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MULTIPLICATION RINGS VIA THEIR TOTAL QUOTIENT RINGS

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1. Introduction. In the following paper ring will always mean commutative ring which may or may not have an identity. We use the letter N exclusively for nilpotents of the ring A.

A ring such that, given any two ideals L and M with $L \subseteq M$ there exists an ideal Q such that $L = QM$ is called a *multiplication ring*. For references to early papers on multiplication rings by Krull and Mori the reader is referred to [**2**]. A ring in which every regular ideal is invertible is called a *Dedekind ring*. It is easy to see that a multiplication ring with an identity is a Dedekind ring. Just as a Dedekind domain is defined by valuations on a field, a Dedekind ring is defined by valuations on its total quotient ring. Studying multiplication rings is reduced by this approach to studying the total quotient rings of multiplication rings. Among the results we obtain by this approach is the following: *A commutative ring A is a multiplication ring if and only if it is generated by idempotents, and for any prime ideal P, P is invertible, or A_P is a field, or P is maximal and A_P is a special principal ideal ring and there exists an idempotent contained in all prime ideals of A except P.*

We devote the remainder of the introduction to definitions and results not directly related to multiplication rings that will be used subsequently. A convenient reference for details is [**3**] as this contains further references for most of the concepts.

A ring A is said to be *generated by idempotents* if for every $a \in A$ there exists $e = e^2 \in A$ such that $ea = a$. Such rings have many of the properties of rings with units; for example, if for some ideal M, $a \in M = A$, then there exists $e = e^2 \in A \backslash M$ with $ea = e$, and M is contained in a maximal proper ideal which is prime. Suppose that A is generated by idempotents, $ea = a$ and $e = e^2$; a is called e-regular if there exists $b \in A$ such that $ab = e$. An ideal of A is called *regular* if it contains an e-regular element for every idempotent $e \in A$. Note that an ideal consisting entirely of zero divisors may be regular according to this definition! If A is a ring *generated by idempotents* it is a regular ideal, even though unless A has an identity, it consists entirely of zero divisors.

It is possible to define the total quotient ring K of A. Briefly, the set of idempotents E of A is ordered by $e \leqq f$ if $ef = e$ and

$$K = \varinjlim_{e \in E} K_e$$

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where K_e is the total quotient ring of Ae. There is a one-to-one correspondence between the contracted ideals (respectively prime or primary) of A and the ideals (respectively prime or primary) of K. If P is a non regular prime ideal then A_P is naturally isomorphic to K_{KP}.

The word valuation means valuation on a commutative ring. If v is a valuation on R then $v^{-1}(\infty)$ denotes the prime ideal of R that maps to ∞ under the valuation. A Prufer ring is a ring generated by idempotents in which every finitely generated ideal is invertible; clearly Dedekind rings are Prufer. Both rings are defined by a family of valuations on the total quotient rings, the family of valuations being determined by the maximal ideals. In the case of a Dedeking ring, these valuations are discrete of rank one and only finitely many of them are non zero at a given regular element.

PROPOSITION 1. *Let A be generated by idempotents. The following three conditions are equivalent:*

(1) *A has dimension zero, i.e. every prime ideal is maximal;*

(2) *for all $x \in A$, $x^{2n}y = x^n$ for some $y \in A$ and n;*

(3) *for all $x \in A$, $x^{n+1}b = x^n$ for some $b \in A$ and n.*

Proof. The proof in [**9**] generalizes to rings generated by idempotents.

COROLLARY. *Let A be a zero dimensional ring generated by idempotents. Let $a \in A$ and $e = e^2 \in A$ be such that $ae = a$.*

(i) *There exists $b \in A$ and n such that $(a - ba^2)^n = 0$.*

(ii) *There exists $d \in A$ such that $ed = d$, $a + d$ is e-invertible and ad is nilpotent.*

(iii) *There exists $f = f^2 \in A$ contained in exactly those prime ideals of A which contain a.*

(iv) *A is its own total quotient ring.*

Proof. (i) Let $b \in A$ be such that $a^{n+1}b = a^n$; then $(a - ba^2)^n = a^n(e - ba)^n = 0$ by the binomial theorem.

(ii) Let $d = e - ab$; ad is certainly nilpotent. To show that $a + d$ is a unit of Ae we must show that $a + d$ is in no prime P of the zero dimensional ring Ae. If $a \in P$ then $ab \in P$ so $d = e - ab \notin P$, and since ad is nilpotent, either $a \in P$ or $d \in P$. Thus $a + d \notin P$.

(iii) By (ii) there exists $c \in A$ such that $(a + (e - ab))c = e$. So

$$e = e^n = (a + (e - ab))^n c^n = a^n c^n + (e - ab)hc^n = f + g$$

where $fg = a^n c^n (e - ab))hc^n \in (a^n - a^{n+1}b) = 0$, and $ef = f$. Thus $f = fe = f(f + g) = f^2$. If $a \in P$ then $f \in P$, and if $a \notin P$ then since $ae = a$, $e \notin P$ so by (ii) $e - ab \in P$; consequently $g \in P$ and $f \notin P$.

(iv) Suppose a is e-regular and d is as in (ii), so $a^n d^n = 0$. Using the e-regularity of a, $0 = a^n d^n e = \ldots = ad^n e = d^n e = d^n$. But $(a + d)b = e$; so $(a + d)^n b^n = e^n = e$, i.e., $e = b^n(a^n + \ldots + nad^{n-1} + d^n) = ab^n t$ so that a is e-invertible.

A commutative ring A is called *von Neumann regular* if, for all $y \in A$ there exists $x \in A$ such that $xy^2 = y$. It follows from the above results that A is von Neumann regular if and only it if is zero dimensional without nilpotents.

2. Preliminary results on multiplication rings. A ring A is called an *AM-ring* if whenever L and M are ideals with L properly contained in M then $L = QM$ for some ideal Q. Clearly a multiplication ring is an AM ring.

In this section we gather together and shorten proofs of some known results on AM-rings. In all the following lemmas the ideals are those of an AM-ring A. N denotes the nilpotents of A.

LEMMA 2. (i) *If the prime ideal P is properly contained in M, then $P = PM$.*

(ii) *If $L = LQ$ then for each $a \in L$, $(a) = aQ$, so $a = xa$ with $x \in Q$. If $Q \subseteq N$ then $L = 0$.*

(iii) *Every power of a prime ideal P is p-primary.*

(iv) *An idempotent ideal L is a multiplication ring and if $x \in L$ there exists $y \in L$ such that $xy = x$.*

(v) *Let P be prime properly contained in M. If $D = \bigcap_{n=1}^{\infty} M^n$, then $P = PD$.*

Proof. (i) $P = QM$; since $M \not\subseteq P$, $Q \subseteq P$ so $P \subseteq PM \subseteq P$; thus $P = PM$.

(ii) Either $(a) = Q'L = Q'LQ = aQ$ or $(a) = L = LQ = aQ$. If $x^n = 0$ then $a = ax = ax^n = 0$.

(iii) If $ab \in P^n$, $a \notin P$ and $b \notin P^n$; then for some r, $1 \leqq r < n$, $b \in P^r \backslash P^{r+1}$. Since $P \subsetneq P + (a)$, $P = P(P + (a))$ by (i), so by (ii), $(b) = b(P + (a)) = bP + (ba) \subseteq P^{r+1}$, a contradiction.

(iv) Let $x \in L = L^2$; by (ii) $x = xy$, $y \in L$ so for any ideal $Q \subseteq L$, $Q = QL$.

(v) Since $P = PM = PM^2 = \ldots = PM^n$, $P \subseteq \bigcap PM^n \subseteq D$; so by (i), $P = PD$ unless $P = D$. Thus we assume $D = P$ and $D \neq D^2$ and obtain a contradiction. Let $a \in D \backslash D^2$, $D = DM^2$; so by (ii), $a = xa$ with $x \in M^2 \backslash D$. Since $D \neq D^2$, $M \neq M^2$. Let $y \in M \backslash M^2$; then $a(y - xy) = 0 \in D^2$; $y - xy \in M \backslash M^2$ so $y - xy \notin D \subseteq M^2$. By (iii), $D^2 = P^2$ is P-primary so $a \in D^2$, a contradiction.

