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COEFFICIENTS IN EXPANSIONS OF CERTAIN RATIONAL FUNCTIONS

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1. Introduction. The constant term of certain rational functions has attracted much attention recently. For example the Dyson conjecture; that the constant term of

$$\prod_{1 \leqq i \neq j \leqq n} (1 - x_i/x_j)^{a_i}$$

is the multinomial coefficient

$$\binom{a_1 + \ldots + a_n}{a_1, \ldots, a_n};$$

has spawned many generalizations (see [**2**], [**7**]). In this paper we consider some other families of rational functions which have interesting constant terms. For example, Corollary 4 states that the constant term of

$$(1.1) \qquad h(z) = (1 - z^{A+B})^{A+B}(1 - z^A)^{-A}(1 - z^B)^{-B}z^{-AB}$$

is $\binom{A+B}{B}$. Here, and throughout this paper, A and B denote fixed positive integers.

In order to prove this result, we consider the rational function in two variables

$$f(w, x) = (1 - wx)^{A+B}(1 - w)^{-A}(1 - x)^{-B}.$$

In Theorem 1 we give all of the coefficients of $f(w, x)$. Corollary 4 will then follow easily. We use the Lagrange inversion formula in two variables to prove Theorem 1.

The original problem which focused our attention on (1.1) was a problem of Mallows [**8**]. Using probabilistic techniques, he proved that if $t, \alpha, \beta \geqq 0$ and $\alpha + \beta = 1$, then

$$(1.2) \qquad \frac{1}{\pi}\int_0^{\pi} \left[\frac{\sin u}{(\sin \alpha u)^{\alpha}(\sin \beta u)^{\beta}}\right]^t du = \frac{\Gamma(t+1)}{\Gamma(\alpha t + 1)\Gamma(\beta t + 1)}.$$

He remarked that a direct proof of (1.2) would be interesting. We give

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a direct proof in Theorem 6, as one of the several applications of Theorem 1. The relation between (1.1) and (1.2) is given in the proof of Theorem 6.

Two q-analogues of Theorem 1 are stated in Theorem 9 and Theorem 10. However, the resulting q-analogues of Corollary 4 are not so simple. We use a transformation formula for a terminating basic hypergeometric ${}_3\phi_2$ series to prove these results. It would be interesting to find a proof by q-Lagrange inversion in two variables. Unfortunately, the principle of q-Lagrange inversion is not yet fully understood, although progress has been made in [**1**], [**4**], and [**5**]. There is as yet no multivariable q-Lagrange inversion formula.

Interesting multivariable extensions of Theorems 1, 9, or 10 would be much desired. For example the q-Dyson conjecture in several variables has been formulated by Andrews [**2**]. It has not been settled yet. It is not clear to us what the multivariable analogues of our results are.

Because of the suggestive nature of Corollary 4, one could ask for a direct combinatorial proof. Zeilberger [**13**] has given such a proof for Dyson's conjecture.

2. Notation and preliminaries. We use common notation for rising factorials and hypergeometric series [**9**]. Thus, for any integer m,

$$(2.1) \qquad (a)_m = \Gamma(a+m)/\Gamma(a),$$

and

$$(2.2) \qquad {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| z\right) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m z^m}{m!(c)_m}$$

for complex a, b, c. Sometimes we will encounter the case when a, b and c are negative integers and $c < \max(a, b)$. In this case we agree to terminate the sum when $m = \min(-a, -b)$. Vandermonde's theorem [**9**, p. 69] states that

$$(2.3) \qquad {}_2F_1\left(\begin{matrix} a, -n \\ c \end{matrix} \middle| 1\right) = \frac{(c-a)_n}{(c)_n},$$

whenever the finite series on the left is well-defined.

In Section 5, where basic hypergeometric series [**11**] are involved, we write, for any integer m,

$$(2.4) \qquad (a)_m = (a;q)_m = \prod_{r=0}^{\infty} \frac{(1-aq^r)}{(1-aq^{m+r})},$$

and

$$(2.5) \qquad {}_3\phi_2\left(\begin{matrix} a, b, d \\ c, e \end{matrix} \middle| q, z\right) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m (d)_m z^m}{(q)_m (c)_m (e)_m}.$$

(The abbreviation $(a)_m$ should cause no confusion, since no ordinary

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hypergeometric series are involved in Section 5.) Again, if some of the numerator parameters are negative integral powers of q with the largest negative integer exponent denoted by $-n$, then we agree to replace the upper summation index ∞ by n.

