ [a_n, b_n] = [x, y]$$

every $[a_i, b_i]$ is weakly perspective into $[a_{i+1}, b_{i+1}]$ $(i = 0, \ldots, n-1)$, then $[u, v]$ is *weakly projective* into $[x, y]$ *(cf.* [9]).

LEMMA 2.1. *Given* $\zeta \in \Xi(L)$ *and* $\theta \in \Theta(L)$, *let* Q *be the set of all intervals* $[x, y]$ *such that* $(u, v) \in \theta$ *whenever* $(u, v) \in \zeta$ *and* $[u, v]$ *is weakly projective into* $[x, y]$. *Then* ξ_Q *(defined as in Lemma 1.2) is a congruence relation and the relative pseudocomplement of* ζ *in* θ.

Proof. Clearly, L is covered by Q and (7) is true. To verify (10), let $[x, y], [y, z] \in Q$ and assume that $(u, v) \in \zeta$ and $[u, v]$ is weakly projective into $[x, z]$. An easy induction shows that there exists $t \in [u, v]$ such that $[u, t]$ and $[t, v]$ are weakly projective into $[x, y]$ and $[y, z]$, respectively; for a single instance of weak perspectivity this is pictured in Figure 3. Consequently, (u, t) and (t, v) belong to θ, whence (u, v) must be in θ. We conclude that ξ_Q is a congruence relation. By construction ξ_Q is the largest tolerance relation with $\xi_Q \cap \zeta \subseteq \theta$, and we are done.
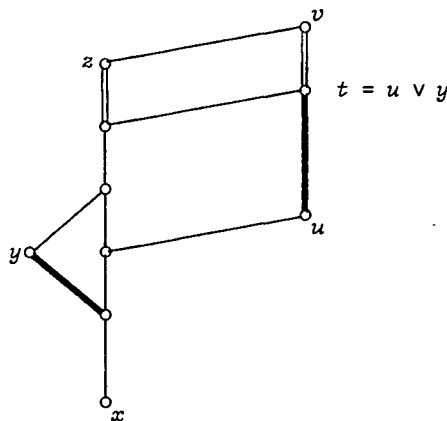

FIGURE 3

In the presence of weak modularity, the pseudocomplement of a tolerance relation can be described in a more convenient way. A lattice L is called *weakly modular* if for every proper interval $[r, s]$ which is weakly projective into $[x, y]$, there exists a proper subinterval $[u, v]$ of $[x, y]$ which is weakly projective into $[r, s]$ (*cf.* [9]). As in the proof of [9, Theorem III.4.9] one obtains

LEMMA 2.2. *Let ξ be a tolerance relation on a weakly modular lattice L . The pseudocomplement ξ^* of ξ is a congruence relation and consists of all pairs (x, y) such that $(u, v) \notin \xi$ for every proper subinterval $[u, v]$ of $[x \wedge y, x \vee y]$.*

Proof. If $[x, y]$ is an interval which contains no pair $u < v$ with $(u, v) \in \xi$, then by weak modularity for every interval $[r, s]$ which is weakly projective into $[x, y]$, we must have $(r, s) \notin \xi$. Hence Lemma 2.1 finishes the proof.

LEMMA 2.3. *Let ξ_0 and ξ_1 be tolerance relations on L such that $\xi_0 \cap \xi_1 = \omega$ (that is, $\xi_1 \subseteq \xi_0^*$). Then*

$$\xi = \Big\{(x, y) \in L^2 \mid \text{for } i = 0, 1, \text{ there exists } z_i \in L \text{ such that}$$

$$x \wedge y = z_0 \wedge z_1,\ x \vee y = z_0 \vee z_1,\ \big(x \wedge y,\ z_i\big) \in \xi_i\Big\}$$

is the join of ξ_0 and ξ_1 in $\Xi(L)$.