LEMMA 3. *Let M be a prime or A itself. $D = \bigcap M^n$ is idempotent. Either $D = M^s$ or D is prime. If $M \neq D$ and $P \subsetneq M$ is a prime, then $P = D$.*

Proof. If $M^s = M^{s+1}$ then $D = M^s = M^{s+1} = \ldots = M^{2s} = D^2$; otherwise D is prime, for if $M^n \neq M^{n+1}$ for all n and $x, y \notin D$, then $x \in M^p \backslash M^{p+1}$ and $y \in M^m \backslash M^{m+1}$. $(x) = LM^p$ or $(x) = M^p$ and $(y) = QM^m$ or $(y) = M^m$, with $L, Q \not\subseteq M$. Consequently $(xy) = M^{p+m}$ or $(xy) = RM^{p+m}$ with $R \not\subseteq M$ and $xy \in M^{p+m} \backslash M^{p+m+1}$ implying $xy \notin D$. By (v) above $D = D^2$, and $P = PD \subseteq D$; suppose $d \in D \backslash P$. Then by (ii), $d - xd = 0$ for some $x \in D$. Let $y \in M \backslash D$. $d(y - xy) = 0$ so $y - xy \in P \subseteq D$, implying $y \in D$, a contradiction. Thus $P = D$.

PROPOSITION 4. *An AM-ring $A \neq A^2$ has every ideal principal and power of A,*

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and either $A^n = 0$ or A is the maximal ideal of a discrete rank one valuation domain with residue class field of order a prime.

Proof. Let $D = \bigcap_{n=1}^{\infty} A^n$. By Lemma 3, $D = D^2$ and A contains at most one prime ideal. Thus $D = D^2 \subseteq N$ and by Lemma 2, (ii), $D = (0)$. Suppose that $M \neq (0)$, and that n is minimal such that $M \not\subseteq A^{n+1}$. If $M \neq A^n$, then $M = QA^n \subseteq A^{n+1}$, a contradiction. Thus $M = A^n$. Applying this to $x \in A \backslash A^2$, $(x) = A$, so every ideal is principal. If for some integer n, $A^n = D = (0)$ the proof is complete; otherwise A is a domain and the conclusion follows as in Theorem 4 of [**1**].

LEMMA 5. *If $L \subsetneq M$ then either there exists a prime ideal P such that $L \subseteq P^{n+1}$ but $M \not\subseteq P^{n+1}$ or $L = A^m$.*

Proof. By the previous proposition we may suppose $A = A^2$. Let $d \in M \backslash L$ by (ii) of Lemma 2, $yd = d$ for some $y \in A$. $y^n \notin L:(d)$ for any positive integer n, so $L:(d)$ is contained in some prime ideal P such that $y \notin P$. Thus $L = (L:M)M \subseteq (L:(d))M \subseteq PM$, and if $M \subseteq P^n$, $M \not\subseteq P^{n+1}$ we are finished. Otherwise, $M \subseteq P^n$ for all n so

$$M \subseteq D = \bigcap_{n=1}^{\infty} P^n;$$

by Lemma 3 and Lemma 2 (ii), $M = QD = QD^2 = MD$ so $d = xd$ with $x \in D$. Thus $(y - xy)d = 0 \in L$ and $y - xy \in L:(d) \subseteq P$ implying $y \in P$, a contradiction.

COROLLARY. *Every ideal Q of an AM-ring is equal to the intersection of its primary components M.*

Proof. If $A \neq A^2$ this is clear. Clearly $Q \subseteq M$; if $Q \neq M$, there exists P^n with $Q \subseteq P^n$ and $M \not\subseteq P^n$, but P^n is P-primary, which gives a contradiction.

THEOREM 6 [**2**]. *A ring A is an AM-ring if and only if the following two conditions are satisfied:*

(i) *every ideal with prime radical is primary and a prime power;*

(ii) *if P is a proper prime ideal of A and M is such that $M \subseteq P^n$, $M \not\subseteq P^{n+1}$ then there exists $y \in A \backslash P$ such that $P^n = M:(y)$.*

Note. A itself is considered an improper prime ideal.

Proof. (i) Let $P = \sqrt{Q}$. If $Q \subseteq P^m$ for all m, then $Q \subseteq D = \bigcap P^n$, and if $P^n \neq P^{n+1}$ for some n, then D is prime and $\sqrt{Q} \subseteq D \neq P$. Thus for some n, $Q \subseteq P^n$ and $Q \not\subseteq P^{n+1}$ or $P^n = P^{n+1}$. Suppose $Q \neq P^n$. By Lemma 5, $Q \subseteq M^m$, $P^n \not\subseteq M^m$ for some prime M. By choice of $n, P \neq M$; by Lemma 2(i) if $P \subsetneq M$ then $P \subseteq M^n$ so $P^n \subseteq M^m$, a contradiction; thus we must have $P \not\subseteq M$. Let $a \in P$, $a \notin M$; since $\sqrt{Q} = P$, $a^t \in Q \subseteq M$ so $a \in M$, a contradiction. Thus $Q = P^n$; by Lemma 2(iii), Q is primary.

$\Rightarrow$ (ii) Since $0 \neq P \neq A$, $A = A^2$ and thus $M = BP^n$. Let $y \in B$, $y \notin P$. Since $yP^n \subseteq M$, $P^n \subseteq M:(y)$. If $x \in M:(y)$ then $xy \in M \subseteq P^n$; since $y \notin P$ and P^n is primary, $x \in P^n$.

$\Rightarrow$ [**2**, Theorem 13].

3. Total quotient rings of multiplication rings. Let S be any subset of the multiplication ring A; we use $S^\perp$ to denote the set $(N:S) = \{x \in A | xS \subseteq N\}$; N is the nilpotents. The following lemma uses ideas due to Mori.

LEMMA 7. *Let x be an element of the multiplication ring A; then $(x^\perp)^\perp \cap x^\perp = N$ and there exists $e = e^2 \in (x^\perp)^\perp$ such that $x^n = ex^n$ for some n.*

Proof. Let $S \subseteq A$. If $y \in S \cap S^\perp$ then $y^2 \in N$ so $S \cap S^\perp \subseteq N$; thus $x^\perp \cap (x) \subseteq N = x^\perp \cap (x^\perp)^\perp$. Since

$$(x) \subseteq N + (x) + x^\perp,\ (x) = D(N + (x) + x^\perp)$$

for some ideal D, and $Dx^\perp \subseteq (x) \cap x^\perp \subseteq N$ so $D \subseteq (x^\perp)^\perp$, and consequently

$$(x) \subseteq (x^\perp)^\perp(N + (x) + (x^\perp)) \subseteq N + (x)(x^\perp)^\perp.$$

Thus for some q, $q \in (x^\perp)^\perp$, $x - qx \in N$, so $(q - q^2)x \in N$ and $q - q^2 \in x^\perp$; thus $q - q^2 \in x^\perp \cap (x^\perp)^\perp = N$. For some integer m, $0 = (q - q^2)^m = q^m - dq^{m+1}$, and $q^m = (dq)^m q^m = eq^m$ where $e = d^m q^m = d^m eq^m = e^2$. $x - q^m x = x - qx + q(x - qx) + \ldots + q^{m-1}(x - qx) \in N$ so $x - ex = x - q^m x + eq^m x - ex = x - q^m x - e(x - q^m x) \in N$. Also for any integer t,

$$(x - ex)(x^t - ex^t) = x^{t+1} - ex^{t+1} - ex^{t+1} + e^2x^{t+1} = x^{t+1} - ex^{t+1},$$

so that for some integer n, $0 = (x - ex)^n = x^n - ex^n$.

COROLLARY [**7**]. *A multiplication ring is generated by idempotents.*

Proof. Let $y \in A$. Since $A = A^2$ there exists $x \in A$ such that $xy = y$ by Lemma 2(iv). Let e and n be as above; then $ye = yx^n e = yx^n = y$.