A useful transformation formula for a terminating ${}_3\phi_2$ series is an iterate of [**10**, equ. (4.4)]

$$(2.6)\qquad (c)_n(e)_n\ {}_3\phi_2\left(\begin{matrix}a, b, q^{-n}\\ c, e\end{matrix}\,\middle|\, q, q\right)$$
$$= (e/a)_n(c/a)_n a^n\ {}_3\phi_2\left(\begin{matrix}a, q^{-n}, q^{1-n}ab/ce\\ q^{1-n}a/e, q^{1-n}a/c\end{matrix}\,\middle|\, q, q\right).$$

We will be interested in the above transformation formula when a, b, c and e are of the form q^{-j} for non-negative integral values of j. In these cases formula (2.6) reduces to the polynomial identity

$$(2.7)\qquad \sum_{j=0}^{n} (q^{-n})_j(a)_j(b)_l(cq^j)_{n-j}(eq^j)_{n-j}(q)_j^{-1}q^j$$
$$= \left(\frac{ec}{a}\right)^n q^{n^2-n} \sum_{l=0}^{n} (q^{-n})_l(a)_l(q^{1-n}ab/ec)_l(q^{1-n+l}a/e)_{n-l}$$
$$\times (q^{1-n+l}a/c)_{n-l}(q)_l^{-1}q^l.$$

A version of Lagrange inversion in two variables states that for a formal Laurent series $F(w, x)$ about the origin,

$$(2.8)\qquad \text{Coefficient of } (xw)^{-1} \text{ in } F(w, x) = \text{Coefficient of } (z_1z_2)^{-1} \text{ in } \mathcal{J} \cdot F(g, h),$$

where

$$(2.9)\qquad \begin{cases} g = g(z_1, z_2) = z_1 \cdot \{\text{power series with constant term } 1\}, \\ h = h(z_1, z_2) = z_2 \cdot \{\text{power series with constant term } 1\}, \end{cases}$$

and $\mathcal{J}$ is the Jacobian

$$(2.10)\qquad \mathcal{J} = \begin{vmatrix} \partial g/\partial z_1 & \partial g/\partial z_2 \\ \partial h/\partial z_1 & \partial h/\partial z_2 \end{vmatrix}.$$

Although this result has been attributed to modern authors, it is in the work of Jacobi [**6**].

A theorem of Carlson [**12**, p. 186] states that if $f(z)$ is analytic and is $O(e^{k|z|})$ in the half-plane Re $(z) \geqq 0$, where k is a constant $<\pi$, then

$$(2.11)\qquad f(z) \equiv 0 \text{ if } f(n) = 0 \text{ for every } n = 0, 1, 2, \ldots.$$

3. Coefficients of rational functions.

THEOREM 1. *For each pair of integers u, $v \geqq 0$, the coefficient $K(u, v)$*

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of $w^u x^v$ *in*

$$f(w, x) = (1 - wx)^{A+B}(1 - w)^{-A}(1 - x)^{-B} \tag{3.1}$$

is

$$K(u, v) = (B - u + 1)_{v-1}(A - v + 1)_{u-1}(AB - Au - Bv)/(u!v!), \tag{3.2}$$

with the interpretation

$$K(0, v) = \binom{B + v - 1}{v}, \quad K(u, 0) = \binom{A + u - 1}{u}.$$

Proof. The result is clear when $uv = 0$, so let $uv > 0$. Apply Lagrange inversion (2.8) with

$$F(w, x) = f(w, x)w^{-u-1}x^{-v-1}$$

and with the propitious change of variables

$$g = z_1/(1 - z_2), \quad h = z_2/(1 - z_1).$$

Since $\mathscr{J} = (1 - z_1 - z_2)(1 - z_1)^{-2}(1 - z_2)^{-2}$, we see that $K = K(u, v)$ equals the coefficient of $(z_1 z_2)^{-1}$ in

$$(1 - z_1 - z_2)(1 - z_1)^{v-1-A}(1 - z_2)^{u-1-B}z_1^{-1-u}z_2^{-1-v}.$$

Thus K is the coefficient of $z_1^u z_2^v$ in

$$(1 - z_1 - z_2)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{A - v + m}{m}\binom{B - u + n}{n}z_1^m z_2^n,$$

so

$$K = \binom{A - v + u}{u}\binom{B - u + v}{v} - \binom{A - v + u - 1}{u - 1}\binom{B - u + v}{v} - \binom{A - v + u}{u}\binom{B - u + v - 1}{v - 1}.$$

Combining the first two terms, we obtain

$$K = \binom{A + u - v - 1}{u}\binom{B + v - u}{v} - \binom{A + u - v}{u}\binom{B + v - u - 1}{v - 1},$$

and the result easily follows.