Proof. Since ξ_0, $\xi_1 \subseteq \xi \subseteq \xi_0 \vee \xi_1$, it suffices to verify that ξ is a tolerance relation. We will check properties (1) through (5). (1) is trivial. Since $\xi_0 \cap \xi_1 = \omega$, $(x, y) \in \xi$ if and only if there is some selfdual diagram as given by Figure 4.
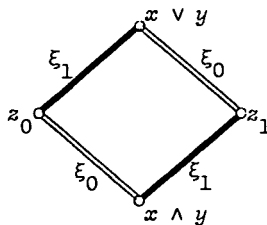

FIGURE 4

Hence by duality we need only check (2) and (4). If (t, x), $(t, y) \in \xi$
and $t \leq x, y$, then there are (t, u_i), $(t, v_i) \in \xi_i$ $(i = 0, 1)$ such
that

$$t = u_0 \wedge u_1 = v_0 \wedge v_1 , \quad x = u_0 \vee u_1 , \quad y = v_0 \vee v_1 .$$

Then $(t, u_i \vee v_i) \in \xi_i$ $(i = 0, 1)$ and $x \vee y = u_0 \vee v_0 \vee u_1 \vee v_1$.
Since $(t, (u_0 \vee v_0) \wedge (u_1 \vee v_1)) \in \xi_0 \cap \xi_1 = \omega$, we must have
$t = (u_0 \vee v_0) \wedge (u_1 \vee v_1)$, whence $(t, x \vee y) \in \xi$, which proves (2).
Now, if $(x, t) \in \xi$ and $x \leq t, y$, then there are $(x, z_i) \in \xi_i$
$(i = 0, 1)$ such that $x = z_0 \wedge z_1$, and $t = z_0 \vee z_1$. Then
$(y, y \vee z_i) \in \xi_i$ $(i = 0, 1)$ and $t \vee y = y \vee z_0 \vee z_1$. Since
$(y, (y \vee z_0) \wedge (y \vee z_1)) \in \xi_0 \cap \xi_1 = \omega$, we must have
$y = (y \vee z_0) \wedge (y \vee z_1)$, whence $(y, t \vee y) \in \xi$, which proves (4). By
Lemma 1.1, the proof is complete.

A lattice L is called 0-*modular* if it contains no N_5 with bottom
element 0 , that is, for $x, y, z \in L$, $x \leq y$ and $y \wedge z = 0$ imply that
$x = (x \vee z) \wedge y$ (*cf.* [15]).

THEOREM 2.4. *For every lattice L , the tolerance lattice $\Xi(L)$ is
0-modular, pseudocomplemented and algebraic.*

Proof. It remains to check 0-modularity. Let $\xi_0, \xi_1, \xi_2 \in \Xi(L)$
such that $\xi_1 \subseteq \xi_2$, $\xi_0 \cap \xi_2 = \omega$, and $\xi_0 \vee \xi_1 = \xi_0 \vee \xi_2$. If
$(x, z) \in \xi_2$ and $x < z$, then $(x, z) \in \xi_0 \vee \xi_1$, whence by Lemma 2.3
there are $y_0, y_1 \in L$ such that $x = y_0 \wedge y_1$, $z = y_0 \vee y_1$, and
$(x, y_i) \in \xi_i$ $(i = 0, 1)$. Since $(x, y_0) \in \xi_0 \cap \xi_2 = \omega$, we have $x = y_0$
and thus $y_1 = z$. Hence $(x, z) \in \xi_1$, and therefore $\xi_1 = \xi_2$.
Consequently, $\Xi(L)$ is 0-modular.

The lattice $\Delta(L)$ of all compatible reflexive relations is isomorphic
to the square $(\Xi(L))^2$ (see [1]). Hence $\Delta(L)$ is also 0-modular and
pseudocomplemented.

Note that every finite distributive lattice is isomorphic to a tolerance lattice. This is a corollary to [9, Theorem II.3.17]: every finité distributive lattice is isomorphic to the congruence lattice of a sectionally complemented lattice. Indeed, on a sectionally complemented lattice there are no tolerance relations other than congruence relations (*cf.* [1]).

3. Examples

We will now demonstrate that there are several canonical tolerance relations on any modular lattice. On a distributive lattice one can find even more.

First some terminology. A lattice L is *atomistic* if every element of L is the join of atoms; L is *relatively atomistic* if every interval of L is an atomistic lattice. L is *discrete* if every bounded chain is finite. L is *weakly atomic* if every proper interval contains two elements u and v such that v covers u ; if L does not contain such a pair, then L is *dense*.

