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Small Prime Solutions of Quadratic Equations

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Abstract. Let $b_1, \ldots, b_5$ be non-zero integers and n any integer. Suppose that $b_1 + \cdots + b_5 \equiv n \pmod{24}$ and $(b_i, b_j) = 1$ for $1 \le i < j \le 5$. In this paper we prove that

(i) if all b_j are positive and $n \gg \max\{|b_j|\}^{41+\varepsilon}$, then the quadratic equation $b_1 p_1^2 + \cdots + b_5 p_5^2 = n$ is soluble in primes p_j, and

(ii) if b_j are not all of the same sign, then the above quadratic equation has prime solutions satisfying $p_j \ll \sqrt{|n|} + \max\{|b_j|\}^{20+\varepsilon}$.

1 Introduction

For any integer n, we consider quadratic equations in the form

$$b_1 p_1^2 + \cdots + b_5 p_5^2 = n, \tag{1.1}$$

where p_j are prime variables and the coefficients b_j are non-zero integers. A necessary condition for the solubility of (1.1) is

$$b_1 + \cdots + b_5 \equiv n \pmod{24}. \tag{1.2}$$

We also suppose

$$(b_i, b_j) = 1, \quad 1 \le i < j \le 5, \tag{1.3}$$

and write $B = \max\{2, |b_1|, \ldots, |b_5|\}$. The main results in this paper are the following two theorems.

Theorem 1 *Suppose* (1.2) *and* (1.3)*. If $b_1, \ldots, b_5$ are not all of the same sign, then* (1.1) *has solutions in primes p_j satisfying*

$$p_j \ll \sqrt{|n|} + B^{20+\varepsilon},$$

where the implied constant depends only on ε.

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Theorem 2 *Suppose* (1.2) *and* (1.3). *If* $b_1, \dots, b_5$ *are all positive, then* (1.1) *is soluble whenever*

$$n \gg B^{41+\varepsilon},$$

where the implied constant depends only on ε.

Theorem 2 with $b_1 = \cdots = b_5 = 1$ is a classical result of Hua [7] in 1938. The quadratic equation (1.1) in general was first studied by M. C. Liu and Tsang [13], who obtained a qualitative bound B^A, in the place of $B^{20+\varepsilon}$ and $B^{41+\varepsilon}$ in Theorems 1 and 2 above, without the explicit values of the constant A.

Our investigation on (1.1) is not only motivated by [7] and [13], but also by the following work on small prime solutions of the equation

$$b_1 p_1 + b_2 p_2 + b_3 p_3 = n, \tag{1.4}$$

where b_1, b_2, b_3, n are non-zero integers satisfying some necessary conditions. This problem was first raised and investigated by Baker in his well-known work [1], and was later settled qualitatively by M. C. Liu and Tsang [12]. In this problem, the constant A corresponding to the 20 in our Theorem 1 is called Baker's constant. The first author [3] proved that Baker's constant is ≤ 4190, and M. C. Liu and Wang [14] improved this to 45.

We prove our theorems by the circle method, and the idea will be explained in Section 2. At this stage, we only point out that in contrast to the earlier works [3], [12], [13], [14] which treat the enlarged major arc by the Deuring-Heilbronn phenomenon, we show that in the context of this paper, the possible existence of Siegel zero does not have special influence and hence the Deuring-Heilbronn phenomenon can be avoided. This observation enables us to get better results without numerical computations.

Notation As usual, $\varphi(n)$, $\mu(n)$, and $\Lambda(n)$ stand for the functions of Euler, Möbius, and von Mangoldt respectively, $d(n)$ is the divisor function. We use $\chi \bmod q$ and $\chi^0 \bmod q$ to denote a Dirichlet character and the principal character modulo q, and $L(s,\chi)$ is the Dirichlet L-function. $r \sim R$ means $R < r \leq 2R$. The letters c and c_j denote absolute positive constants, but the value of c without subscript may vary at different places. The letter ε denotes a positive constant which is arbitrarily small.

In mathematical formulae, we will write "s.t." for "similar terms". For example, "$A_1B_2C_3D_4E_5+$ s.t." means the sum of all possible terms $A_\alpha B_\beta C_\gamma D_\delta E_\iota$ with $(\alpha, \dots, \iota)$ being any permutation of $(1, \dots, 5)$.

2 Outline of the Method

Denote by $r(n)$ the weighted number of solutions of (1.1), *i.e.*,

$$r(n) = \sum_{\substack{n = b_1 p_1^2 + \cdots + b_5 p_5^2 \\ M < |b_j| p_j^2 \leq N}} (\log p_1) \cdots (\log p_5),$$

where $M = N/200$. We will investigate $r(n)$ by the circle method. To this end, we set

$$P = (N/B)^{1/8-\varepsilon}, \quad Q = N/(PL^{9000}), \quad \text{and} \quad L = \log N. \tag{2.1}$$

By Dirichlet's lemma on rational approximation, each $\alpha \in [1/Q, 1 + 1/Q]$ may be written in the form

$$\alpha = a/q + \lambda, \quad |\lambda| \le 1/(qQ) \tag{2.2}$$

for some integers a, q with $1 \le a \le q \le Q$ and $(a, q) = 1$. We denote by $\mathfrak{M}(a, q)$ the set of α satisfying (2.2), and define the major arcs $\mathfrak{M}$ and the minor arcs $\mathfrak{m}$ as follows:

$$\mathfrak{M} = \bigcup_{q\le P} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \mathfrak{M}(a, q), \quad \mathfrak{m} = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \setminus \mathfrak{M}. \tag{2.3}$$

It follows from $2P \le Q$ that the major arcs $\mathfrak{M}(a, q)$ are mutually disjoint. Let

$$S_j(\alpha) = \sum_{M<|b_j|p^2\le N} (\log p)e(b_j p^2\alpha).$$

Then we have

$$r(n) = \int_0^1 S_1(\alpha)\cdots S_5(\alpha)e(-n\alpha)\,d\alpha = \int_{\mathfrak{M}} + \int_{\mathfrak{m}}. \tag{2.4}$$

The integral on the major arcs $\mathfrak{M}$ causes the main difficulty, which is solved by the following:

Theorem 3 *Assume* (1.3). *Let* $\mathfrak{M}$ *be as in* (2.3) *with* P, Q *determined by* (2.1). *Then we have*

$$\int_{\mathfrak{M}} S_1(\alpha)\cdots S_5(\alpha)e(-n\alpha)\,d\alpha = \frac{1}{32}\mathfrak{S}(n, P)\mathfrak{J}(n) + O\left(\frac{N^{3/2}}{|b_1\cdots b_5|^{1/2}L}\right), \tag{2.5}$$

where $\mathfrak{S}(n, P)$ *and* $\mathfrak{J}(n)$ *are defined in* (2.6) *and* (2.7) *respectively.*

The proof of this theorem forms the bulk of this paper, Sections 3–6. From (2.1) one sees that our major arcs are quite large. Historically, enlarged major arcs are treated by the Deuring-Heilbronn phenomenon. But here we observe that under the assumption (1.3), we can save the factor $B^{5/2}$ in Lemma 3.1 below (in Lemma 3.8 in [13], there is an extra factor of $B^{5/2}$ on the right-hand side). With this saving, (2.5) can be derived from the large sieve inequality, Gallagher's lemma and classical results on the distribution of zeros of L-functions. This approach has also been used by Bauer, M. C. Liu, and Zhan [2], and by M. C. Liu, Zhan, and the second author [10], [11].

To derive Theorems 1 and 2 from Theorem 3, we need to bound $\mathfrak{S}(n,P)$ and $\mathfrak{I}(n)$ from below. For χ mod q, we define

$$C(\chi,a)=\sum_{h=1}^{q}\bar{\chi}(h)e\Big(\frac{ah^2}{q}\Big),\quad C(q,a)=C(\chi^0,a).$$

If $\chi_1,\dots,\chi_5$ are characters mod$q$, then we write

$$B(n,q,\chi_1,\dots,\chi_5)=\sum_{\substack{h=1\\(h,q)=1}}^{q}e\Big(-\frac{hn}{q}\Big)C(\chi_1,b_1h)\cdots C(\chi_5,b_5h),$$

$$B(n,q)=B(n,q,\chi^0,\dots,\chi^0),\quad A(n,q)=\frac{B(n,q)}{\varphi^5(q)},\quad \mathfrak{S}(n,x)=\sum_{q\le x}A(n,q). \tag{2.6}$$

Lemma 2.1 *Assuming* (1.2)*, we have* $\mathfrak{S}(n,P)\gg(\log\log B)^{-c_1}$ *for some constant* $c_1>0$.