LEMMA 8. *Let x be an element of the multiplication ring A such that $x^\perp \subseteq N$; then x is not a zero divisor.*

Proof. Let $F = 0:(x)$ and $G = 0:(x^2)$. Suppose $F \neq G$. Then $F \subset G$ and by Lemma 5, $F \subseteq P^n$, $G \nsubseteq P^n$ for some prime P. Let $G \subseteq P^m$, $G \nsubseteq P^{m+1}$, $m \geqq 1$ since $G \subseteq N$. By Theorem 6, $G:(y) = P^m$ for some $y \notin P$, so that $xyP^m \subseteq xG \subseteq F \subseteq P^{m+1}$. Since $P^{m+1} \neq P^m$ is primary $xy \in P$ and consequently $x \in P$, so $yx^m \in G$ and $yx^{m+2} = 0$, implying that $y \in (x^\perp) \subseteq N \subseteq P$, a contradiction. It follows that $F = G$.

Since $F \subseteq F + (x)$, $F = Q(F + (x))$ for some Q, and since $Qx \subseteq F$, $Q \subseteq G = F$, so $F \subseteq F(F + (x)) \subseteq F \subseteq N$. So $F = F(F + x)$ and by Lemma 2(ii), $F = 0$. Consequently x is not a zero divisor.

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LEMMA 9. *Let A be a multiplication ring. Let x and $g = g^2$ be such that $gx = x$; then there exists $e = e^2$ and an integer n such that $ex^n = 0$ and $e + x$ is g-regular.*

Proof. By Lemma 7 there exists $f = f^2 \in (x^\perp)^\perp$ such that $fx^n = x^n$, and consequently $fx - x \in N$. Let $e = g - fg$. Then $ex^n = (g - fg)x^n = x^n - fx^n = 0$ and $g(e + x) = ge + gx = e + x$, so it remains only to show $y(x + e) = 0$ implies $yg = 0$.

Suppose $z(x + e) \in N$. Then $0 = z^m(x + g - fg)^m = z^m g^m + z^m(x - fg)d$ so that $z^m g \in (x - fg) \subseteq (x^\perp)^\perp$, since $x, f \in (x^\perp)^\perp$; hence $zg \in (x^\perp)^\perp$. Also $fz(x + e) = z(fx + fg - f^2 g) = zfx = z(fx - x + x)$ so that $zx = fz(x + e) - z(fx - x)$. Since $z(x + e) \in N$ and $fx - x \in N$, $zx \in N$ so $zg \in x^\perp \cap (x^\perp)^\perp = N$. Thus in the multiplication ring Ag, $(x + e)^\perp \subseteq Ng$ so by the previous lemma $e + x$ is a nonzero divisor, i.e. $(e + x)gy = 0$ implies $gy = 0$. Thus $e + x$ is g-regular in A.

COROLLARY. *Any element x of a multiplication ring may be written $x = fx + q$ where $f = f^2$, fx is f-regular, $fq = 0$, and q is nilpotent. f and q are determined uniquely.*

Proof. Let g be any idempotent such that $gx = x$. Let e be as in the preceding Lemma. Define $f = g - e$ and $q = ex$. q is nilpotent. $f^2 = f$ and $ef = 0$, so $fq = 0$. Suppose $fxy = 0$. Since $x + e$ is g-regular and $f(x + e)y = 0$, $0 = gfy = fy$ implying that fx is f-regular. Let $x = f'x + q'$ be a second decomposition and let $g = f - ff'$. $gx = g(f'x + q') = ff'x - ff'^2x + gq' = gq'$. Thus $0 = gq'^n = gx^n$ and using the f-regularity of fx repeatedly $0 = fxgx^{n-1} = fxgx^{n-2} = \ldots = fg$. So $0 = f(f - ff') = f - ff'$, i.e., $f = ff'$. By symmetry $f = ff' = f'$. Finally $q' = x - f'x = fx + q - f'x = q$.

We call $x = fx + q$ the *idempotent-nilpotent decomposition* of x.

PROPOSITION 10. *Let M be an ideal of a multiplication ring properly containing a prime ideal P; then M is a regular ideal.*

Proof. If $M = A$, M is regular since A is generated by idempotents. Assume $M \neq A$. Let f be any idempotent. We show M contains an f-regular element. Let $x \in M \backslash P$ and $h = h^2 \in A \backslash M$ with $hx = x$; since $h(f - hf) = 0$, $g = h + f - fh \in A \backslash M$ and $gx = x$. Choose e as in Lemma 9. Since $ex^n = 0 \in P$, $e \in P$ so $e + x \in M$ and $e + x$ is g-regular so $(e + x)f$ is f-regular.

COROLLARY. *Let A be a multiplication ring. Then every prime ideal which is not regular is minimal, and A is a total quotient ring if and only if it is zero dimensional.*

Proof. This follows by the above proposition and because every zero dimensional ring is a total quotient ring.

LEMMA 11. *In a multiplication ring, an ideal is idempotent if and only if each of its primary components is idempotent; any intersection of idempotent ideals is idempotent.*

Proof. Let $I = \bigcap P_i^{n_i}$ be any intersection of idempotent primary ideals indexed by i. We show $I = I^2$ by proving that their primary components are equal.

Let P^m be any primary component of I^2; since $I^2 \subseteq P$, $I \subseteq P$. Clearly we need only show $I \subseteq P^m$. There are three possible cases:

(i) $P = P_i$ for some i; then $I \subseteq P_i^{n_i} = P_i^{2n_i} = P^{2n_i}$, so $I \subseteq P^m$;

(ii) $P \subsetneq P_i$ for some i so that $P = P^2$ and $I \subseteq P^m = P$;

(iii) $P \nsubseteq P_i$ for all i. Suppose that $I \nsubseteq P^m$; then $I \subseteq P^n$, $I \nsubseteq P^{n+1}$, so by Theorem 6 (ii), there exists $b \notin P$ such that $bP^n \subseteq I \subseteq P_i^{n_i}$, and since $P \nsubseteq P_i$, $b \in P_i^{n_i}$. Since this holds for each i, $b \in \bigcap P_i^{n_i} = I \subseteq P$, a contradiction.

In particular any ideal with idempotent primary components is idempotent.

Let P^n be any primary component of the idempotent ideal I. $I = I^2 \subseteq P^n I \subseteq P^{2n} \subseteq P^{n+1}$. Thus $P^n = P^{n+1} = P^{2n}$.

The final result now follows by writing each idempotent ideal as the intersection of its primary components, all of which are idempotent.

LEMMA 12. *Let A be a Prufer ring whose total quotient ring K is a multiplication ring. Let Ω be the family of valuations of K corresponding to the maximal regular ideals of A, and suppose that if $v \in \Omega$ then $v^{-1}(\infty) = [v^{-1}(\infty)]^2$. Let Q, D be ideals of A with $Q \subseteq D$ and D not regular; then there exists an ideal E such that $Q = ED$.*

Proof. Let $D' = KD$, $Q' = KQ$ and let E' be the ideal of K such that $Q' = E'D'$. Let

$$E = \{a \in E' | v(a) \geqq v(b) \text{ for some } b \in Q \text{ and for all } v \in \Omega\}.$$

$DE \subseteq Q$. It suffices to show $(DE)^e \subseteq Q^e$ (where ^{e} denotes the extension to A_M) for all maximal ideals M. If M is not regular this follows from $DEK \subseteq D'E' = Q'$ for in this case $A_M \cong K_{KM}$. If M is regular it corresponds to a valuation v. Let

$$N(v) = \{x \in A | xa = 0 \text{ for some } a \in A \backslash M\}$$

and

$$N'(v) = \{x \in K | xa = 0 \text{ for some } a \in A \backslash M\}.$$

We first prove that $N'(v) = v^{-1}(\infty)$. Clearly $N'(v) \subseteq v^{-1}(\infty)$. Suppose $v(x) = \infty$. Let $x = fx + q$ be the nilpotent-idempotent decomposition. $fxd = f$ so $v(f) = \infty$ and since $f = f^2$, $f \in A$, so $f \in M$. Let $e = e^2 \in A \backslash M$; then $e - ef \in A \backslash M$ and $f(e - ef) = 0$ so $f \in N'(v)$; thus $x^n = fx^n \in N'(v)$ so the radical of $N'(v)$ is $v^{-1}(\infty)$. Since K is a multiplication ring $N'(v) = (v^{-1}(\infty))^m = v^{-1}(\infty)$.