THEOREM 3.1. *For a modular lattice L , define*

$$\theta = \{(x,\, y) \in L^2 \mid [x \wedge y,\, x \vee y] \text{ is of finite length}\} \, ,$$

$$\xi = \{(x,\, y) \in L^2 \mid [x \wedge y,\, x \vee y] \text{ is complemented}\} \, .$$

Then θ is a congruence relation and ξ is a tolerance relation on L .
The blocks of

    (i) θ ,

   (ii) θ^* ,

  (iii) θ^{**} ,

   (iv) ξ ,

    (v) $\theta^{**} \cap \xi$

are the maximal convex sublattices of L which are

    (i) *discrete,*

   (ii) *dense,*

  (iii) *weakly atomic,*

 (iv) relatively complemented,

 (v) relatively atomistic,

respectively.

Proof. It is not hard to see that θ is a congruence relation and
that the blocks of θ, θ^*, θ^{**} are of the asserted form. To prove that ξ
is a tolerance relation, let a, b, c, $x \in L$ such that $a \leq b$, c and
$a \leq x \leq b \vee c$. If u is a relative complement of $b \wedge x$ in $[a, b]$ and
if v is a relative complement of $c \wedge (u \vee x)$ in $[a, c]$, then $u \vee v$
is a relative complement of x in $[a, b \vee c]$. Indeed, by modularity, we
have

$$(u \vee v) \wedge x = (u \vee v) \wedge (u \vee x) \wedge x = \left[u \vee \left(v \wedge (u \vee x)\right)\right] \wedge x$$
$$= u \wedge x = a ,$$

$$u \vee v \vee x = [u \vee (b \wedge x)] \vee \left[v \vee \left(c \wedge (u \vee x)\right)\right] = b \vee c .$$

Hence, whenever $[a, b]$ and $[a, c]$ are complemented, so is $[a, b \vee c]$.
Dually, $[a \wedge b, c]$ is complemented if $[a, c]$ and $[b, c]$ are
complemented. By modularity, intervals which are weakly perspective into
complemented intervals are complemented. Thus, applying Lemma 1.2, we get
that ξ is a tolerance relation. The blocks of ξ are necessarily the
maximal relatively complemented convex sublattices of L . It is routine
to check that a complemented modular lattice is atomistic if and only if it
is weakly atomic. Hence the blocks of $\theta^{**} \cap \xi$ are the maximal relatively
atomistic convex sublattices of L .

For a modular lattice L of finite length, the blocks of the relation
ξ in Theorem 3.1 are the maximal atomistic intervals of L . These
intervals were studied in $[10]$, $[13]$. Some of the results in these papers
follow (with remarks in $[1]$) from the fact that ξ is a tolerance
relation. If L is a distributive lattice, then ξ can be characterized
as the intersection of all tolerance relations with exactly two overlapping
blocks (*cf.* $[2]$).

Completeness conditions in distributive lattices also give rise to
natural tolerance relations. A lattice L is called *<α-complete* if for
every nonempty subset A of L with fewer than α elements, the join
$\vee A$ and the meet $\wedge A$ exist. For a cardinal α , α^+ denotes its
successor cardinal. Then L is *α-complete* if and only if L is

$<\alpha^+$-complete. L is *conditionally* $<\alpha$-*complete* if every interval of L is
$<\alpha$-complete. L is called $<\alpha$-*homogeneous* if L is conditionally
$<\alpha$-complete and does not contain any α-complete proper interval.

     PROPOSITION 3.2. *Let L be a distributive lattice and let α be an
uncountable cardinal. Then*

$$\xi_\alpha = \{(x,\ y) \in L^2 \mid [x \wedge y,\ x \vee y] \text{ is } <\alpha\text{-}complete\}$$

is a tolerance relation. The blocks of ξ_α and $\xi_\alpha \cap (\xi_{\alpha^+})^*$ are the

*maximal conditionally $<\alpha$-complete convex sublattices and the maximal
$<\alpha$-homogeneous convex sublattices of L , respectively.*

     Proof. Let $[x,\ y]$ and $[x,\ z]$ be $<\alpha$-complete intervals. Assume
that A is a nonempty subset of $[x,\ y \vee z]$ with fewer than α elements.
Then

$$u = \bigvee_{a \in A} y \wedge a \quad \text{and} \quad v = \bigvee_{a \in A} z \wedge a$$

exist. Since L is distributive, $u \vee v$ is an upper bound of A . If w
is any upper bound of A , then $u \le y \wedge w$ and $v \le z \wedge w$, whence
$u \vee v \le w$. We conclude that $u \vee v = \vee A$. Dually, if $[x,\ z]$ and
$[y,\ z]$ are $<\alpha$-complete, then $[x \wedge y,\ z]$ is $<\alpha$-complete. It is readily
seen that an interval which is weakly perspective into an $<\alpha$-complete
interval is also $<\alpha$-complete. Thus an application of Lemmas 1.2 and 2.2
completes the proof.