Lemma 2.2 *Suppose* (1.3) *and*

(i) b_j*'s are not all of the same sign and* $N\ge 10|n|$*; or*
(ii) all b_j*'s are positive and* $n=N$.

Then we have

$$\mathfrak{I}(n):=\sum_{\substack{b_1m_1+\cdots+b_5m_5=n\\ M<|b_j|m_j\le N}}(m_1\cdots m_5)^{-1/2}\asymp\frac{N^{3/2}}{|b_1\cdots b_5|^{1/2}}. \tag{2.7}$$

The proofs of these two lemmas will be given in Section 7.
We now derive Theorems 1 and 2 from Theorem 3 and Lemmas 2.1 and 2.2.

Proofs of Theorems 1 and 2 We start from (2.4) and let $N_j=N/|b_j|$. To estimate the integral on $\mathfrak{m}$, one appeals to the estimate on p. 151 in [13]:

$$\begin{aligned}S_5(\alpha)&\ll N_5^{1/2+\varepsilon}(|b_5|P^{-1}+N_5^{-1/4}+QN_5^{-1})^{1/4}\\&\ll N_5^{1/2+\varepsilon}(|b_5|/P)^{1/4}\ll N^{1/2+\varepsilon}(|b_5|P)^{-1/4}.\end{aligned} \tag{2.8}$$

Also, we have the following mean-value estimate for $S_j(\alpha)$:

$$\int_0^1|S_j(\alpha)|^4\,d\alpha\ll L^4\sum_{\substack{m_1^2+m_2^2=m_3^2+m_4^2\\ m_\nu^2\le N_j,\nu=1,\dots,4}}1\ll N_j^{1+\varepsilon},$$

which in combination with Hölder's inequality gives

$$\int_0^1|S_1(\alpha)\cdots S_4(\alpha)|\,d\alpha\ll\frac{N^{1+\varepsilon}}{|b_1\cdots b_4|^{1/4}}. \tag{2.9}$$

It therefore follows from (2.8) and (2.9) that

$$\left|\int_{\mathfrak{m}}\right| \ll \frac{N^{3/2+\varepsilon}}{|b_1\cdots b_5|^{1/4}P^{1/4}}. \tag{2.10}$$

The contribution from the major arcs can be handled by Theorem 3, which together with (2.10) gives

$$r(n) = \frac{1}{32}\mathfrak{S}(n,P)\mathfrak{I}(n) + O\biggl(\frac{N^{3/2}}{|b_1\cdots b_5|^{1/2}L} + \frac{N^{3/2+\varepsilon}}{|b_1\cdots b_5|^{1/4}P^{1/4}}\biggr).$$

Now assume the conditions (i) or (ii) in Lemma 2.2. Applying Lemmas 2.1 and 2.2 to the above formula, we conclude

$$r(n) \gg |b_1\cdots b_5|^{-1/2}N^{3/2}(\log\log B)^{-c_1}$$

provided that $P \gg N^{\varepsilon}|b_1\cdots b_5|$, or equivalently $N \gg B^{1+\varepsilon}|b_1\cdots b_5|^8$. This proves Theorems 1 and 2.

3 An Explicit Expression

In this section, we establish an explicit expression for the integral in Theorem 3 (see Lemma 3.2 below), from which and the estimates in Sections 4–6 we can derive Theorem 3 at the end of Section 6.

Lemma 3.1 *Let $\chi_j \bmod r_j$ with $j = 1,\ldots,5$ be primitive characters, $r_0 = [r_1,\ldots,r_5]$, and χ^0 the principal character* $\bmod q$. *Then*

$$\sum_{\substack{q\le x\\ r_0|q}} \frac{1}{\varphi^5(q)}|B(n,q,\chi_1\chi^0,\ldots,\chi_5\chi^0)| \ll r_0^{-1+\varepsilon}\log^{2^{15}} x.$$

Proof Lemma 3.1(c) of [13] asserts that for any character $\chi \bmod p^{\alpha}$ with $\alpha \ge 0$,

$$|C(\chi,a)| \le 2(2,p)(a,p^{\alpha})^{1/2}p^{\alpha/2}.$$

Therefore for characters $\chi_1,\ldots,\chi_5 \bmod p^{\alpha}$,

$$|B(n,p^{\alpha},\chi_1,\ldots,\chi_5)| \le p^{\alpha}\bigl(2(2,p)p^{\alpha/2}\bigr)^5\prod_{j=1}^{5}(b_j,p^{\alpha})^{1/2} \le 2^{10}p^{4\alpha},$$

where in the last inequality we have used the condition (1.3); in fact $\prod_{j=1}^{5}(b_j,p^{\alpha})^{1/2} \le p^{\alpha/2}$. Since for $\chi_1,\ldots,\chi_5 \bmod q$, the function $|B(n,q,\chi_1,\ldots,\chi_5)|$ is multiplicative with respect to q (in the sense as Lemma 3.2 in [13]), we have

$$|B(n,q,\chi_1,\ldots,\chi_5)| \le q^4 2^{10\omega(q)} \le q^4 d^{10}(q),$$

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where $\omega(q)$ denotes the number of distinct prime divisors of q. Thus, for the characters in the lemma, we have

$$\sum_{\substack{q\le x\\ r_0|q}} \frac{1}{\varphi^5(q)}|B(n,q,\chi_1\chi^0,\dots,\chi_5\chi^0)| \ll \sum_{\substack{q\le x\\ r_0|q}} \frac{q^4 d^{10}(q)}{\varphi^5(q)} \ll r_0^{-1+\varepsilon}\sum_{q\le x}\frac{d^{15}(q)}{q}.$$

The desired result now follows from Lemma 2.4 in [8]. ∎

For $j=1,\dots,5$, recall $N_j = N/|b_j|$, and set

$$M_j = M/|b_j|, \quad V_j(\lambda) = \sum_{M<|b_j|m^2\le N} e(b_j m^2\lambda),$$

and

$$W_j(\chi,\lambda) = \sum_{M<|b_j|p^2\le N} (\log p)\chi(p)e(b_j p^2\lambda) - \delta_\chi \sum_{M<|b_j|m^2\le N} e(b_j m^2\lambda), \tag{3.1}$$

where $\delta_\chi = 1$ or 0 according as χ is principal or not. Also, define

$$J_j = \sum_{r\le P} r^{-1/5+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \max_{|\lambda|\le 1/(rQ)} |W_j(\chi,\lambda)|,$$

and

$$K_j = \sum_{r\le P} r^{-1/5+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \left(\int_{-1/(rQ)}^{1/(rQ)} |W_j(\chi,\lambda)|^2\, d\lambda\right)^{1/2},$$

where $\sum_{\chi \bmod r}{}^*$ is over all the primitive characters modulo r.

Now we state the main result of this section.

Lemma 3.2 *Let $\mathfrak{M}$ be as in* (2.3). *Then*

$$\begin{aligned}
\int_{\mathfrak{M}} S_1(\alpha)\cdots S_5(\alpha)e(-n\alpha)\,d\alpha - \frac{1}{32}\mathfrak{S}(n,P)\mathfrak{J}(n)
&\ll (J_1J_2J_3K_4K_5)L^{c_2} + (J_1J_2J_3K_4|b_5|^{-1/2} + \text{s.t.})L^{c_2}\\
&\quad + (J_1J_2J_3|b_4|^{-1/2}|b_5|^{-1/2} + \text{s.t.})L^{c_2}\\
&\quad + (J_1J_2N_3^{1/2}|b_4|^{-1/2}|b_5|^{-1/2} + \text{s.t.})L^{c_2}\\
&\quad + (J_1N_2^{1/2}N_3^{1/2}|b_4|^{-1/2}|b_5|^{-1/2} + \text{s.t.})L^{c_2}\\
&\quad + |b_1\cdots b_5|^{-1/2}N^{3/2}L^{-1},
\end{aligned}$$

where $c_2 = 2^{15}+1$ and "s.t." means similar terms as explained at the end of Section 1.