Consequently $N(v) = N'(v) \cap A = v^{-1}(\infty) \cap A$. Let $\phi: A \to A/N(v)$. It suffices to show if $a \in E$, i.e. if $v(a) \geqq v(b)$ for some $b \in Q$, then $\phi(a) \in Q^e$.

Let $a = af + q$ and $b = bf' + q'$ be the nilpotent-idempotent decompositions of a and b. $v(q) = v(q') = \infty$ and so $q \in N(v)$. If $v(f) = \infty$ then $f \in N(v)$ and $\phi(a) = 0 \in Q^e$; otherwise $v(f) = 0$, and so $v(f') = 0$, aff' and bff' are ff' regular; $v(aff') = v(a) \geqq v(b) = v(bff')$ so $aff' = bff'c/d$ and $a(dff') = b(cff')$ where $dff' \in A\backslash M$, so $\phi(a) = k\phi(b) \in Q^e$.

$Q \subseteq DE$. Let $q \in Q$. Since K is zero dimensional there exists $a \in K$ and n such that $0 = (q - aq^2)^n$. Thus for each $v \in \Omega$, $v(aq - a^2q^2) = v(q - aq^2) = \infty$, so $v(aq) = v(a^2q^2) = 2v(aq)$, so $v(aq) = 0$ or $v(aq) = \infty$. Thus $aq \in A$, and since $aq \in Q' \subseteq D'$, $aq \in D = D' \cap A$. Thus $aq^2 \in DE$, since $Q \subseteq E$. Let

$$F = \bigcap_{v\in\Omega} v^{-1}(\infty);$$

since $v^{-1}(\infty)$ is idempotent, so is F by Lemma 11. Let N' be the nilpotents of K; then $N' \subseteq F$; so by Lemma 2 (ii), since $p = q - aq^2 \in N' \subseteq F$, $p = pf$ for some $f \in F$, and $v(f) = \infty$ for all $v \in \Omega$. Since $p = q - aq^2 \in Q \subseteq QK = D'E'$,

$$p = \sum_{1\leqslant i\leqslant n} a_i b_i,$$

$a_i \in D'$ and $b_i \in E'$. Thus

$$p = pf = pf^2 = \sum_{1\leqslant i\leqslant n} a_i f b_i f,$$

and $v(a_i f) = v(b_i f) = \infty$ for all $v \in \Omega$. Thus $p = q - aq^2 \in DE$, and since $aq^2 \in DE$, so does q.

THEOREM 13. *Let A be a ring generated by idempotents with total quotient ring K. A is a multiplication ring if and only if the following conditions are satisfied*:

(i) *K is a multiplication ring;*

(ii) *A is a Dedekind ring;*

(iii) *non-maximal prime ideals of A are idempotent.*

Proof. $\Rightarrow$ (i) Let L, M be ideals of K with $L \subseteq M$; L and M are the extensions of their contractions to A, L' and M'. Since $L' \subseteq M'$ there exists Q' such that $L' = Q'M'$; then $L = KQ'M$.

(ii) We show that for any idempotent e the regular ideals of Ae are invertible. Let $x = xe \in Ae$ be a regular element of the ideal Q of Ae. Since $(x) \subseteq Q$ there is an ideal B of A such that $(x) = BQ = eBQ$; thus $eB(x^{-1})Q = Ae$.

(iii) This holds by Lemma 3.

$\Leftarrow$ Suppose L, M are ideals of A with $L \subseteq M$. If M is regular then since A is a Dedekind ring there exists a fractionary ideal M' such that $MM' = A$ so $L = LM'M$, with $LM' \subseteq MM' = A$. Otherwise the result follows from Lemma 12, for if v is a valuation defining A and $P = v^{-1}(\infty) \cap A$, then P is a non maximal prime so $P = P^2$; consequently $v^{-1}(\infty) = KP = KP^2 = KP \cdot KP = [v^{-1}(\infty)]^2$.

Note: The proof also shows that the third condition above may be replaced by:

for each valuation v of K corresponding to a regular ideal of A, $v^{-1}(\infty) = [v^{-1}(\infty)]^2$

COROLLARY 1. *Let A be a ring without nilpotents having total quotient ring K. The following conditions are equivalent:*

(1) *A is a multiplication ring;*

(2) *A is a Dedekind ring and all non regular prime ideals of A are minimal;*

(3) *K is zero dimensional and A is Dedekind.*

Proof. (1) $\Rightarrow$ (2) by the corollary to Proposition 10 and Theorem 13.

(2) $\Rightarrow$ (3) by the correspondence between non-regular prime ideals of A and those of K.

(3) $\Rightarrow$ (1). Since K is zero dimensional without nilpotents it is a von Neumann regular ring, and each ideal is idempotent. Thus K is a multiplication ring in which all prime ideals are idempotent and the result follows by the above note.

A ring is called *semi-hereditary* if every finitely generated ideal is projective. A is semi-hereditary if and only if A is Prufer and K is von Neumann regular (Endo) [**3**]. Thus:

COROLLARY 2. *A multiplication ring is semi-hereditary if and only if it has no nilpotents.*

A ring is called *hereditary* if every ideal is projective.

COROLLARY 3. *A commutative ring generated by idempotents is hereditary if and only if it is a multiplication ring without nilpotents such that every ideal of the total quotient ring is a direct sum of principal ideals.*

Proof. This follows from the above and results of Kaplansky [**4**]. See also Marot [**6**].

PROPOSITION 14. *Let A be a multiplication ring with total quotient ring K. If $x \in A$, then $v(x)$ is zero or infinity for all but a finite number of the valuations of K defining A.*

Proof. Let g be an idempotent such that $gx = x$. Since $v(g) = \infty$, implies $v(x) = \infty$, we need consider only valuations such that $v(g) = 0$. By Lemma 9, there exists $e = e^2$ such that $ex^n = 0$ and $e + x$ is g-regular. So $v(e + x) = 0$ for all but a finite number of these valuations since A is Dedekind. Finally $(e + x)x^n = ex^n + x^{n+1} = x^{n+1}$, so that if $v(x) \neq \infty$ then $v(e + x) = v(x)$.

4. Zero dimensional multiplication rings. We introduce the definition of level of a prime ideal.

Let Ω be the family of prime ideals of a zero dimensional ring A. $P \in \Omega$ is said to have *level zero* if

$$\bigcap_{Q \in \Omega \setminus \{P\}} Q \not\subseteq P.$$

This is clearly equivalent to the existence of an element in all primes except P. By the Corollary to Proposition 1 this element may be assumed idempotent. Higher level primes are defined inductively. Let Ω_n denote the primes of level n. Let

$$S_n = \Omega/\left(\bigcup_{i=0}^{n} \Omega_i\right);$$

then $P \in \Omega_{n+1}$ if $\bigcap_{Q \in S_n \setminus \{P\}} Q \not\subseteq P$. Primes in S_n for all integers n are said to have *infinite level.*

Example 1. A von Neumann regular ring in which all primes have infinite level: Let A be the ring of all characteristic functions of subsets of the real interval $[0, 1]$ generated by intervals of the form $[0, b)$, $0 < b \leqq 1$. Since $a \in A$ implies that $a^2 = a$, A is von Neumann regular. $P_x = \{ f \in A \mid f(x) = 0\}$ for some x, $0 \leqq x < 1$, is a prime ideal. Since for any $y \in [0, 1)$,

$$\bigcap_{x \in [0,1) \setminus \{y\}} P_x = (0) \subseteq P$$

for any prime P, there are no primes of level zero; thus all primes have infinite level.