     Proposition 3.2 applies to the results in [11], §§1, 3, which are
proved for relatively complemented distributive lattices. In this case, of
course, all the ξ_α are congruence relations. For arbitrary distributive
lattices this does not hold (see [11], Example 1.8). For modular lattices,
ξ_α may even fail to be a tolerance relation, as our final example shows.
Consider the modular lattice $K = \mathrm{FM}\left({}_D M_3\right)$ of [14], where D denotes the
chain of integers with bounds added (see Figure 5, p. 380). By [14],
Hilfssatz 2, $D \cong [0,\ a] \cong [a,\ 1]$, and hence a is doubly irreducible.
Thus $L = K - \{a\}$ is a noncomplete modular lattice. Since $[0,\ b]$ and
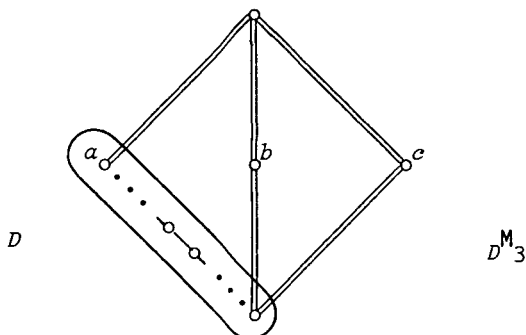$[0,\ c]$ are complete and $b \vee c = 1$, $\xi_{\aleph_1}$ is not a tolerance relation of

L .

FIGURE 5

References .

[1] Hans-J. Bandelt, "Local polynomial functions on lattices", *Houston J. Math.* (to appear).

[2] Hans-J. Bandelt, "Tolerances on median algebras", submitted.

[3] Ivan Chajda, "Notes on lattice congruences", *Časopis Pěst. Mat.* 103 (1978), 255-258.

[4] Ivan Chajda, "A characterization of distributive lattices by tolerance lattices", *Arch. Math. (Brno)* 15 (1979), 203-204.

[5] Ivan Chajda and Bohdan Zelinka, "Lattices of tolerances", *Časopis Pěst. Mat.* 102 (1977), 10-24.

[6] Ivan Chajda and Bohdan Zelinka, "Minimal compatible tolerances on lattices", *Czechoslovak Math. J.* 27 (102) (1977), 452-459.

[7] Ivan Chajda and Bohdan Zelinka, "Tolerances and convexity", *Czechoslovak Math. J.* 29 (104) (1979), 584-587.

[8] E. Fried and E.T. Schmidt, "Standard sublattices", *Algebra Universalis* 5 (1975), 203-211.

[9] George Grätzer, *General lattice theory* (Birkhäuser Verlag, Basel and Stuttgart, 1978).

[10] Christian Herrmann, "*S*-verklebte Summen von Verbänden", *Math. Z.* 130 (1973), 255-274.

[11]   J. Jakubík, "Conditionally α-complete sublattices of a distributive
          lattice", *Algebra Universalis* 2 (1972), 255-261.

[12]   M. Kindermann, "Über die Äquivalenz von Ordnungspolynomvoll-
          ständigkeit und Toleranzeinfachheit endlicher Verbände",
          Contributions to general algebra, 145-149 (Proc. Klagenfurt
          Conf., Klagenfurt, 1978.   Verlag Johannes Heyn, Klagenfurt,
          1979).

[13]   Г.Ч. Куринной [G.Č. Kurinnoǐ], "Новое доказательство теоремы
          Дилуорса" [A new proof of a theorem of Dilworth], *Vestnik
          Har'kov. Gos. Univ.* No. 93 *Mat. Vyp.* 38 (1973), 11-15, 1.

[14]   Aleit Mitschke und Rudolf Wille, "Freie modulare Verbände $\mathrm{FM}\!\left({}_{D}M_3\right)$ ",
          *Proceedings of the University of Houston Lattice Theory
          Conference*, Houston, Texas, 1973, 383-396 (Department of
          Mathematics, University of Houston, Houston, Texas, 1973).

[15]   J.C. Varlet, "A generalization of the notion of pseudo-
          complementedness", *Bull. Soc. Roy. Sci. Liège* 37 (1968),
          149-158.

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