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Proof Introducing Dirichlet characters, we can rewrite the exponential sum $S_j(\alpha)$ as (see for example [4, Section 26, (2)])

$$S_j\Big(\frac{h}{q}+\lambda\Big) = \frac{C(q,b_jh)}{\varphi(q)}V_j(\lambda) + \frac{1}{\varphi(q)}\sum_{\chi \bmod q} C(\chi,b_jh)W_j(\chi,\lambda) =: T_j + U_j,$$

say. Thus,

$$\int_{\mathfrak{M}} S_1(\alpha)\cdots S_5(\alpha)e(-n\alpha)\,d\alpha = I_0+\cdots+I_5, \tag{3.2}$$

where I_ν denotes the contribution from those products with ν pieces of U_j and $5-\nu$ pieces of T_j, *i.e.*,

$$I_\nu = \sum_{q\le P}\sum_{\substack{h=1\\(h,q)=1}}^{q} e\Big(-\frac{hn}{q}\Big)\int_{-1/(qQ)}^{1/(qQ)}(U_1\cdots U_\nu T_{\nu+1}\cdots T_5 + \text{s.t.})e(-n\lambda)\,d\lambda.$$

We will prove that I_0 gives the main term and $I_1,\ldots,I_5$ the error term.

We begin with I_5. Reducing the characters in I_5 into primitive characters, we have

$$\begin{aligned}|I_5| &= \Bigg|\sum_{q\le P}\sum_{\chi_1 \bmod q}\cdots\sum_{\chi_5 \bmod q}\\ &\qquad\frac{B(n,q,\chi_1,\ldots,\chi_5)}{\varphi^5(q)}\int_{-1/(qQ)}^{1/(qQ)}W_1(\chi_1,\lambda)\cdots W_5(\chi_5,\lambda)e(-n\lambda)\,d\lambda\Bigg|\\ &\le \sum_{r_1\le P}\cdots\sum_{r_5\le P}\sideset{}{^*}\sum_{\chi_1 \bmod r_1}\cdots\sideset{}{^*}\sum_{\chi_5 \bmod r_5}\sum_{\substack{q\le P\\ r_0|q}}\frac{|B(n,q,\chi_1\chi^0,\ldots,\chi_5\chi^0)|}{\varphi^5(q)}\\ &\qquad\times\int_{-1/(qQ)}^{1/(qQ)}|W_1(\chi_1\chi^0,\lambda)|\cdots|W_5(\chi_5\chi^0,\lambda)|\,d\lambda,\end{aligned}$$

where χ^0 is the principal character modulo q and $r_0=[r_1,\ldots,r_5]$. For $q\le P$ and $M<|b_j|p^2\le N$, we have $(q,p)=1$. Using this and (3.1), we have $W_j(\chi_j\chi^0,\lambda)=W_j(\chi_j,\lambda)$ for the primitive characters χ_j above. Consequently by Lemma 3.1, we obtain

$$\begin{aligned}
|I_5| &\le \sum_{r_1\le P}\cdots\sum_{r_5\le P}\sum_{\chi_1 \bmod r_1}{}^{*}\cdots\sum_{\chi_5 \bmod r_5}{}^{*}\int_{-1/(r_0Q)}^{1/(r_0Q)}|W_1(\chi_1,\lambda)|\cdots|W_5(\chi_5,\lambda)|\,d\lambda\\
&\quad\times\sum_{\substack{q\le P\\ r_0|q}}\frac{|B(n,q,\chi_1\chi^0,\dots,\chi_5\chi^0)|}{\varphi^5(q)}\\
&\ll L^{c_2}\sum_{r_1\le P}\cdots\sum_{r_5\le P}r_0^{-1+\varepsilon}\sum_{\chi_1 \bmod r_1}{}^{*}\cdots\sum_{\chi_5 \bmod r_5}{}^{*}\\
&\quad\times\int_{-1/(r_0Q)}^{1/(r_0Q)}|W_1(\chi_1,\lambda)|\cdots|W_5(\chi_5,\lambda)|\,d\lambda.
\end{aligned}$$

Applying the inequality $r_0^{-1+\varepsilon}\le r_1^{-1/5+\varepsilon}\cdots r_5^{-1/5+\varepsilon}$ to the above quantity and then using Cauchy's inequality, we get

$$\begin{aligned}
|I_5| &\ll L^{c_2}\Big\{\sum_{r_1\le P}r_1^{-1/5+\varepsilon}\sum_{\chi_1 \bmod r_1}{}^{*}\max_{|\lambda|\le 1/(r_1Q)}|W_1(\chi_1,\lambda)|\Big\}\\
&\qquad\times\cdots\times\Big\{\sum_{r_3\le P}r_3^{-1/5+\varepsilon}\sum_{\chi_3 \bmod r_3}{}^{*}\max_{|\lambda|\le 1/(r_3Q)}|W_3(\chi_3,\lambda)|\Big\}\\
&\qquad\times\Big\{\sum_{r_4\le P}r_4^{-1/5+\varepsilon}\sum_{\chi_4 \bmod r_4}{}^{*}\Big(\int_{-1/(r_4Q)}^{1/(r_4Q)}|W_4(\chi_4,\lambda)|^2\,d\lambda\Big)^{1/2}\Big\}\\
&\qquad\times\Big\{\sum_{r_5\le P}r_5^{-1/5+\varepsilon}\sum_{\chi_5 \bmod r_5}{}^{*}\Big(\int_{-1/(r_5Q)}^{1/(r_5Q)}|W_5(\chi_5,\lambda)|^2\,d\lambda\Big)^{1/2}\Big\}\\
&= J_1J_2J_3K_4K_5L^{c_2}.
\end{aligned}\tag{3.3}$$

To bound $I_4,\dots,I_1$, we need the estimates $V_j(\lambda)\ll N_j^{1/2}$ and

$$\begin{aligned}
H_j^2 := \int_{-1/Q}^{1/Q}|V_j(\lambda)|^2\,d\lambda &= \sum_{M_j^{1/2}<m_1,m_2\le N_j^{1/2}}\int_{-1/Q}^{1/Q}e\big(b_j(m_1^2-m_2^2)\lambda\big)\,d\lambda\\
&\ll \sum_{M_j^{1/2}<m\le N_j^{1/2}}Q^{-1}+|b_j|^{-1}\sum_{M_j^{1/2}<m_1\ne m_2\le N_j^{1/2}}|m_1^2-m_2^2|^{-1}\\
&\ll N_j^{1/2}Q^{-1}+|b_j|^{-1}L^2\ll |b_j|^{-1}L^2.
\end{aligned}$$

Thus similarly to (3.3), we have

$$\left.\begin{aligned}
|I_4| &\ll (J_1J_2J_3K_4H_5+\text{s.t.})L^{c_2}\ll (J_1J_2J_3K_4|b_5|^{-1/2}+\text{s.t.})L^{c_2},\\
|I_3| &\ll (J_1J_2J_3H_4H_5+\text{s.t.})L^{c_2}\ll (J_1J_2J_3|b_4b_5|^{-1/2}+\text{s.t.})L^{c_2},\\
|I_2| &\ll (J_1J_2N_3^{1/2}H_4H_5+\text{s.t.})L^{c_2}\ll (J_1J_2N_3^{1/2}|b_4b_5|^{-1/2}+\text{s.t.})L^{c_2},\\
|I_1| &\ll (J_1N_2^{1/2}N_3^{1/2}H_4H_5+\text{s.t.})L^{c_2}\ll (J_1N_2^{1/2}N_3^{1/2}|b_4b_5|^{-1/2}+\text{s.t.})L^{c_2}.
\end{aligned}\right\}\tag{3.4}$$