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COROLLARY 2. *Suppose that $A = ad$, $B = bd$ for integers a, b, $d > 0$. For each integer s, the coefficient K_s of $w^{b(d-s)}x^{as}$ in*

$$(3.3) \quad f(w, x) = (1 - wx)^{A+B}(1 - w)^{-A}(1 - x)^{-B}$$

is given by

$$(3.4) \quad K_0 = \binom{A + B - 1}{B}, \; K_d = \binom{A + B - 1}{A}, \; K_s = 0 \quad \text{for } 0 < s < d.$$

Proof. Apply Theorem 1 with $u = b(d - s)$, $v = as$.

COROLLARY 3. *Fix nonzero complex numbers λ, μ. Then the constant term of*

$$(1 - \lambda\mu z^{A+B})^{A+B}(1 - \lambda z^A)^{-A}(1 - \mu z^B)^{-B}z^{-AB}$$

is

$$\binom{A + B - 1}{B}\lambda^B + \binom{A + B - 1}{A}\mu^A.$$

Proof. Apply Corollary 2 with $d = (A, B)$, $w = \lambda z^A$, and $x = \mu z^B$, to see that the constant term equals

$$\sum_{s=0}^{d} \lambda^{b(d-s)}\mu^{as}K_s = \binom{A + B - 1}{B}\lambda^B + \binom{A + B - 1}{A}\mu^A.$$

COROLLARY 4. *The constant term in*

$$h(z) = (1 - z^{A+B})^{A+B}(1 - z^A)^{-A}(1 - z^B)^{-B}z^{-AB}$$

is

$$\binom{A + B}{A}.$$

Proof. Set $\lambda = \mu = 1$ in Corollary 3.

4. Evaluation of integrals and series. Let C be the contour $\{e^{i\theta} : 0 \leqq \theta \leqq 2\pi\}$ and let E be the arc $\{e^{i\theta} : 0 \leqq \theta \leqq 2\pi/(A + B)\}$. Corollary 4 states that

$$\frac{1}{2\pi i}\int_{C'} h(z)\,\frac{dz}{z} = \binom{A + B}{A},$$

where C' is C except with inward indentations around the poles of $h(z)$ on C. We now prove the surprising fact that if one integrates only along E, i.e., along the first $1/(A + B)$-th of C, the result is $\frac{1}{A + B}\binom{A + B}{A}$. Note that $h(z)$ has no poles on E.

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THEOREM 5. *We have*

$$(4.1)\qquad I: = \frac{A+B}{2\pi i}\int_E h(z)\frac{dz}{z} = \binom{A+B}{A}.$$

Proof. For $0 < \rho < 1$, let

$$I_\rho = \frac{A+B}{2\pi i}\int_{\rho E} h(z)\frac{dz}{z},$$

where $\rho E = \{\rho e^{i\theta} : 0 \leqq \theta \leqq 2\pi/(A+B)\}$. In the notation of Theorem 1,

$$f(w, x) = \sum_{u,v\geqq 0} K(u, v)w^u x^v,$$

so

$$(4.2)\qquad h(z) = z^{-AB}f(z^A, z^B) = \sum_{u,v\geqq 0} K(u, v)z^{Au+Bv-AB}.$$

Since $\binom{A+B}{A}$ is the constant term of $h(z)$ by Corollary 4, it follows from (4.2) that

$$D_\rho: = I_\rho - \binom{A+B}{A}$$

$$= \frac{(A+B)}{2\pi i}\sum_{\substack{u,v\geqq 0\\ Au+Bv\neq AB}} \frac{K(u, v)}{Au+Bv-AB}\rho^{Au+Bv-AB}$$

$$\times (e^{2\pi i(Au+Bv-AB)/(A+B)} - 1)$$

$$= -\frac{(A+B)}{2\pi i}\rho^{-AB}\sum_{R=1}^{A+B-1}(e^{2\pi iR/(A+B)} - 1)S_\rho(R),$$

where

$$(4.3)\qquad S_\rho(R) = \sum_{\substack{u,v\geqq 0\\ Au+Bv-AB\equiv R(\mathrm{mod}\, A+B)}} \frac{K(u, v)}{AB-Au-Bv}\rho^{Au+Bv}.$$