A ring in which each ideal is a power of a nilpotent principal maximal ideal is called a *special principal ideal ring* or *SPIR*. It is easily seen that a multiplication ring with only one prime ideal is a *SPIR*; in particular if A is a zero dimensional multiplication ring then $A_P \cong A/P^n$ is a *SPIR*. We call A_P the *SPIR corresponding to P*.

THEOREM 15. *let A be a ring generated by idempotents. The following conditions are equivalent*:

(1) *A is a multiplication ring which is its own total quotient ring.*

(2) *A is a zero dimensional ring in which every primary ideal is a prime power and every non idempotent prime is of level zero.*

(3) *If P is a prime such that A_P is not a field then P is of level zero and A_P is a SPIR.*

Proof. (1) $\Rightarrow$ (2) That A is of dimension zero and that primary ideals are prime powers, follow by the corollary to Proposition 10 and by Theorem 6 respectively. Let P be a prime ideal; let I index the remaining prime ideals and suppose $P \supseteq D = \bigcap_{i \in I} P_i$. We show that $P = P^2$. Let $PQ = D$; then for each $i \in I$, $PQ \subseteq P_i$ and since $P \not\subseteq P_i$, $Q \subseteq P_i$. Thus $Q \subseteq D$ so $D = PQ \subseteq PD \subseteq P^2D \subseteq P^2$.

Suppose $a \in P \backslash P^2$. Let $e = e^2 \notin P$ with $ea = a$. By (iii) of the Corollary to Proposition 1, there exists b in A such that ab is nilpotent, $be = b$ and $(a + b)d = e$. Since $e \notin P$, $a + b \notin P$ so $b \notin P$. However $ab \in N \subseteq D \subseteq P^2$, and since P^2 is P primary this means $a \in P^2$, a contradiction.

(2) $\Rightarrow$ (1) Since A is zero dimensional it is a total quotient ring.

Since A is generated by idempotents any ideal with a maximal ideal as radical is primary, and consequently since A is zero dimensional any ideal with prime radical is primary and thus a prime power. It remains to show that condition (ii) of Theorem 6 holds. Let Q have primary component P^n with $P^n \neq P^{n+1}$. Let B and D be the intersection of the remaining primary components $P_i^{n_i}$ and their prime ideals respectively. Since $P \neq P^2$, $D \not\subseteq P$ and there exists $a \in D$, $a \notin P$. Let $e = e^2$ be such that $ea = a$, and let b, d be such that $(ab)^m = 0$, $(a + b)d = e$ (by the Corollary of Proposition 1). If $b \in P_i$, $a + b \in P_i$ so $e \in P_i$ and $a = ae^{n_i} \in P_i^{n_i}$; if $b \notin P_i$, then since $(ab)^m \in P_i^{n_i}$, $a^m \in P_i^{n_i}$ so that $a^m \in B$ and $a^m P^n \subseteq BP^n \subseteq Q$ so $P^n \subseteq (Q{:}a^m)$. If $x \in Q{:}a^m$, then $a^m x \in P^n$, but $a^m \notin P$ and P^n is primary so $x \in P^n$; thus $P^n \supseteq (Q{:}a^m)$.

(1) and (2) $\Rightarrow$ (3) It follows from (1) by the remarks preceding the theorem that A_P is a $SPIR$; the rest follows from (2), since if $P \neq P^2$ then A_P is a field.

(3) $\Rightarrow$ (2) The one-to-one correspondence between the P-primary ideals of A and those of A_P shows that every P-primary ideal is a prime power. Since each A_P is of dimension zero, so is A. If $P \neq P^2$, then A_P is not a field so P is of level zero.

Note that in a zero dimensional ring generated by idempotents the existence of an element contained in all prime ideals but one implies the existence of a primitive idempotent contained in all prime ideals but one, by the Corollary of Proposition 1. ($e = e^2$ is *primitive* if $f = f^2$ and $ef = f$ imply $e = f$ or $f = 0$.)

LEMMA 16. *Let I and J be ideals generated by idempotents. Let Q be an ideal generated by nilpotents. $I + J$ is idempotent and if $I = J + Q$, then $Q \subseteq I = J$.*

Proof. $I + J = I^2 + J^2 \subseteq I^2 + IJ + J^2 = (I + J)^2 \subseteq I + J$.

Let e be any idempotent in $I = J + Q$. Then $e = f + q$ with $f \in J$, $q \in Q$ and $q^n = 0$. Thus $e = e^n = (f + q)^n = fb + q^n = fb \in J$. Thus $I \subseteq J$.

We use the following notation: A is a zero dimensional multiplication ring. Φ_0 and Φ_∞ denote the families of primary components of zero corresponding to prime ideals of level zero and infinity respectively; h denotes the natural homomorphism of A into R where

$$R = \prod_{Q \in \Phi_0 \cup \Phi_\infty} A/Q;$$

$F = \bigoplus_{Q \in \Phi_0} A/Q$; N denotes nilpotents of A.

PROPOSITION 17. *A zero dimensional multiplication ring A is a subdirect product of SPIR's. With the above notation, identifying A with its image in R, all but a finite number of components of any element are either zero or units and in particular, $N \subseteq F \subseteq A \subseteq R$.*

Proof. Let $Q \in \Phi_0 \cup \Phi_\infty$. Since A maps onto A/Q, and each A/Q is an $SPIR$, to show the first part we need only show h injective i.e.,

$$D = \bigcap_{Q \in \Phi_0 \cup \Phi_\infty} Q = 0.$$

Suppose that a is a non zero element of D. Let M be a P-primary component of zero to which a does not belong. Since $M \notin \Phi_0$, $P = P^2$ so $M = P$ and $a \notin P$, and since $M \notin \Phi_\infty$, P has finite level. Let P' be the prime ideal of minimal level to which a does not belong. By definition of level, there exists $b \in A$ such that b belongs to all those prime ideals of A which have level no less than the level of P' except P' itself. Thus ab is in all prime ideals of A except P', so P' is of level zero, a contradiction, since this implies $D \subseteq P'$.

Let $a \in A$ have idempotent-nilpotent decomposition $a = ae + q$. If the component corresponding to P is neither zero nor a unit then $ae + q \in P \backslash Q$. Since $q \in P$, $ea \in P$ so $e = ead \in P$ and $ea = e^n a \in P^n = Q$. It follows that $a \in P \backslash Q$ only if $q \in P \backslash Q$ and we need only prove the special case $N \subseteq F \subseteq A$. It is clear that $F \subseteq A$, for if P is the prime corresponding to $Q \in \Phi_0$ then by the Corollary to Proposition 1, there exists an idempotent in all primes but P; the collection of such idempotents generate F. Let $N + F = E$. If $P \neq P^2$ then $P^m \in \Phi_0$ and so the only prime ideals containing E are idempotent. Thus E is an intersection of idempotent ideals and $E = E^2$. Since E is a multiplication ring it is generated by idempotents. Thus by Lemma 16, $N \subseteq F$.

A ring is called *T-nilpotent* if, given any sequence of elements $a_1, a_2, a_3, \ldots, a_n, \ldots$ in the Jacobson radical, there exists an integer m such that $a_1 a_2 a_3 \ldots a_m = 0$.

COROLLARY 1. *A zero dimensional multiplication ring is T-nilpotent.*

Proof. Let $a_1, a_2, a_3, \ldots$ be a sequence of elements in the Jacobson radical. Since A is zero dimensional $a_i \in N$. Since $a_1 \in N \subseteq F$, a_1 has only finitely many non zero components. Let m be the maximal order of nilpotence occurring in the rings containing non zero components of a_1; then $a_1 a_2 a_3 \ldots a_m = 0$.

COROLLARY 2. *There are only a finite number of SPIR's of characteristic p^n, $n > 1$ and p a fixed prime number, associated with a multiplication ring with identity.*

Proof. By going to the total quotient ring we may work with a zero dimensional ring. If P is a prime ideal such that A_P has characteristic p^n, $n > 1$, then p (the sum of p identity elements) is in $P \backslash P^2$, so has a component at that point which is neither zero nor a unit. Since there are only a finite number of such components there are only a finite number of $SPIR$'s of characteristic p^n, $n > 1$.