It remains to compute I_0. By the partial summation formula,

$$
\begin{aligned}
V_j(\lambda) &= \int_{M_j^{1/2}}^{N_j^{1/2}} e(b_j\lambda u^2)\,du + O(1+|\lambda|N) \\
&= \frac{1}{2}\sum_{M<|b_j|m\le N} m^{-1/2}e(b_jm\lambda) + O(1+|\lambda|N).
\end{aligned} \tag{3.5}
$$

Also we have the following elementary bound:

$$
\begin{aligned}
\sum_{M<|b_j|m\le N} m^{-1/2}e(b_jm\lambda) &\ll \min(N_j^{1/2}, M_j^{-1/2}|b_j\lambda|^{-1}) \\
&\ll |b_j|^{-1/2}\min(N^{1/2}, M^{-1/2}|\lambda|^{-1}).
\end{aligned} \tag{3.6}
$$

Substituting (3.5) into I_0, we have

(3.7)

$$
\begin{aligned}
I_0 = \frac{1}{32}\sum_{q\le P}\frac{B(n,q)}{\varphi^5(q)}\int_{-1/(qQ)}^{1/(qQ)}\prod_{j=1}^{5}\Big\{\sum_{M<|b_j|m\le N}\frac{e(b_jm\lambda)}{m^{1/2}}\Big\}e(-n\lambda)\,d\lambda \\
+ O\Big\{\sum_{q\le P}\frac{|B(n,q)|}{\varphi^5(q)}\int_{-1/(qQ)}^{1/(qQ)}\Big(\prod_{j=1}^{4}\Big|\sum_{M<|b_j|m\le N}\frac{e(b_jm\lambda)}{m^{1/2}}\Big|(1+|\lambda|N)+\text{s.t.}\Big)\,d\lambda\Big\}.
\end{aligned}
$$

By (3.6) and Lemma 3.1 with $r_0 = 1$,

$$
\begin{aligned}
&\sum_{q\le P}\frac{|B(n,q)|}{\varphi^5(q)}\int_{-1/(qQ)}^{1/(qQ)}\Big|\sum_{M<|b_j|m\le N}\frac{e(b_jm\lambda)}{m^{1/2}}\Big|^4(1+|\lambda|N)\,d\lambda \\
&\quad\ll \frac{1}{b_j^2}\sum_{q\le P}\frac{|B(n,q)|}{\varphi^5(q)}\Big\{\int_0^{1/\sqrt{MN}}N^2\,d\lambda + \int_{1/\sqrt{MN}}^{1/Q}M^{-2}N|\lambda|^{-3}\,d\lambda\Big\} \ll \frac{NL^{c_2}}{b_j^2}.
\end{aligned}
$$

So by Hölder's inequality,

$$
\begin{aligned}
\sum_{q\le P}\frac{|B(n,q)|}{\varphi^5(q)}\int_{-1/(qQ)}^{1/(qQ)}\prod_{j=1}^{4}\Big|\sum_{M<|b_j|m\le N}\frac{e(b_jm\lambda)}{m^{1/2}}\Big|(1+|\lambda|N)\,d\lambda &\ll \frac{NL^{c_2}}{|b_1\cdots b_4|^{1/2}} \\
&\ll \frac{N^{3/2}}{|b_1\cdots b_5|^{1/2}L}.
\end{aligned}
$$

The other error terms in (3.7) can be treated similarly and they are $\ll |b_1\cdots b_5|^{-1/2}N^{3/2}L^{-1}$. Now we extend the integral in the main term of (3.7) to $[-1/2, 1/2]$; by Lemma 3.1 and (3.6), the resulting error is

$$
\ll \frac{L^{c_2}}{|b_1\cdots b_5|^{1/2}}\int_{1/(PQ)}^{1/2}M^{-5/2}|\lambda|^{-5}\,d\lambda \ll \frac{(PQ)^4L^{c_2}}{|b_1\cdots b_5|^{1/2}M^{5/2}} \ll \frac{N^{3/2}}{|b_1\cdots b_5|^{1/2}L},
$$

where we have used (2.1). Thus (3.7) becomes

$$I_0 = \frac{1}{32}\mathfrak{S}(n,P)\mathfrak{J}(n) + O\left(\frac{N^{3/2}}{|b_1\cdots b_5|^{1/2}L}\right). \tag{3.8}$$

Therefore, Lemma 3.2 now follows from (3.2), (3.3), (3.4), and (3.8). ■

4 Estimation of J

We have

$$J_j \ll L \max_{R\le P} J_j(R)$$

where $J_j(R)$ is defined similarly to J_j except that the sum is over $r \sim R$. The estimation of $J_j(R)$ falls naturally into two cases according as R is small or large. For $R > L^C$, where C is some positive constant, one appeals to contour integration, mean-value estimates for the Dirichlet L-functions or their derivatives, the large sieve inequality, and Heath-Brown's identity. While for $R \le L^C$, one uses the classical zero-density estimates and zero-free region for the Dirichlet L-functions.

We first establish the following result for large R. In Lemma 4.3 we shall consider small R.

Lemma 4.1 *Let $A > 0$ be arbitrary. Then there exists a constant $C = C(A) > 0$ such that when $L^C < R \le P$,*

$$J_j(R) \ll N_j^{1/2} L^{-A},$$

where the implied constant depends at most on A.

To prove this result, it suffices to show that

$$\sum_{r\sim R}\ \sideset{}{^*}\sum_{\chi \bmod r}\ \max_{|\lambda|\le 1/(rQ)} |W_j(\chi,\lambda)| \ll R^{1/5-\varepsilon} N_j^{1/2} L^{-A} \tag{4.1}$$

holds for $L^C < R \le P$ and arbitrary $A > 0$. Let

$$\hat{W}_j(\chi,\lambda) = \sum_{M<|b_j|m^2\le N} \big(\Lambda(m)\chi(m) - \delta_\chi\big)\, e(b_j m^2 \lambda). \tag{4.2}$$

Then

$$W_j(\chi,\lambda) - \hat{W}_j(\chi,\lambda) = -\sum_{m\ge 2}\ \sum_{M<|b_j|p^{2m}\le N} (\log p)\chi(p) e(b_j p^{2m}\lambda) \ll N_j^{1/4}. \tag{4.3}$$

Thus (4.1) is a consequence of the estimate

$$\sum_{r\sim R}\ \sideset{}{^*}\sum_{\chi \bmod r}\ \max_{|\lambda|\le 1/(rQ)} |\hat{W}_j(\chi,\lambda)| \ll R^{1/5-\varepsilon} N_j^{1/2} L^{-A}, \tag{4.4}$$

where $L^C < R \le P$ and $A > 0$ is arbitrary.

Let $M_j^{1/2} < u \le N_j^{1/2}$, and let $D_1, \dots, D_{10}$ be positive numbers such that

$$2^{-10} M_j^{1/2} \le D_1 \cdots D_{10} < u, \quad \text{and} \quad 2D_6, \dots, 2D_{10} \le u^{1/5}.$$

For $\nu = 1, \dots, 10$ let

$$a_\nu(m) = \begin{cases} \log m & \text{if } \nu = 1; \\ 1 & \text{if } \nu = 2, 3, 4, 5; \\ \mu(m) & \text{if } \nu = 6, 7, 8, 9, 10. \end{cases}$$

We define the following functions of a complex variable s:

$$f_\nu(s) = f_\nu(s, \chi) = \sum_{m \sim D_\nu} \frac{a_\nu(m)\chi(m)}{m^s}, \quad F(s) = F(s, \chi) = f_1(s) \cdots f_{10}(s).$$