It remains to show that

$$(4.4)\qquad \lim_{\rho\to 1} S_\rho(R) = 0 \quad (0 < R < A+B),$$

for then $\lim_{\rho\to 1} D_\rho = 0$, so

$$I = \lim_{\rho\to 1} I_\rho = \binom{A+B}{A}.$$

Let $d = (A, B)$ and write $A = da$, $B = db$, $\sigma = a + b$. If d does not divide R, then $S_\rho(R) = 0$, so assume that $R = dr$ for an integer r. Note that $0 < r < \sigma$ and $(a, b) = (a, \sigma) = 1$. By the formula for $K(u, v)$

given in Theorem 1, (4.3) becomes

$$(4.5)\qquad S_\rho(R) = \sum_{\substack{u,v\geqq 0\\ au+bv-dab\equiv r(\mathrm{mod}\ \sigma)}} \frac{(B-u+1)_{v-1}(A-v+1)_{u-1}\rho^{Au+Bv}}{u!v!}.$$

The congruence in (4.5) can be restated in the form

$$u - v \equiv g(\mathrm{mod}\ \sigma)$$

with fixed $g \equiv B + ra^{-1}(\mathrm{mod}\ \sigma)$, where a^{-1} denotes the inverse of a $(\mathrm{mod}\ \sigma)$. Note that $g \not\equiv B(\mathrm{mod}\ \sigma)$. Now write

$$\begin{aligned} S_\rho(R) &= S_\rho^{+} + S_\rho^{-} - S_\rho^{0} \\ &= \left(\sum_{\substack{u\geqq v\geqq 0\\ u-v\equiv g(\mathrm{mod}\ \sigma)}} + \sum_{\substack{v\geqq u\geqq 0\\ u-v\equiv g(\mathrm{mod}\ \sigma)}} - \sum_{\substack{u=v\geqq 0\\ u-v\equiv g(\mathrm{mod}\ \sigma)}}\right) \\ &\qquad\qquad \times \frac{(B-u+1)_{v-1}(A-v+1)_{u-1}\rho^{Au+Bv}}{u!v!}. \end{aligned}$$

It is easily seen that

$$(4.6)\qquad S_\rho^{-} \text{ is obtained from } S_\rho^{+} \text{ by the replacements } \left\{\begin{array}{l} A \to B \\ B \to A \\ g \to -g \end{array}\right\}.$$

The condition $u - v \equiv g(\mathrm{mod}\ \sigma)$ is equivalent to $u = v + \tau$, where $\tau = g + m\sigma$ for integers m. Thus,

$$S_\rho^{+} = \sum_{\substack{m\\ \tau\geqq 0}} \sum_{v\geqq 0} \frac{(1+B-\tau-v)_{v-1}(A-v+1)_{v+\tau-1}}{(v+\tau)!v!} \rho^{A\tau+(A+B)v}.$$

In view of the formula $(x - v)_v = (-1)^v(1 - x)_v$ we obtain

$$\begin{aligned} S_\rho^{+} &= \sum_{\substack{m\\ \tau\geqq 0}} \sum_{v>0} \frac{(A)_\tau\rho^{A\tau}(-A)_v(\tau-B)_v\rho^{(A+B)v}}{\tau!A(B-\tau)v!(\tau+1)_v} \\ &= \sum_{\substack{m\\ \tau\geqq 0}} \frac{(A)_\tau\rho^{A\tau}}{\tau!A(B-\tau)}\, {}_2F_1\left(\left.\begin{matrix}\tau-B, -A\\ \tau+1\end{matrix}\right| \rho^{A+B}\right). \end{aligned}$$

This ${}_2F_1$ terminates after at most $A + 1$ terms, so, using (2.3), we see that

$${}_2F_1\left(\left.\begin{matrix}\tau-B, -A\\ \tau+1\end{matrix}\right| \rho^{A+B}\right) = \frac{(B+1)_A}{(\tau+1)_A}\{1 + 0(1-\rho)\},$$

where the implied constant depends only on A and B. Thus by Abel's theorem for power series,

$$\lim_{\rho\to 1} S_\rho^+ = \sum_{\substack{m\\ \tau\geqq 0}} \frac{(A)_\tau (B+1)_A}{\tau! A(B-\tau)(\tau+1)_A}$$

$$= -\frac{(B+1)_A}{A!} \sum_{\substack{m\\ \tau\geqq 0}} \frac{1}{(\tau+A)(\tau-B)}$$

$$= -\binom{A+B}{A} \sum_{\substack{m\\ \sigma m+g\geqq 0}} \frac{1}{(\sigma m+g+A)(\sigma m+g-B)}.$$