Consideration of $\bigoplus_{n=1}^{\infty} Z/(4)$ shows that the same result does not hold for multiplication rings without unit.

Let e and f be idempotents; we use $e \vee f$ to denote $e + f - ef$.

PROPOSITION 18. *A zero dimensional multiplication ring is a Bezout ring,* i.e. *every finitely generated ideal is principal.*

Proof. Let $a_1, a_2, \ldots, a_n$ generate an ideal of the zero dimensional multiplication ring A. Let $a_i = e_i a_i + q_i$ be the nilpotent-idempotent decomposition for each i, $1 \leqq i \leqq n$. Let $e = e_1 \vee e_2 \vee \ldots \vee e_n$. Let $f_1, \ldots, f_m$ be the primitive idempotents corresponding to the non zero components of $q_1, q_2, \ldots, q_n$. Let $f' = f_1 \vee f_2 \vee \ldots \vee f_m$, and let $f = f' - ef'$. Af is the direct sum of a finite number of *SPIR*'s so that every ideal of Af is principal; in particular the ideal generated by $(q_1f, q_2f, \ldots, q_nf)$ is generated by a nilpotent q. It is easily seen that $(a_1, a_2, \ldots, a_n) = (e, q) = (e + q)$.

PROPOSITION 19. *Every ideal in a zero dimensional multiplication ring A can be expressed as the direct sum of an idempotent ideal and a nilpotent ideal. Such an expression is unique.*

Proof. Let M be any ideal of A. By Lemma 16, M contains a largest idempotent ideal I. Let G be the ideal generated by those primitive idempotents corresponding to P-primary ideals $Q \in \Phi_0$ such that $M \subseteq P$ but $M \not\subseteq Q$. Let $L = M \cap G \cap N$. Let $a \in I \cap L$. If $a \neq 0$ then, since $a \in G$, there is a primitive idempotent e in G such that $ae \neq 0$. Since I is generated by idempotents there is $f = f^2$ such that $fa = a$. Thus $efa = ea \neq 0$ so $ef \neq 0$ so $e = ef \in I$, a contradiction. Thus $I \cap L = 0$.

Clearly $I + L \subseteq M$. Suppose $I + L \neq M$; then by Lemma 3, there exists a prime ideal P and a smallest integer m such that $I + L \subseteq P^m$ but $M \not\subseteq P^m$. If $m > 1$, then let $e = e^2 \notin P$ be primitive and $b \in M$, $b \notin P^m$. Then $eb \notin P^m$ but $eb \in M \cap G \cap N$ so $L \not\subseteq P^m$, a contradiction. Otherwise $m = 1$, so $M \not\subseteq P$. Let $a \in M \backslash P$ have idempotent-nilpotent decomposition $a = ae + q$; then $e \in M \backslash P$, implying $e \in I$ since $e = e^2$. Thus $I \not\subseteq P$, a contradiction.

Suppose $M = I + L = I' + L'$; then $I = I^2 = I(I + L) = II' + IL'$ and $I = II'$ by Lemma 16. By symmetry $I = I'$. Let $q = f + q' \in L$ with $q' \in L'$, $f \in I' = I$ and let $e = e^2 \in I$ be such that $ef = f$. Then $q - q' = f = fe = (q - q')e = 0$ so $q = q'$ and $L \subseteq L'$. Thus $L = L'$.

A ring A is called an *almost multiplication ring* if every ideal with prime radical is a prime power. This is equivalent to the condition that A_M is either a discrete rank one valuation domain or an *SPIR* for each maximal ideal M [**5**]. A domain with the same properties is called an *almost Dedekind domain*. The following examples exhibit almost multiplication rings which fail to be multiplication rings.

Example 2. An almost multiplication ring without nilpotents which is a total quotient ring but is not a multiplication ring: Let A be an almost Dedekind domain with zero Jacobson radical which is not a Dedekind domain [**5**, p. 220]. Let R be the total quotient ring like A as constructed in [**3**]. R is an almost multiplication ring but is not von Neumann regular and hence is not a multiplication ring.

Example 3. A zero dimensional almost multiplication ring which is not a multiplication ring: Let

$$A = \prod_{i=1}^{\infty} Z/(4).$$

For each a in A, $a^2a^2 = a^2$, so by Proposition 1, A is zero dimensional. Let $\phi: A \to A_P$ where P is any prime ideal. We prove that A_P is a *SPIR* by showing that if a and b are in P and are such that $\phi(a) \neq 0$ and $\phi(b) \neq 0$, then $\phi(a) = \phi(b)$. Let a' and b' be elements of A which have zero components where a and b are not 2 and have all other components units. Thus $a(1 - a^2 + a') = 0$, since $\phi(a) \neq 0$, $1 - a^2 + a' \in P$ and since $1 - a^2 \notin P$, $a' \notin P$. Similarly $b' \notin P$. Now $(a - b)a'b' = 0$, so $\phi(a - b) = 0$, i.e., $\phi(a) = \phi(b)$. Since there are nilpotents with an infinite number of non zero components, A is not a multiplication ring.

5. Some particular cases of multiplication rings. We begin with an example which illustrates a method of constructing a multiplication ring with unit from a family of zero dimensional multiplication rings.

Example 4. Let A_i, $i \in I$ be an infinite family of zero dimensional multiplication rings with unit, each of which contains the field k. Let A be the subring of $\prod_{i \in I} A_i$ generated by

$$G = \bigoplus_{i \in I} A_i$$

and the functions on I with constant values in k. A is a zero dimensional multiplication ring. The prime ideals of A are $G = G^2$ and ideals $P_{i,j}$ corresponding to primes P_j in A_i as follows: the elements in $P_{i,j}$ have the ith component in P_j and all other components arbitrary. Since $P_{i,j}$ inherits the properties of P_j, A is a multiplication ring.

If the prime ideals of A_i, for all but a finite number of $i \in I$ have level less than n, and an infinite number of the A_i's have primes of level $n - 1$, then G is a prime ideal of level n. Thus it is possible to build multiplication rings with prime ideals of any level.

By choosing each $A_i = k[X]/(X^2)$ for $i \in I$ and I infinite, we see that there is no analogy to the second corollary to Proposition 17 for *SPIR*'s of characteristic p (where p is a prime or zero.)

A zero dimensional ring is called *simple of level one* if it has a unit, only one prime ideal of level one, and no prime ideals of higher level. A multiplication ring is called *simple of level one* if its total quotient ring is simple of level one. If, in addition, it has only one idempotent prime ideal it is called *special simple of level one.*

LEMMA 20. *Let A be a zero dimensional multiplication ring which is simple of level one; then every element is the sum of a unit and an element in*

$$F = \bigoplus_{Q \in \Phi_0} A/Q.$$

Proof. Let M be the prime ideal of level one. Since $M = M^2$ is the only prime ideal containing F, $F = M$. Let $a \in A \backslash F$; then $(a) + F = A$, so $ba + f = 1$.

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f is in all but a finite number of prime ideals so ba and hence a belong to only a finite number of prime ideals. Thus for some $d \in F$, $a + d$ is contained in no prime ideals and hence is a unit.

The above lemma suggests analogies to representation theorems for complete local rings. However results in this direction are limited to the following: if the characteristic of the field A/F is $p \neq 0$, then $A = A_1 \oplus D$ where A_1 contains a subfield of characteristic p and D is a direct sum of a finite number of $SPIR$'s (for p in A maps to zero in A/F, so $p + d = 0$ where $d \in F$). The next example shows that such a subfield need not exist in the zero characteristic case and that little improvement can be made to the secondary corollary to Proposition 17.