Now we recall Heath-Brown's identity (see Lemma 1 in [6]) for $k = 5$, which states that

$$\frac{\zeta'}{\zeta}(s) = \sum_{\nu=1}^{5} \binom{5}{\nu} (-1)^{\nu-1} \zeta'(s) \zeta^{\nu-1}(s) G^\nu(s) + \frac{\zeta'}{\zeta}(s) \big(1 - \zeta(s) G(s) \big)^5,$$

where $\zeta(s)$ is the Riemann zeta-function, and $G(s) = \sum_{m \le u^{1/5}} \mu(m) m^{-s}$. The reason why we choose $k = 5$ is that the identity with $k \le 4$ will give weaker results, and when $k \ge 6$ it produces the same estimate as the case $k = 5$. Equating coefficients of the Dirichlet series on both sides provides an identity for $-\Lambda(m)$. Also, for $m \le u$ the coefficient of m^{-s} in $-(\zeta'/\zeta)(s)\big(1 - \zeta(s)G(s)\big)^5$ is zero. Thus,

$$\Lambda(m) = \sum_{\nu=1}^{5} \binom{5}{\nu} (-1)^{\nu-1} \sum_{\substack{m_1 \cdots m_{2\nu} = m \\ m_{\nu+1}, \dots, m_{2\nu} \le u^{1/5}}} (\log m_1) \mu(m_{\nu+1}) \cdots \mu(m_{2\nu}).$$

Applying this identity to the sum

$$\sum_{M_j^{1/2} < m \le u} \Lambda(m) \chi(m), \tag{4.5}$$

one finds that (4.5) is a linear combination of $O(L^{10})$ terms, each of which is of the form

$$\sigma(u; \mathbf{D}) = \sum_{\substack{m_1 \sim D_1 \\ M_j^{1/2} < m_1 \cdots m_{10} \le u}} \cdots \sum_{m_{10} \sim D_{10}} a_1(m_1)\chi(m_1) \cdots a_{10}(m_{10})\chi(m_{10})$$

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where $\mathbf{D}$ denotes the vector $(D_1, \ldots, D_{10})$. By using Perron's summation formula (see for example, Lemma 3.12 in [16]) and then shifting the contour to the left, the above $\sigma(u; \mathbf{D})$ is

$$\begin{aligned}
&= \frac{1}{2\pi i} \int_{1+1/L-iT}^{1+1/L+iT} F(s, \chi) \frac{u^s - M_j^{s/2}}{s}\, ds + O\Big(\frac{N_j^{1/2} L^2}{T}\Big) \\
&= \frac{1}{2\pi i} \Big\{ \int_{1+1/L-iT}^{1/2-iT} + \int_{1/2-iT}^{1/2+iT} + \int_{1/2+iT}^{1+1/L+iT} \Big\} + O\Big(\frac{N_j^{1/2} L^2}{T}\Big),
\end{aligned}$$

where T is a parameter satisfying $2 \leq T \leq N_j^{1/2}$. The integral on the two horizontal segments above can be easily estimated as

$$\ll \max_{1/2 \leq \sigma \leq 1+1/L} |F(\sigma \pm iT, \chi)| \frac{u^\sigma}{T} \ll \max_{1/2 \leq \sigma \leq 1+1/L} N_j^{(1-\sigma)/2} L \frac{u^\sigma}{T} \ll \frac{N_j^{1/2} L}{T}$$

on using the trivial estimate

$$\begin{aligned}
F(\sigma \pm iT, \chi) &\ll |f_1(\sigma \pm iT, \chi)| \cdots |f_{10}(\sigma \pm iT, \chi)| \\
&\ll (D_1^{1-\sigma} L) D_2^{1-\sigma} \cdots D_{10}^{1-\sigma} \ll N_j^{(1-\sigma)/2} L.
\end{aligned}$$

Thus,

$$\sigma(u; \mathbf{D}) = \frac{1}{2\pi} \int_{-T}^{T} F\Big(\frac{1}{2} + it, \chi\Big) \frac{u^{\frac{1}{2}+it} - M_j^{\frac{1}{2}(\frac{1}{2}+it)}}{\frac{1}{2} + it}\, dt + O\Big(\frac{N_j^{1/2} L^2}{T}\Big).$$

Since $R > L^C$ (so $\chi \neq \chi^0$), we have in (4.2) that

$$\begin{aligned}
\hat{W}_j(\chi, \lambda) &= \sum_{M < |b_j| m^2 \leq N} \Lambda(m) \chi(m) e(b_j m^2 \lambda) \\
&= \int_{M_j^{1/2}}^{N_j^{1/2}} e(b_j u^2 \lambda)\, d\Big\{ \sum_{M_j^{1/2} < m \leq u} \Lambda(m) \chi(m) \Big\},
\end{aligned}$$

and consequently $\hat{W}(\chi, \lambda)$ is a linear combination $O(L^{10})$ terms, each of which is of the form

$$\begin{aligned}
\int_{M_j^{1/2}}^{N_j^{1/2}} e(b_j u^2 \lambda)\, d\sigma(u; \mathbf{D}) &= \frac{1}{2\pi} \int_{-T}^{T} F\Big(\frac{1}{2} + it, \chi\Big) \int_{M_j^{1/2}}^{N_j^{1/2}} u^{-1/2+it} e(b_j u^2 \lambda)\, du\, dt \\
&\quad + O\Big(\frac{N_j^{1/2} L^2}{T} (1 + |\lambda| N)\Big).
\end{aligned}$$

By taking $T = N_j^{1/2}$ and changing variables in the inner integral, we deduce from the above formulae that

(4.6)
$$|\hat{W}_j(\chi,\lambda)| \ll L^{10} \max_{\mathbf{D}} \left| \int_{-T}^{T} F\left(\frac{1}{2}+it,\chi\right) \int_{M_j}^{N_j} v^{-3/4} e\left(\frac{t}{4\pi}\log v + b_j\lambda v\right) dv\, dt \right| \\ + PL^{9012},$$

where the maximum is taken over all $\mathbf{D} = (D_1, \ldots, D_{10})$. Since

$$\frac{d}{dv}\left(\frac{t}{4\pi}\log v + b_j\lambda v\right) = \frac{t}{4\pi v} + b_j\lambda, \quad \frac{d^2}{dv^2}\left(\frac{t}{4\pi}\log v + b_j\lambda v\right) = -\frac{t}{4\pi v^2},$$

by Lemmas 4.4 and 4.3 in [16], the inner integral in (4.6) can be estimated as

(4.7)
$$\begin{aligned} &\ll M_j^{-3/4} \min\left\{ \frac{N_j}{(|t|+1)^{1/2}}, \frac{N_j}{\min\limits_{M_j<v\le N_j} |t+4\pi b_j\lambda v|} \right\} \\ &\ll \begin{cases} N_j^{1/4}(|t|+1)^{-1/2} & \text{if } |t| \le T_0; \\ N_j^{1/4}|t|^{-1} & \text{if } T_0 < |t| \le T; \end{cases} \end{aligned}$$

where $T_0 = 8\pi N/(RQ)$. Here the choice of T_0 is to ensure that $|t + 4\pi b_j\lambda v| > |t|/2$ whenever $|t| > T_0$; in fact,

$$|t+4\pi b_j\lambda v| \ge |t| - 4\pi|b_j v|/(rQ) > |t|/2 + T_0/2 - 4\pi N/(RQ) = |t|/2.$$

It therefore follows from (4.6) and (4.7) that the lemma (more precisely, (4.4)) is a consequence of the following two estimates: For $0 < T_1 \le T_0$, we have

(4.8)
$$\sum_{r\sim R} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_1}^{2T_1} \left| F\left(\frac{1}{2}+it,\chi\right) \right| dt \ll R^{1/5-\varepsilon} N_j^{1/4} (T_1+1)^{1/2} L^{-A},$$

while for $T_0 < T_2 \le T$, we have

(4.9)
$$\sum_{r\sim R} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_2}^{2T_2} \left| F\left(\frac{1}{2}+it,\chi\right) \right| dt \ll R^{1/5-\varepsilon} N_j^{1/4} T_2 L^{-A}.$$

Both (4.8) and (4.9) are deduced from the following bound, which is Lemma 5.2 in [10].

Lemma 4.2 *Let $F(s,\chi)$ be defined as above. Then for any $R \ge 1$ and $T_3 > 0$,*

$$\sum_{r\sim R} \sideset{}{^*}\sum_{\chi \bmod r} \int_{T_3}^{2T_3} \left| F\left(\frac{1}{2}+it,\chi\right) \right| dt \ll (R^2 T_3 + R T_3^{1/2} N_j^{3/20} + N_j^{1/4}) L^c.$$

Now we can complete the proof of Lemma 4.1 immediately.