By (4.6),

$$\lim_{\rho\to 1} S_\rho^- = -\binom{A+B}{A} \sum_{\substack{m\\ \sigma m-g\geqq 0}} \frac{1}{(\sigma m-g+B)(\sigma m-g-A)}$$

$$= -\binom{A+B}{A} \sum_{\substack{m\\ \sigma m+g\leqq 0}} \frac{1}{(\sigma m+g-B)(\sigma m+g+A)},$$

where the last equality follows from the change of variable $m \to -m$. Thus

$$\lim_{\rho\to 1} S_\rho(R) = \lim_{\rho\to 1} (S_\rho^+ + S_\rho^- - S_\rho^0)$$

$$= -\binom{A+B}{A} \sum_m \frac{1}{(\sigma m+g-B)(\sigma m+g+A)} = 0,$$

because the series $\sum_m (\sigma m+g-B)^{-1}(\sigma m+g+A)^{-1}$ converges,

$$\frac{A+B}{(\sigma m+g-B)(\sigma m+g+A)} = \frac{1}{\sigma m+g-B} - \frac{1}{\sigma m+g+A},$$

and

$$\sigma m+g-B = \sigma m'+g+A$$

if and only if

$$m' = m-(A+B)/\sigma.$$

This proves (4.4).

THEOREM 6. *If* $t, \alpha, \beta \geqq 0$ *and* $\alpha+\beta = 1$, *then*

$$L = \frac{1}{\pi}\int_0^\pi \left(\frac{\sin u}{(\sin \alpha u)^\alpha (\sin \beta u)^\beta}\right)^t du = \frac{\Gamma(t+1)}{\Gamma(\alpha t+1)\Gamma(\beta t+1)}. \tag{4.7}$$

Proof. Recall that

$$C = \{e^{i\theta} : 0 \leqq \theta \leqq 2\pi\}, \quad E = \{e^{i\theta} : 0 \leqq \theta \leqq 2\pi/(A+B)\}.$$

We have

$$(4.8)\qquad L = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1-e^{iu}}{(1-e^{iu\alpha})^{\alpha}(1-e^{iu\beta})^{\beta}e^{iu\alpha\beta}}\right)^t du$$
$$= \frac{1}{2\pi i}\int_C \left(\frac{1-w}{(1-w^{\alpha})^{\alpha}(1-w^{\beta})^{\beta}w^{\alpha\beta}}\right)^t \frac{dw}{w}.$$

By continuity, it suffices to prove (4.7) for each fixed rational α, $0 < \alpha < 1$. Let A and B be positive integers which vary in such a way that $\alpha = A/(A+B)$ stays fixed. If we put $t = A + B$, $\alpha = A/(A+B)$, $\beta = B/(A+B)$, $w = z^{A+B}$ in (4.8), it follows from Theorem 5 that

$$L = \frac{A+B}{2\pi i}\int_E h(z)\frac{dz}{z} = \binom{A+B}{A}.$$

Thus (4.7) is true for our fixed $\alpha = A/(A+B)$ and all values of t of the form $A+B$, i.e., all positive integers t which are multiples of the reduced denominator of α. It remains to show that (4.7) is true for our fixed α and all real $t \geqq 0$.

In the half-plane $\operatorname{Re}(t) \geqq 0$, both members of (4.7) are analytic functions of t. The integrand in (4.7) takes its maximum value $(\alpha^{\alpha}\beta^{\beta})^{-t}$ when $u = 0$, so for $\operatorname{Re}(t) \geqq 0$,

$$\frac{\Gamma(\alpha t+1)\Gamma(\beta t+1)L}{\Gamma(t+1)} = O\left(\frac{\Gamma(\alpha t+1)\Gamma(\beta t+1)}{\Gamma(t+1)\alpha^{\alpha t}\beta^{\beta t}}\right) = O(\sqrt{t+1}),$$

by Stirling's formula [9, p. 31]. Thus (4.7) holds for all $t \geqq 0$ by Carlson's theorem (2.11).