Example 5. Let $\{p_i, i = 1, 2, \ldots\}$ be the set of prime numbers. For each i let $n_{i,j}$, $j = 1, 2, \ldots, s_i$ be integers. Let Z^* be the canonical image of the integers in the ring

$$R = \prod_{i=1}^{\infty} \left(\bigoplus_{1 \leqq j \leqq s_i} Z/P_i^{n_{i,j}} \right).$$

Let F be the subring consisting of elements with all but a finite number of components zero, and let A be the total quotient ring of the subring of R generated by Z^* and F. It is not difficult to check that A is a zero dimensional multiplication ring which is special simple of level one and A/F is isomorphic to the rationals, but A has no direct summand with a subfield of characteristic zero.

The same type of construction but with $k[X]$ in place of Z shows that even if A/F is not of zero characteristic we cannot choose a subfield isomorphic to A/F.

LEMMA 21. *Let A be a zero dimensional ring with at most a countable number of primes of level one, and no primes of higher level; then A is a direct sum of $SPIR$'s and rings which are simple of level one.*

Proof. Let $P_1, P_2, \ldots$ be the prime ideals of level one. For each i, $i = 1, 2, 3, \ldots$ choose an idempotent e_i' such that $e_i' \notin P_i$ but $e_i' \in P_j$, $j \neq i$, $j = 1, 2, 3, \ldots$. Define e_i inductively as follows:

$$e_1 = e_1', \quad e_i = e_i' - e_i' \left(\sum_{1 \leqq j \leqq i} e_j \right),$$

so that $e_i e_j = 0$ if $i \neq j$. Let H be the family of prime ideals containing $\bigoplus_{i=1}^{\infty} A e_i$. If $P \in H$ then P has level zero, so there exists $f = f^2 \notin P$ but contained in all other prime ideals. These idempotents form a family J. Since $e_i \in \bigoplus A e_i \in P$, $e_i f$ is an idempotent in all prime ideals; so is zero. Then

$$A = \bigoplus_{i=1}^{\infty} A e_i \oplus \left(\bigoplus_{f \in J} A f \right),$$

for this direct sum is contained in no prime ideal of A. Ae_i is simple of level one and Af is $SPIR$.

Note that if A has a unit the direct sum must be finite.

LEMMA 22. *Let K_i, $i \in I$ be rings generated by idempotents. Let $F = \bigoplus_{i\in I} K_i$ and $B = \prod_{i\in I} K_i$. Let R be a ring such that $F \subseteq R \subseteq B$. Let Ω be a family of valuations v on R defining a ring A, such that $F \not\subseteq v^{-1}(\infty)$. Let M_i denote the elements of F with ith component zero. Let $\Omega_i = \{v \in \Omega | v^{-1}(\infty) \supseteq M_i\}$ and let A_i be defined by the restriction of valuations in Ω_i to K_i. Then*

$$A = R \cap \left(\prod_{i\in I} A_i\right)$$

and if $R = \bigoplus_{i\in I} K_i$ then $A = \bigoplus_{i\in I} A_i$.

Proof. Since $v^{-1}(\infty) \not\supseteq F$ there exists $e \in K_i$ and $e_i \in R$ with ith component e and all other components zero such that $v(e_i) \neq \infty$. Let P_i denote the elements of R with ith component zero. If $f \in P_i$ then $e_i f = 0$ so $v(f) = \infty$ and $v^{-1}(\infty) \supseteq P_i \supseteq M_i$ so $v \in \Omega_i$ and $\Omega = \bigcup_{i\in I} \Omega_i$. Let $a \in R$. For each $i \in I$ let a_i denote the element of R with the ith component equal to the ith component of a and all other components zero. Then $a \notin \prod A_i$ if and only if for some i, $a_i \notin A_i$ so $v(a_i) < 0$ for some $v \in \Omega_i$, and $a - a_i \in P_i \subseteq v^{-1}(\infty)$ so $v(a) = v(a_i) < 0$ and $a \notin A$. Since $A \subseteq R$ it follows that $A = R \cap (\prod A_i)$.

Let A be a simple level one multiplication ring which is defined as a subring of its total quotient ring R by a family of valuations Ω. Let K_i, $i \in I$ be the $SPIR$'s associated with primes of level zero so that

$$\bigoplus_{i\in I} K_i = F \subseteq R \subseteq \prod_{i\in I} K_i.$$

Let $\Omega_\infty = \{w \in \Omega | w^{-1}(\infty) = F\}$ and let Ω_i and A_i be as in the preceding lemma. Let $B = \bigcap_{w\in\Omega_\infty} A_w$. By the lemma $A = B \cap (\prod_{i\in I} A_i)$. We make the following additional observations:

(i) Ω_∞ gives rise to a Dedekind domain on R/F.

(ii) A_i is either a Dedekind domain or an $SPIR$.

(iii) Let $a \in R \backslash F$. There is a finite set $S \subseteq I$ such that $w(a) = \infty$ if and only if $w \in \Omega_i$ with $i \in S$. $w(a) = 0$ for all but a finite number of the remaining valuations.

(iv) If A is a special simple level one then $A = B$.

The following result is now deduced easily from the preceding lemmas.

PROPOSITION 23. *Let A be a multiplication ring with total quotient ring K. If K has at most a countable number of prime ideals of level one and no prime ideals of higher level, then A is a direct sum of simple level one multiplication rings, Dedekind domains and SPIR's. If A has only a finite number of idempotent prime ideals then it is a direct sum of SPIR's and a finite number of Dedekind domains and special simple level one multiplication rings. If A has all its primes of* *level zero then it is a direct sum of Dedekind domains and SPIR's. If A has no idempotent prime ideals then it is a direct sum of SPIR's none of which are fields. The existence of an identity in A causes all the direct sums involved to be finite.*

PROPOSITION 24. *Let A be a multiplication ring with total quotient ring a direct sum $K_1 \oplus K_2$ where K_2 has no nilpotent elements. Then $A = A_1 \oplus A_2$ where A_1 has quotient ring K_1 and A_2 has no nilpotents. In particular a multiplication with only a finite number of non-idempotent minimal prime ideals is a direct sum of a multiplication ring without nilpotents and a finite number of SPIR's.*

Proof. This follows at once from Lemma 22.

There exist multiplication rings with idempotent primes such that $A_2 = 0$ in any decomposition with the above properties, e.g. the last part of Example 4.

A complete study of multiplication rings seems to require the use of ringed spaces generalizing the work of Pierce [**8**]. The ringed spaces involved are rather special for if (X, R) is a ringed space corresponding to a zero dimensional multiplication ring, the only stalks which are not fields are *SPIR*'s and each *SPIR* lies over a point of X which is both open and closed.

6. Further results on multiplication rings. Our study of the total quotient rings of multiplication rings enables us to strengthen certain results. The first shows that given any element of a multiplication ring there is a smallest idempotent which fixes it.

PROPOSITION 25. *Let x be any element of a multiplication ring A. There exist orthogonal idempotents f and h such that $(f + h)x = x$, fx is f-regular, hx is nilpotent, and if y is an idempotent such that $yx = x$ then $y(f + h) = f + h$; f and h are unique.*

Proof. Let $x = fx + q$ be the idempotent-nilpotent decomposition. To each nonzero component of q there corresponds a primitive idempotent. Let h be the (finite) sum of these primitive idempotents, so $q = hx$. If $e = e^2$ then $ehx = 0$ implies $eh = 0$. In particular $fh = 0$. Let $g = f + h - (f + h)y$. $0 = gx = fgx$, so since x is f-regular, $gf = 0$. Also $g = g^2$ and $hgx = 0$ so $hg = 0$. Thus $0 = (f + h)g = (f + h) - (f + h)y$, implying $y(f + h) = f + h$. The uniqueness now follows immediately.

Note that the above result does not hold if y is not idempotent (take $Z/(4)$ with $x = 2$, $h = 1$, $y = 3$).

The following is a generalization of an earlier characterization of the non-idempotent primary components of a multiplication ring, Theorem 6.