Proof of Lemma 4.1 By taking $T_3 = T_1$ in Lemma 4.2, the left-hand side of (4.8) is now

$$\ll (R^2 T_1 + RT_1^{1/2} N_j^{3/20} + N_j^{1/4}) L^c \ll R^{1/5-\varepsilon} N_j^{1/4} (T_1 + 1)^{1/2} L^{-A},$$

provided that $L^C < R \le P = (N/B)^{1/8-\varepsilon}$ with a sufficiently large C. Here $L^C < R$ guarantees that $N_j^{1/4} L^c$ is dominated by the quantity on the right-hand side. This establishes (4.8). Similarly we can prove (4.9) by taking $T_3 = T_2$ in Lemma 4.2. Lemma 4.1 now follows. ■

Now we treat the case $R \le L^C$.

Lemma 4.3 *Let $A > 0$ and $C > 0$ be arbitrary. Then for $R \le L^C$, we have*

$$J_j(R) \ll N_j^{1/2} L^{-A},$$

where the implied constant depends at most on C.

Proof We use the explicit formula (see [4, pp. 109 and 120])

$$\sum_{m \le u} \Lambda(m)\chi(m) = \delta_\chi u - \sum_{|\gamma| \le T} \frac{u^\rho}{\rho} + O\Big\{ \Big(\frac{u}{T} + 1 \Big) \log^2(quT) \Big\} \tag{4.10}$$

where $\rho = \beta + i\gamma$ is a non-trivial zero of the function $L(s,\chi)$, and $2 \le T \le u$ is a parameter. Taking $T = N_j^{1/6}$ in (4.10), and then inserting it into $\hat{W}_j(\chi,\lambda)$, we get by $M_j^{1/2} < u \le N_j^{1/2}$, $M_j = N_j/200$, and (4.2) that

$$\begin{aligned} \hat{W}_j(\chi,\lambda) &= \int_{M_j^{1/2}}^{N_j^{1/2}} e(b_j u^2 \lambda)\, d\Big\{ \sum_{n \le u} \big(\Lambda(m)\chi(m) - \delta_\chi \big) \Big\} \\ &= -\int_{M_j^{1/2}}^{N_j^{1/2}} e(b_j u^2 \lambda) \sum_{|\gamma| \le N_j^{1/6}} u^{\rho-1}\, du + O\big(N_j^{1/3} (1 + |\lambda| N) L^2 \big) \\ &\ll N_j^{1/2} \sum_{|\gamma| \le N_j^{1/6}} N_j^{(\beta-1)/2} + O(N_j^{1/3} P L^{9002}). \end{aligned}$$

Now we need Satz VIII.6.2 in Prachar [15], which states that $\prod_{\chi \bmod q} L(s,\chi)$ is zero-free in the region $\sigma \ge 1 - \eta(T), |t| \le T$ except for the possible Siegel zero, where $\eta(T) = c_3 \log^{-4/5} T$. But by Siegel's theorem (see for example [4, Section 21]) the Siegel zero does not exist in the present situation, since $r \le L^C$. We also need the zero-density estimate (see *e.g.* Huxley [9]):

$$N^*(\alpha, q, T) \ll (qT)^{12(1-\alpha)/5} \log^c(qT),$$

where $N^*(\alpha, q, T)$ denotes the number of zeros of $\prod^*_{\chi \bmod q} L(s, \chi)$ in the region $\operatorname{Re} s \geq \alpha$, $|\operatorname{Im} s| \leq T$. Thus,

$$\sum_{|\gamma| \leq N_j^{1/6}} N_j^{(\beta-1)/2} \ll L^c \int_0^{1-\eta(N_j^{1/6})} (N_j^{1/6})^{12(1-\alpha)/5} N_j^{(\alpha-1)/2} \, d\alpha$$

$$\ll L^c N_j^{-\eta(N_j^{1/6})/10} \ll \exp(-c_4 L^{1/5}).$$

Consequently,

$$\sum_{r \sim R} \sideset{}{^*}\sum_{\chi \bmod r} \max_{|\lambda| \leq 1/(rQ)} |\hat{W}_j(\chi, \lambda)| \ll N_j^{1/2} L^{-A}, \tag{4.11}$$

where $R \leq L^C$, and $A > 0$ is arbitrary. Lemma 4.3 now follows from (4.11) and (4.3). ■

5 Estimation of K

In this section, we estimate K by establishing the following Lemma 5.1. We remark that in proving Lemma 5.1 we need not distinguish the two cases $R > L^C$ and $R \leq L^C$ as in Lemmas 4.1 and 4.3, since we need not save a factor L^{-A} on the right-hand side of (5.1).

Lemma 5.1 *We have*

$$K_j \ll |b_j|^{-1/2} L^c \tag{5.1}$$

where $c > 0$ is some absolute constant.

Proof By the definition of K_j and (4.3), we have

$$K_j \ll L \max_{R \leq P} \sum_{r \sim R} r^{-1/5+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \left(\int_{-1/(rQ)}^{1/(rQ)} |W_j(\chi, \lambda)|^2 \, d\lambda \right)^{1/2}$$

$$\ll L \max_{R \leq P} \sum_{r \sim R} r^{-1/5+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \left(\int_{-1/(rQ)}^{1/(rQ)} |\hat{W}_j(\chi, \lambda)|^2 \, d\lambda \right)^{1/2} + |b_j|^{-1/2}.$$

Thus to establish (5.1), it suffices to show that

$$\sum_{r \sim R} \sideset{}{^*}\sum_{\chi \bmod r} \left(\int_{-1/(rQ)}^{1/(rQ)} |\hat{W}_j(\chi, \lambda)|^2 \, d\lambda \right)^{1/2} \ll |b_j|^{-1/2} R^{1/5-\varepsilon} L^c \tag{5.2}$$

holds for $R \leq P$ and some $c > 0$.

By Gallagher's lemma (see [5, Lemma 1]), we have

(5.3)
$$\int_{-1/(rQ)}^{1/(rQ)} |\hat{W}_j(\chi,\lambda)|^2\,d\lambda \ll \Big(\frac{1}{RQ}\Big)^2 \int_{-\infty}^{\infty}\Big|\sum_{\substack{\nu<|b_j|m^2\le\nu+rQ\\ M<|b_j|m^2\le N}}\big(\Lambda(m)\chi(m)-\delta_\chi\big)\Big|^2\,d\nu$$
$$\ll \Big(\frac{1}{RQ}\Big)^2 \int_{M-rQ}^{N}\Big|\sum_{\substack{\nu<|b_j|m^2\le\nu+rQ\\ M<|b_j|m^2\le N}}\big(\Lambda(m)\chi(m)-\delta_\chi\big)\Big|^2\,d\nu.$$

Let $X=\max(\nu,M)/|b_j|$ and $Y=\min(\nu+rQ,N)/|b_j|$. Then the sum in (5.3) can be written as

$$\sum_{X<m^2\le Y}\big(\Lambda(m)\chi(m)-\delta_\chi\big). \tag{5.4}$$

Before estimating (5.4), we observe first that, for any $0<\beta<1$,

$$Y^\beta - X^\beta \ll \frac{(\nu+rQ)^\beta-\nu^\beta}{|b_j|^\beta} = \frac{\nu^\beta\{(1+rQ/\nu)^\beta-1\}}{|b_j|^\beta} \ll \frac{rQ}{|b_j|^\beta M^{1-\beta}}, \tag{5.5}$$

where in the last step we have used $M-rQ\le\nu\le N$ and $rQ\le 2RQ\le 2PQ\ll ML^{-9000}$.

In the case $\chi=\chi^0$ mod 1, the quantity in (5.4) is

$$\ll Y^{1/2}-X^{1/2}\ll |b_j|^{-1/2}M^{-1/2}Q$$

by (5.5) with $r=1$. This contributes to (5.3) acceptably.