THEOREM 7. *Let R be an integer $\not\equiv 0 \pmod{A+B}$. Then*

$$\sum_{\substack{j,k\geqq 0\\ Aj+Bk-AB\equiv R \,(\mathrm{mod}\, A+B)}} \frac{(A)_j(B)_k}{j!k!\left(\dfrac{Aj+Bk-AB}{A+B}\right)_{A+B+1}} = 0.$$

Proof. Let $H(z)$ be an antiderivative of

$$z^{-1}\left(h(z) - \binom{A+B}{A}\right)$$

in a neighborhood of a point z with $0 < |z| < 1$. By Corollary 4,

$$H(z) = \sum_{\substack{j,k,n\geqq 0\\ A(j+n)+B(k+n)\neq AB}} \binom{A+j-1}{j}\binom{B+k-1}{k}\binom{A+B}{n}$$
$$\times (-1)^n \frac{z^{A(j+n)+B(k+n)-AB}}{A(j+n)+B(k+n)-AB}.$$

For $0 < \rho < 1$, let $T_{\rho}(R)$ denote the sum of the coefficients of those powers z^m in the expansion of $H(\rho z)$ for which $m \equiv R(\bmod A + B)$. Thus,

$$T_\rho(R) = \frac{1}{A+B} \sum_{\substack{j,k \geqq 0 \\ Aj+Bk-AB \equiv R (\bmod A+B)}} \rho^{Aj+Bk-AB} \binom{A+j-1}{j} \times \binom{B+k-1}{k} \sum_{n \geqq 0} \frac{\binom{A+B}{n}(-1)^n \rho^{(A+B)n}}{(n+\delta)},$$

where

$$\delta = \frac{Aj + Bk - AB}{A+B}.$$

The inner sum on n equals

$$\delta^{-1} {}_2F_1\left(\begin{matrix}\delta, \; -A-B \\ \delta+1\end{matrix} \middle| \rho^{A+B}\right),$$

and it can be seen with the aid of (2.3) that

$$\text{(4.9)} \quad \lim_{\rho \to 1} T_\rho(R) = (A+B-1)! \times \sum_{\substack{j,k \geqq 0 \\ Aj+Bk-AB \equiv R (\bmod A+B)}} \frac{(A)_j (B)_k}{j!k! \left(\frac{Aj+Bk-AB}{A+B}\right)_{A+B+1}}.$$

On the other hand, the definition of $T_\rho(R)$ together with (4.2) and (4.3) show that

$$T_\rho(R) = -\rho^{-AB} S_\rho(R).$$

The result thus follows from (4.4) and (4.8).

Even the special case $A = 1$, $R = B$ of Theorem 7 appears to be non-trivial. We conclude this section by discussing this case.

COROLLARY 8. *For every positive integer* B,

$$\sum_{m=0}^{\infty} \frac{(1 + m + [m/B])_B}{(m - B/(B+1))_{B+2}} = 0,$$

where $[x]$ *denotes the greatest integer* $\leqq x$.

Proof. Put $A = 1$, $R = B$ in Theorem 7 and write $j + Bk = m(B+1)$ for $m = 0, 1, 2, \ldots$. Since $m(B+1) - Bk = j \geqq 0$, we have $k \leqq m + [m/B]$ so Theorem 7 yields

$$\sum_{m=0}^{\infty} \sum_{k=0}^{m+[m/B]} \frac{(B)_k}{k!(m - B/(B+1))_{B+2}} = 0.$$

For any integer $N \geqq 0$,

$$\sum_{k=0}^{N} \frac{(B)_k}{k!} = \frac{(N+1)_B}{B!},$$

so

$$\sum_{m=0}^{\infty} \frac{(1+m+[m/B])_B}{B!(m-B/(B+1))_{B+2}} = 0.$$

5. A q-analogue of theorem 1. The following result is beautifully analogous to Theorem 1. Recall that in this section we use the notation

$$(a)_m = \prod_{r=0}^{\infty} (1-aq^r)/(1-aq^{m+r}).$$

THEOREM 9. *For each pair of integers $u, v \geqq 0$, the coefficient $K_q(u, v)$ of $w^u x^v$ in*

$$f_q(w, x) = \frac{(wx)_{A+B}}{(w)_A(qx)_B} \tag{5.1}$$

is

$$K_q(u, v) = \frac{q^{uv-u}(q^{B-u+1})_{v-1}(q^{A-v+1})_{u-1}\{(1-q^A)(1-q^B) - (1-q^A)(1-q^u) - (1-q^B)(1-q^v)\}}{(q)_u(q)_v} \tag{5.2}$$