LEMMA 26. *If Q is an isolated primary component of the ideal M belonging to a prime ideal P such that $P \neq P^2$ or P is of level zero, then there exists $a \in A \backslash P$ such that $(M:a) = Q$.*

Proof. This has already been proved except in the case that $Q = P^n = P^{n+1}$. Thus P has level zero and there is a primitive idempotent e in the total quotient ring belonging to all primes but the one corresponding to P. Since all valuations must be infinite or zero at e, $e \in A$. $P^n e = 0$ so $(M:e) \supseteq P^n$ and if $ex \in M \subseteq P^n$, then $x \in P^n$ so $(M:e) \subseteq P^n$. Thus $(M:e) = Q$.

Note. Every primary component in a multiplication ring is not of the form $(M:a)$. Consider the prime $M = P_{1/2}$ of Example 10. If $P_{1/2} = (0:f)$ then $f(x) = 0$ if $x \neq 1/2$ and $f(1/2) = 1$. Such a function is not in the ring.

LEMMA 27. *Let Q_i, $i \in I$ be a family of primary ideals. If Q is a primary component of $M = \bigcap_{i \in I} Q_i$ then $Q = Q_i$ or $Q = Q^2$ is prime.*

Proof. Let P_i denote the prime ideal corresponding to Q_i, $i \in I$. Suppose $Q = P^n$ with $P \neq P^2$, and n such that $P^n \neq P^{n-1}$.

By the previous lemma there exists $a \in A \backslash P$ such that $(M:a) = P^n$, so that for each $i \in I$, $aP^n \subseteq M \subseteq Q_i$. Let $J = \{i \in I | P = P_i\}$. If $Q_i = P^n$ for some $i \in J$ the proof is finished; otherwise there is a maximum integer $m < n$ such that $Q_i = P^m$, taking $m = 0$ if $J = \phi$. If $P \neq P_i$ then there exists $b \in P^n$ such that $b \notin P_i$ (for P is maximal since $P \neq P^2$), and it follows from $ab \in M \subseteq Q_i$ that $a \in Q_i$. Thus $aP^m \subseteq \bigcap Q_i = M$, so that $(M:a) = P^m \neq P^n$, a contradiction. The result follows.

LEMMA 28. *Let F_i, $1 \leqq i \leqq s$ be a finite partition of the set of primary components of an ideal M of a multiplication ring A. Let*

$$M_i = \bigcap_{Q \in F_i} Q, \quad 1 \leqslant i \leqslant s.$$

Then $M = M_1 M_2 \dots M_s$.

Proof. Clearly $B = M_1 M_2 \dots M_s \subseteq M$. Suppose $B \neq M$; then there exists a prime ideal P such that $B \subseteq P^n$, $M \nsubseteq P^n$. Since $B \subseteq P$, $M_i \subseteq P$ for some i, so $M \subseteq M_i \subseteq P$. It follows that $P \neq P^2$, and M has a P-primary component P^m, $m < n$. By the previous lemma only one M_i, say M_1, can be contained in P and this must have primary component P^m. Thus there exists $a \notin P$ such that $(a:M_1) = P^m$, i.e., $P^m a \subseteq M_1$ so $P^m a M_2 \dots M_s \subseteq B \subseteq P^n$, and $aM_2 \dots M_2 \nsubseteq P$, so $P^m \subseteq P^n$, a contradiction.

LEMMA 29. *Let A be a multiplication ring with total quotient ring K. For each $i \in I$ let Q_i be a P_i-primary ideal of A with no P_i regular. Let $M = \bigcap_{i \in I} Q_i$; then $M = MK \cap A$.*

Proof. Clearly $M \subseteq MK \cap A$. If P^n is a P-primary component of M then by Lemma 27 either $P = P^2$ or $P^n = Q_i$; in either case P is not regular. By the one-to-one correspondence between P-primary ideals of A and K, $KP^n \cap A = P^n$ so $MK \cap A \subseteq KP^n \cap A = P^n$; thus $MK \cap A \subseteq M$.

PROPOSITION 30. *Any ideal M of a multiplication ring A may be written*

$M = M'I + Q$ where $I \cap Q = 0$, $I^2 = I$, $Q \subseteq N$ and M' is the intersection of the regular primary components of M.

Proof. Let K be the total quotient ring of A. Let D be the intersection of the non-regular primary components of M. By the previous lemmas $M = M'D = M'(DK \cap A)$. Now by Proposition 19, $DK = I' + Q$ with $I' = I'^2$, $Q \subseteq N$ and $I' \cap Q = 0$. Let $I = I' \cap A$. Then

$$M = M'((I' + Q) \cap A) = M'(I + Q) = M'I + Q.$$

The following example shows that the M' of the above proposition need not be regular.

Example 6. Let K be the zero dimensional multiplication ring formed by the construction of Example 4, using a countable number of copies of the rationals $\mathbf{Q}$. Let $F = \oplus_{i=1}^{\infty}\mathbf{Q}_i$. Let p_i, $i = 1, 2, 3, \ldots$ be the prime numbers. Define a valuation v_i on K by lifting the p_i-adic valuation on the ith component. It is easily checked that this family of valuations defines a multiplication ring A. Let P_i be the prime ideal of A corresponding to v_i.

$$Q = \bigcap_{i=1}^{\infty} P_i \subseteq F,$$

and it is easily checked that the representation given by the above proposition is $Q = QF$.

PROPOSITION 31. *Any finitely generated ideal in a multiplication ring is generated by two elements.*

Proof. Let A be the multiplication ring with total quotient ring K. Let $M = (a_1, a_2, \ldots, a_n)$. Let g be an idempotent such that $ga_i = a_i, i = 1,2,\ldots,n$. We may restrict ourselves to the case of rings with unit by considering the ring Ag. Let $M = M'I + Q$ be the decomposition given by the previous proposition. $(Q + I)K$ is finitely generated ideal which, since K is a Bezout ring, is principal; it is generated by $e + q$ where e is an idempotent and q is a nilpotent with $eq = 0$. Since $e + q \in A$, $I + Q = (e + q)$.

Let P^m be a regular primary ideal containing M. Let D be the idempotent prime ideal contained in P. Since $a_i \in P^m$, $i = 1, 2, \ldots, n$ and $M \nsubseteq D$, there is some a_i with a regular primary component a power of P. Since each a_i has only a finite number of regular primary components $M' = P_1^{m_1}P_2^{m_2}\ldots P_s^{m_s}$. Let $v_1, v_2, \ldots, v_s$ be the valuations of K defined by $P_1, \ldots, P_s$. Let b be any regular element M' and let $v_{s+1}, \ldots, v_t$ be the valuations belonging to regular primes not containing M' at which $v(b) > 0$. By the approximation theorem on the Prufer ring A [**3**], there exists $c \in A$ such that $v_i(c) = m_i$, $1 \leqq i \leqq s$ and $v_i(c) = 0$, $i = s + 1, \ldots, t$ (since the valuations are rank one they are independent). Thus $(b, c) = M'$ so $M = (b, c)(e) + (q) = (be + q, ce)$.

THEOREM 32. *A commutative ring A is a multiplication ring if and only if the following three conditions hold:*

(i) *A is generated by idempotents;*

(ii) *A is a Dedekind ring;*

(iii) *if P is any non-regular prime ideal then either P is maximal, A_P is a SPIR and there exists an idempotent contained in all prime ideals of A except P, or A_P is a field.*

Proof. Let K be the total quotient ring of A. By the introduction the prime ideals of K are of the form KP where P is a non-regular prime ideal of A and $A_P \cong K_{KP}$.

$\Rightarrow$ (i) and (ii) follow from the corollary to Lemma 7 and Theorem 13 respectively. If P is not regular and A_P is not a field then by Theorem 15, $A_P \cong K_{KP}$ is a *SPIR*, $P \neq P^2$ and there exists $a \in K$ contained in all prime ideals of K but KP. Let e be the corresponding primitive idempotent; then $e = e^2 \in A$ but $e \notin P$.

$\Leftarrow$ K is a multiplication ring by Theorem 15 (since $K_{KP} = A_P$). Let P be a prime ideal of A. If P is regular then P is maximal (A is Dedekind). Thus if P is not maximal then $A_P = K_{KP}$ is a field and it follows that $KP = (KP)^2$. The result now follows by the modified version of Theorem 13.

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