For other χ, we have $\delta_\chi=0$ in (5.4). Using Heath-Brown's identity to this sum, and applying Perron's formula as before, we see that (5.4) is a linear combination of $O(L^{10})$ terms, each of which has the form

$$\frac{1}{2\pi}\int_{-T}^{T} F\Big(\frac12+it,\chi\Big)\frac{Y^{\frac12(\frac12+it)}-X^{\frac12(\frac12+it)}}{\frac12+it}\,dt + O\Big(\frac{N_j^{1/2}L^2}{T}\Big),$$

where $\mathbf{D}$, $F(s,\chi)$ are as in Section 4, and T is a parameter satisfying $2\le T\le N_j^{1/2}$. One easily sees that

$$\frac{Y^{\frac12(\frac12+it)}-X^{\frac12(\frac12+it)}}{\frac12+it} = \frac12\int_X^Y u^{-3/4+it/2}\,du = \frac12\int_X^Y u^{-3/4}e\Big(\frac{t}{4\pi}\log u\Big)\,du.$$

The integral can be easily estimated by (5.5) as $\ll Y^{1/4}-X^{1/4}\ll |b_j|^{-1/4}M^{-3/4}RQ$. On the other hand, one has trivially

$$\frac{Y^{\frac12(\frac12+it)}-X^{\frac12(\frac12+it)}}{\frac12+it} \ll \frac{Y^{1/4}}{|t|} \ll \frac{N_j^{1/4}}{|t|}.$$

Collecting the two upper bounds, we get

$$\frac{Y^{\frac{1}{2}(\frac{1}{2}+it)}-X^{\frac{1}{2}(\frac{1}{2}+it)}}{\frac{1}{2}+it} \ll \min\left(\frac{RQ}{M^{3/4}|b_j|^{1/4}}, \frac{N_j^{1/4}}{|t|}\right) \ll \frac{1}{|b_j|^{1/4}}\min\left(\frac{RQ}{N^{3/4}}, \frac{N^{1/4}}{|t|}\right).$$

Taking $T = N_j^{1/2}$ and $T_0 = N/(QR)$, we see that

$$\sigma(u; \mathbf{D}) \ll \frac{RQ}{|b_j|^{1/4}N^{3/4}}\int_{|t|\le T_0}\left|F\left(\frac{1}{2}+it,\chi\right)\right| dt + \frac{N^{1/4}}{|b_j|^{1/4}}\int_{T_0<|t|\le T}\left|F\left(\frac{1}{2}+it,\chi\right)\right|\frac{dt}{|t|} + O(L^2).$$

And consequently (5.3) becomes

$$\int_{-1/(rQ)}^{1/(rQ)}|\hat{W}(\chi,\lambda)|^2\,d\lambda \ll \frac{L^{20}}{|b_j|^{1/2}N^{1/2}}\max_{\mathbf{D}}\left(\int_{|t|\le T_0}\left|F\left(\frac{1}{2}+it,\chi\right)\right| dt\right)^2 + \frac{N^{3/2}L^{20}}{|b_j|^{1/2}(QR)^2}\max_{\mathbf{D}}\left(\int_{T_0<|t|\le T}\left|F\left(\frac{1}{2}+it,\chi\right)\right|\frac{dt}{|t|}\right)^2 + \frac{NL^{24}}{(QR)^2}.$$

Now the left-hand side of (5.2) is

$$\ll \frac{L^{10}}{|b_j|^{1/4}N^{1/4}}\max_{\mathbf{D}}\sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}\int_{|t|\le T_0}\left|F\left(\frac{1}{2}+it,\chi\right)\right| dt + \frac{N^{3/4}L^{10}}{|b_j|^{1/4}RQ}\max_{\mathbf{D}}\sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}\int_{T_0<|t|\le T}\left|F\left(\frac{1}{2}+it,\chi\right)\right|\frac{dt}{|t|} + \frac{N^{1/2}RL^{12}}{Q}.$$

Thus, to prove (5.2) it suffices to show that the estimate

$$\sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}\int_{T_1}^{2T_1}\left|F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll R^{1/5-\varepsilon}N_j^{1/4}L^c \tag{5.6}$$

holds for $R \le P$ and $0 < T_1 \le T_0$, and

$$\sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}\int_{T_2}^{2T_2}\left|F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll R^{1/5-\varepsilon}\left(\frac{RQ}{|b_j|^{1/4}N^{3/4}}\right)T_2L^c \tag{5.7}$$

holds for $R \le P$ and $T_0 < T_2 \le T$.

The estimates (5.6) and (5.7) follows from Lemma 4.2. The proof of Lemma 5.1 is complete. ■

Proof of Theorem 3 Collecting Lemmas 3.2, 4.1, 4.3 and 5.1, we get Theorem 3. ■

6 Necessary and Sufficient Condition for Congruent Solubility

In this section, we suppose only that $(b_1, \dots, b_5) = 1$ but b_j may not be pairwisely relative prime. For any $q \geq 1$, we define

$$N(q) = \mathrm{Card}\{(m_1, \dots, m_5) : 1 \leq m_i \leq q, (m_i, q) = 1, \\ b_1 m_1^2 + \dots + b_5 m_5^2 \equiv n \pmod{q}\}.$$

A necessary condition for the solubility of the equation (1.1) is the congruence solubility that $N(q) > 0$ for all integer $q \geq 1$. In this section, we prove the necessary and sufficient condition for the congruence solubility below. It will follow that the condition (1.2) is actually sufficient for the congruence solubility under our assumption (1.3). Moreover, we will also obtain an asymptotic estimation for $N(p)$ in Proposition 4 which is useful and essential to the proofs in Section 7.

It is known (see [13, Section 3]) that $N(q)$ is a multiplicative function of q and $N(p^\alpha) \geq 1$ if and only if $N(p) \geq 1$ for odd prime p and $\alpha \geq 1$ and $N(2^\alpha) \geq 1$ if and only if $N(8) \geq 1$ for $\alpha \geq 3$. Thus, it only needs to consider $N(2)$, $N(4)$, $N(8)$ and $N(p)$ for odd prime p.

It is straightforward to verify that

$$N(2^l) = \begin{cases} \varphi(2^l)^5 & \text{if } b_1 + \dots + b_5 \equiv b \pmod{2^l}; \\ 0 & \text{otherwise;} \end{cases}$$

for $l = 1, 2, 3$ and

$$N(3) = \begin{cases} 2^5 & \text{if } b_1 + \dots + b_5 \equiv b \pmod{3}; \\ 0 & \text{otherwise.} \end{cases}$$

Thus it remains to consider $p \geq 5$. We are going to show:

Proposition 4 *Let $b_1, \dots, b_5$ and n be any integers. For convenience, we let $b_6 = -n$. For $p \geq 7$, $N(p) = 0$ if and only if*

(i) p divides exactly 5 of $b_1, \dots, b_6$; or
(ii) p divides exactly 4 of $b_1, \dots, b_6$ (say $p \nmid b_i, b_j$) and $\left(\frac{b_i}{p}\right) = -\left(\frac{-1}{p}\right)\left(\frac{b_j}{p}\right)$.

For the case $p = 5$, $N(5) = 0$ if and only if (i) or (ii) or

(iii) 5 divides exactly 3 of $b_1, \dots, b_6$ (say $5 \nmid b_i, b_j, b_k$) and

$$\left(\frac{b_i}{5}\right) = \left(\frac{b_j}{5}\right) = \left(\frac{b_k}{5}\right).$$

Moreover, if $N(p) > 0$ then $N(p) = p^4 + O(p^3)$ except when p divides exactly 4 of $b_1, \dots, b_6$ (say $p \nmid b_i, b_j$) and $\left(\frac{b_i}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{b_j}{p}\right)$, $N(p) = 2(p-1)^4$. Here $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol.