with the interpretation

$$K_q(u, 0) = (q^A)_u/(q)_u, \quad K_q(0, v) = q^v(q^B)_v/(q)_v. \tag{5.3}$$

Proof. Heine's q-binomial theorem [**11**, p. 92] gives

$$\frac{1}{(w)_A} = \sum_{j \geqq 0} \frac{(q^A)_j w^j}{(q)_j}.$$

Since

$$f_q(w, x) = \{(w)_A(qx)_B(xwq^{A+B})_{-A-B}\}^{-1},$$

expansion by the q-binomial theorem gives

$$f_q(w, x) = \sum_{j,k,n \geqq 0} \frac{(q^A)_j w^j (q^B)_n (qx)^n (q^{-A-B})_k (wxq^{A+B})^k}{(q)_j(q)_n(q)_k}.$$

The coefficient of $w^u x^v$ is obtained by combining the terms with $k = v - n = u - j$. Thus

$$K_q(u, v) = \sum_{k=0}^{M} \frac{(q^A)_{u-k}(q^B)_{v-k}(q^{-A-B})_k q^{v+k(A+B-1)}}{(q)_{u-k}(q)_{v-k}(q)_k},$$

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where $M = \min(u, v, A + B)$. Using the formula

$$(q^A)_{u-k} = \frac{(q^A)_u q^{k(k+1)/2-uk-Ak}(-1)^k}{(q^{1-u-A})_k},$$

we obtain

$$K_q(u, v) = \frac{(q^A)_u(q^B)_v q^v}{(q)_u(q)_v} \sum_{k=0}^{\infty} \frac{(q^{-v})_k(q^{-u})_k(q^{-A-B})_k q^k}{(q^{1-A-u})_k(q^{1-B-v})_k(q)_k},$$

that is

$$(5.4) \quad K_q(u, v) = \frac{(q^A)_u(q^B)_v q^v}{(q)_u(q)_v} {}_3\phi_2\left(\begin{matrix} q^{-v}, q^{-u}, q^{-A-B} \\ q^{1-A-u}, q^{1-B-v} \end{matrix} \middle| q, q\right).$$

When $A + B \leqq \min(u, v)$ we choose $a = q^{-v}$, $b = q^{-u}$, $n = A + B$, $c = q^{1-A-u}$ and $e = q^{1-B-v}$. Hence the ${}_3\phi_2$ in the left the member of (2.6) equals the ${}_3\phi_2$ series in (5.4), while the right member of (2.6) is a multiple of

$$(5.5) \quad \sum_{l=0}^{A+B} \frac{(q^{-A-B})_l(q^{-v})_l}{(q)_l q^{-l}} (q^{-1})_l(q^{l-A})_{A+B-l}(q^{l+u-v-B})_{A+B-l},$$

which vanishes for $B > 0$. The right side of (5.2) also vanishes when $A + B \leqq \min(u, v)$ because one of the factors $(q^{B-u+1})_{v-1}$, $(q^{A-v+1})_{u-1}$ vanishes. We finally turn to the case $u < A + B$, $u \leqq v$. In this case we take $n = u$, $a = q^{-v}$, $b = q^{-A-B}$, $c = q^{1-A-u}$, $e = q^{1-B-v}$ in (2.6) to obtain, from (5.4)

$$K_q(u, v) = \frac{(q^A)_u(q^B)_v(q^{1-B})_u(q^{1-A+v-u})_u}{(q)_u(q)_v(q^{1-A-u})_u(q^{1-B-v})_u} q^{v-uv} \times {}_3\phi_2\left(\begin{matrix} q^{-v}, q^{-1}, q^{-u} \\ q^{B-u}, q^{A-v} \end{matrix} \middle| q, q\right).$$

Hence

$$(5.6) \quad K_q(u, v) = \frac{(q^A)_u(q^B)_v(q^{1-B})_u(q^{1-A+v-u})_u}{(q)_u(q)_v(q^{1-A-u})_u(q^{1-B-v})_u} \times q^{v-uv}\left[\frac{(1-q^{B-u})(1-q^{A-v}) - (1-q^{-u})(1-q^{-v})}{(1-q^{B-u})(1-q^{A-v})}\right].$$