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Proof Among the numbers $b_1, \ldots, b_6$, let m of them be divisible by p and k (respectively l) of them be quadratic residues (respectively non-residues) modulo p. Then from the proof of Lemma 3.6 and (3.8) in [13], we have

$$\varphi(p)^{-5} pN(p) = 1 + A(n, p), \tag{6.1}$$

$m + k + l = 6$ and

$$A(n, p) = \frac{1}{2}\varphi(p)^{m-5}\{(\lambda - 1)^k(-\lambda - 1)^l + (\lambda - 1)^l(-\lambda - 1)^k\} \tag{6.2}$$

where $\lambda = \sqrt{p}$ if $p \equiv 1 \pmod 4$ and $\lambda = i\sqrt{p}$ if $p \equiv -1 \pmod 4$. In view of (6.1), $N(p) = 0$ if and only if $A(n, p) = -1$. It has been proved in Lemma 3.6 of [13] that if $p \geq 7$ (respectively $p = 5$) and p does not divide more than 3 (respectively 2) of the six numbers $b_1, \ldots, b_6$ then $N(p) \geq 1$ and when $N(p) \geq 1$, by direct computation of the term $A(n, p)$ using (6.2), we can prove that $|A(n, p)| \ll p^{-1}$ and hence $N(p) = p^4 + O(p^3)$ by (6.1). It remains to consider cases (ii) and (iii) in the proposition (case (i) is trivial). For case (ii), $m = 4$ and $k + l = 2$ and from (6.2)

$$\begin{aligned} A(n, p) &= \frac{1}{2}(p - 1)^{-1}\{(\lambda - 1)^k(-\lambda - 1)^l + (\lambda - 1)^l(-\lambda - 1)^k\} \\ &= \begin{cases} \frac{p+1}{p-1} & \text{if } p \equiv 1 \pmod 4 \text{ and } (k, l) = (0, 2) \text{ or } (2, 0); \\ -1 & \text{if } p \equiv 1 \pmod 4 \text{ and } (k, l) = (1, 1); \\ \frac{p+1}{p-1} & \text{if } p \equiv -1 \pmod 4 \text{ and } (k, l) = (1, 1); \\ -1 & \text{if } p \equiv -1 \pmod 4 \text{ and } (k, l) = (0, 2) \text{ or } (2, 0). \end{cases} \end{aligned}$$

Thus $A(n, p) = -1$ if and only if $\left(\frac{b_i}{p}\right) = -\left(\frac{-1}{p}\right)\left(\frac{b_j}{p}\right)$ and when $N(p) \geq 1$, then $N(p) = 2(p - 1)^4$. For case (iii), $p = 5$, $m = k + l = 3$ and from (6.2) we have

$$\begin{aligned} A(n, 5) &= \frac{1}{32}\{(\sqrt{5} - 1)^k(-\sqrt{5} - 1)^l + (\sqrt{5} - 1)^l(-\sqrt{5} - 1)^k\} \\ &= \begin{cases} -1 & \text{if } (k, l) = (0, 3) \text{ or } (3, 0); \\ \frac{1}{4} & \text{if } (k, l) = (1, 2) \text{ or } (2, 1). \end{cases} \end{aligned}$$

Thus $N(5) = 0$ if $\left(\frac{b_i}{5}\right) = \left(\frac{b_j}{5}\right) = \left(\frac{b_k}{5}\right)$. ∎

7 Proofs of Lemmas 2.1 and 2.2

Lemma 2.1 is a consequence of the following:

Lemma 7.1

(i) *For $x > 0$,*

$$\sum_{q > x} |A(n, q)| \ll x^{-1} B^{\varepsilon} \log^{60}(x + 2).$$

So the singular series $\mathfrak{S}(n) := \mathfrak{S}(n, \infty)$ is absolutely convergent.

(ii) We have $\mathfrak{S}(n) \gg (\log\log B)^{-c_5}$ *for some constant* $c_5 > 0$.

Proof Let $\sigma = \bigl(\log(x+2)\bigr)^{-1}$. From Lemma 3.2 and Corollary 3.5 (a) in [13], we have

$$\begin{aligned}\sum_{q>x} |A(n,q)| &\le \sum_{q=1}^{\infty} \Bigl(\frac{q}{x}\Bigr)^{1-\sigma} |A(n,q)| = x^{-1+\sigma} \sum_{q=1}^{\infty} q^{1-\sigma} |A(n,q)| \\ &\ll x^{-1} \prod_p \bigl(1 + p^{1-\sigma}|A(n,p)|\bigr)\end{aligned} \tag{7.1}$$

because $x^{\sigma} \ll 1$. Using Lemma 3.7 (a) in [13], we have

$$\begin{aligned}\prod_{p\nmid b_1\cdots b_5} \bigl(1 + p^{1-\sigma}|A(n,p)|\bigr) &\le \prod_{p\nmid b_1\cdots b_5} \Bigl(1 + \frac{60}{p^{1+\sigma}}\Bigr) \le \prod_p (1 - p^{-1-\sigma})^{-60} \\ &= \zeta(1+\sigma)^{60} \ll \sigma^{-60} = \log^{60}(x+2).\end{aligned} \tag{7.2}$$

Using (1.3), (6.1) and Proposition 4, we get

$$\prod_{p\mid b_1\cdots b_5} \bigl(1 + p^{1-\sigma}|A(n,p)|\bigr) \le \prod_{p\mid b_1\cdots b_5} (1 + cp^{-\sigma}) \le d(b_1\cdots b_5)^{\log_2(1+c)} \ll B^{\varepsilon}. \tag{7.3}$$

Now (i) follows from (7.1), (7.2) and (7.3).

Using (1.3) and Proposition 4, we have $N(p) = p^4 + O(p^3)$. It follows from this and Lemma 3.7 (a) of [13] that, for some large constant $c > 60$,

$$\begin{aligned}\mathfrak{S}(n) = \prod_p \bigl(1 + A(n,p)\bigr) &\gg \prod_{\substack{p\mid b_1\cdots b_5\\ p>c}} (1 - cp^{-1}) \prod_{\substack{p\nmid b_1\cdots b_5\\ p>c}} (1 - 60p^{-2}) \\ &\gg \prod_{\substack{p\mid b_1\cdots b_5\\ p>c}} (1 - cp^{-1}) \gg \prod_{p\mid b_1\cdots b_5} (1 + p^{-1})^{-(1+c)}.\end{aligned}$$

The desired estimate in (ii) now follows from the well-known estimate $\prod_{p\mid x}(1 + p^{-1}) \ll \log\log x$. ■

Proof of Lemma 2.2 We easily derive the following inequalities:

$$\begin{aligned}\sum_{\substack{b_1m_1+\cdots+b_5m_5=n\\ M<|b_j|m_j\le N}} 1 &\le \sum_{\substack{n-(b_1m_1+\cdots+b_4m_4)\equiv 0 \pmod{|b_5|}\\ M<|b_j|m_j\le N,\, j=1,\ldots,4}} 1 \\ &= \sum_{\substack{M_j<m_j\le N_j\\ j=1,2,3}} \sum_{\substack{m_4\equiv \overline{b_4}(n-(b_1m_1+\cdots+b_3m_3)) \pmod{|b_5|}\\ M_4<m_4\le N_4}} 1 \\ &\ll N_1N_2N_3\frac{N_4}{|b_5|} \ll \frac{N^4}{|b_1\cdots b_5|},\end{aligned}$$

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where $b_4\overline{b_4} \equiv 1 \pmod{|b_5|}$.

To establish inequalities in the other direction, we first consider case (ii) in which all b_j are positive and $n = N$. If $M < b_j m_j \le N/5$ for $j = 1, \dots, 4$, then

$$M < N/5 = N - 4(N/5) \le N - (b_1 m_1 + \cdots + b_4 m_4) = b_5 m_5 < N.$$

It follows that

$$\sum_{\substack{b_1 m_1 + \cdots + b_5 m_5 = n \\ M < b_j m_j \le N}} 1 \ge \sum_{\substack{n - (b_1 m_1 + \cdots + b_4 m_4) \equiv 0 \pmod{b_5} \\ M < b_j m_j \le N/5, j = 1, \dots, 4}} 1 \gg \frac{N^4}{b_1 \cdots b_5}.$$

The case (i) can be treated similarly. We therefore conclude that

$$\sum_{\substack{b_1 m_1 + \cdots + b_5 m_5 = n \\ M < |b_j| m_j \le N}} 1 \asymp \frac{N^4}{|b_1 \cdots b_5|},$$

from which and the definition of $\mathfrak{I}(n)$ (in (2.7)) the desired result follows. ∎

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