We now apply

$$(\lambda)_u = (-\lambda)^u(q^{1-u}/\lambda)_u q^{u(u-1)/2}$$

to obtain

$$\frac{(q^B)_v(q^{1-B})_u}{(q^{1-B-v})_u} = \frac{(q^{B-u})_u(q^B)_v}{(q^{B+v-u})_u} q^{uv} = \frac{(q^{B-u})_{u+v}}{(q^{B+v-u})_u} q^{uv} = (q^{B-u})_v q^{uv}$$

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and

$$\frac{(q^A)_u(q^{1-A+v-u})_u}{(q^{1-A-u})_u} = \frac{(q^A)_u(q^{A-v})_u}{(q^A)_u} q^{uv}.$$

Combining these relationships with the observation

$$\begin{aligned}(1 - q^{B-u})(1 - q^{A-v}) &- (1 - q^{-u})(1 - q^{-v}) \\ &= q^{-u-v}[(1 - q^A)(1 - q^B) - (1 - q^A)(1 - q^u) \\ &\qquad - (1 - q^B)(1 - q^v)]\end{aligned}$$

we establish (5.2).

Our next result is the following complement to Theorem 9.

THEOREM 10. *For each pair of integers $u, v \geqq 0$ the coefficient $K_q(u, v)$ of $w^u x^v$ in*

$$f_q(w, x) = \frac{(wx)_{A+B}}{(w)_A(x)_B}$$

is

$$\begin{aligned}K_q(u, v) = &\frac{q^{uv}(q^{B-u+1})_{v-1}(q^{A-v+1})_{u-1}}{(q)_u(q)_v} \\ &\times [(1 - q^B)(1 - q^{A-v}) + q^B(1 - q^A)(1 - q^{-u})],\end{aligned}$$

where

$$K_q(u, 0) = (q^A)_u/(q)_u,\ K_q(0, v) = (q^B)_v/(q)_v.$$

The proof is similar to our proof of Theorem 9. We reverse the sum, that is, replace k by $M - k$, in the first formula for $K_q(u, v)$ and again use the transformation for a 2-balanced ${}_3\phi_2$.

We conclude with a q-analogue of Corollary 2 in the case $A = B$. As in [**3**, p. 35], we write

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q)_n}{(q)_m(q)_{n-m}}$$

for the Gaussian binomial coefficient.

COROLLARY 11. *The coefficient $K_q(s)$ of $w^{A-s}x^s$ in $[(wx)_{2A}/(w)_A(x)_A]$ is given by*

$$\begin{aligned}K_q(s) &= q^{s(A-s)} \frac{(q^{s+1})_{s-1}(q^{A-s+1})_{A-s-1}(1 - q^A)(1 - q^s)(1 - q^{A-s})}{(q)_s(q)_{A-s}} \\ &= q^{s(A-s)} \begin{bmatrix} 2s - 1 \\ s \end{bmatrix}_q \begin{bmatrix} 2(A - s) - 1 \\ A - s \end{bmatrix}_q (1 - q^A),\end{aligned}$$

for $0 < s < A$, and

$$K_q(0) = \begin{bmatrix} 2A - 1 \\ A \end{bmatrix}_q,\quad K_q(A) = \begin{bmatrix} 2A - 1 \\ A \end{bmatrix}_q.$$

Proof. Put $A = B$, $u = A - s$, and $v = s$ in Theorem 10.

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6. Concluding remarks. R. Askey has pointed out that Saalschütz's theorem (the summation of a 1-balanced ${}_3F_2$) is equivalent to

$$(6.1)\quad (1 - wx)^{A+B-1}(1 - w)^{-A}(1 - x)^{-B} = \sum_{u,v \geqq 0} \frac{(B - u)_v (A - v)_u}{v!u!} w^u x^v.$$

Our proof of Theorem 1 also gives (6.1). Also, (6.1) clearly implies Theorem 1. Finally, the transformation which takes a k-balanced ${}_3F_2$ to a sum of k terms (the $q = 1$ version of (2.6)) is easily proved by multiplying (6.1) by $(1 - x)^{k-1}$ and equating the coefficients of $w^u x^v$.

For the q-analogue we have

$$(6.2)\quad \frac{(wx)_{A+B-1}}{(w)_A (x)_B} = \sum_{u,v \geqq 0} \frac{(q^{B-u})_v (q^{A-v})_u}{(q)_v (q)_u} q^{uv} w^u x^v.$$

If $x \to qx$ and we multiply (6.2) by $1 - wx$, we have Theorem 9. If we multiply (6.2) by $(1 - wxq^{A+B-1})$, we have Theorem 10. Also (2.6) easily follows from (6.2).

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