CONGRUENCES ON ORTHODOX SEMIGROUPS

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Introduction

A semigroup S is called *regular* if $a \in aSa$ for every element a in S. The elementary properties of regular semigroups may be found in A. H. Clifford and G. B. Preston [1]. A semigroup S is called *orthodox* if S is regular and if the idempotents of S form a subsemigroup of S.

In this paper we investigate congruences on orthodox semigroups. Specifically, we obtain a generalization of kernel normal systems of inverse semigroups, introduced by G. B. Preston [6], to orthodox semigroups. A good account of Preston's kernel normal systems may be found in [2], § 7.4.

We then investigate idempotent-separating congruences on orthodox semigroups, and detemine a necessary and sufficient condition for Green's equivalence $\mathscr{H}$ to be a congruence on an orthodox semigroup. We also determine the maximal idempotent-separating congruence on an orthodox semigroup.

Finally, we investigate inverse semigroup congruences on orthodox semigroups, and determine the minimal such congruence.

1. Some preliminary results

We denote the set of idempotents of S by E_S, and the set of inverses of an element a in S by $V(a)$. Thus an orthodox semigroup is a regular semigroup S for which $E_S E_S \subseteq E_S$. The following three results may be found in the paper by N. R. Reilly and H. E. Scheiblich [8]. They will be used in the sequel without comment.

LEMMA 1.1. *Let a and b be arbitary elements of the orthodox semigroup S and let a' and b' be arbitrary inverses of a and b respectively. Then $b'a' \in V(ab)$.*

LEMMA 1.2. *Let a be any element of the orthodox semigroup S and let a' be an arbitrary inverse of a. Then $a'E_S a \subseteq E_S$.*

LEMMA 1.3. *Let e be any element of the set E_S of idempotents of the orthodox semigroup S. Then $V(e) \subseteq E_S$.*

We now give a brief account of some of the results of Preston, all of which

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may be found in [2], § 7.4. Preston first shows that a congruence ρ on a regular semigroup S is uniquely determined by its kernel; that is ρ is uniquely determined by the set of ρ-classes which contain idempotents. He then proceeds to determine a set of conditions on a set $\mathscr{A} = \{A_i : i \in I\}$ of subsemigroups of an inverse semigroup S under which $\mathscr{A}$ can in fact serve as the kernel of some congruence $\rho_{\mathscr{A}}$ on S, and indeed he derives a construction for the associated congruence $\rho_{\mathscr{A}}$.

We proceed along similar lines for orthodox semigroups. Let $\mathscr{A} = \{A_i : i \in I\}$ be the kernel of a congruence ρ on an orthodox semigroup S. Unlike the situation for inverse semigroups, it is not necessarily true that an element A_i of $\mathscr{A}$ is a regular subsemigroup of S, as the following counter-example, due to T. E. Hall, readily shows.

EXAMPLE 1.4. Let $A = \{a_{ij} : i, j \in \{1, 2\}\}$, and let $B = \{b_{ij} : i, j \in \{1, 2\}\}$. Let $S = A \cup B$, and define a multiplication on S by

$$(1) \qquad \begin{aligned} a_{ij} a_{kl} &= \begin{cases} a_{il}, & j = k \\ b_{il}, & j \neq k \end{cases} \\ b_{ij} b_{kl} &= b_{il} \\ b_{ij} a_{kl} &= a_{ij} b_{kl} = b_{il}. \end{aligned}$$

It is straightforward to verify that with this definition of multiplication, S is an orthodox semigroup. We note that B is a subsemigroup of S : indeed B is a rectangular band. The elements a_{11} and a_{22} of S are idempotents whose only inverses are themselves, and the elements a_{12} and a_{21} of S are mutually inverse elements with unique inverses. Now consider the relation ρ on S which identifies a_{ij} with b_{ij}, for all $i, j \in \{1, 2\}$. It is easy to see that ρ is a congruence on S, and that the set of all ρ-classes,

$$\mathscr{A} = \{\{a_{11}, b_{11}\}, \{a_{12}, b_{12}\}, \{a_{21}, b_{21}\}, \{a_{22}, b_{22}\}\},$$

is the kernel of ρ. The set $\{a_{21}, b_{21}\}$ is an element of the kernel of ρ, but is not a regular subsemigroup of S. Thus we have seen that the kernel of a congruence ρ on an othodox semigroup S does not necessarily consist of regular subsemigroups of S. In § 2 we show that a congruence ρ on an orthodox semigroup S is uniquely determined by the set of maximal regular subsemigroups of the elements of the kernel of ρ.

2. The regular kernel

We make use of the following result due to G. Lallement [5].

LEMMA 2.1. *Let ϕ be a homomorphism of a regular semigroup S onto a (necessarily regular) semigroup S'. If $e' \in E_{S'}$, it follows that $e'\phi^{-1} \cap E_S \neq \square$.*

We now prove the following two lemmas.

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LEMMA 2.2. *A homomorphic image of an orthodox semigroup is orthodox.*

PROOF. Let ϕ be a homomorphism from the orthodox semigroup S onto the semigroup S'. Then since S is regular it follows immediately that S' is regular. (See for example [2], § 7.4). Let e' and f' be arbitrary idempotents of S'. By lemma 2.1, $e'\phi^{-1}$ and $f'\phi^{-1}$ both contain idempotents of S, say $e \in e'\phi^{-1} \cap E_S$ and $f \in f'\phi^{-1} \cap E_S$. Then $ef \in E_S$ since S is orthodox, and so $(ef)\phi = e'f' \in E_{S'}$, and it follows that S' is orthodox.

LEMMA 2.3. *Let S be an orthodox semigroup, ρ a congruence on S, and $\mathscr{A} = \{A_i : i \in I\}$ the kernel of ρ. Then $V(A) \subseteq A$, where $A = \bigcup_{i \in I} A_i$, and $V(A) = \bigcup_{a \in A} V(a)$.*

PROOF. Let a be an arbitrary element of A and let a' be an arbitrary inverse of a. Then $a'\phi$ is an inverse of $a\phi$, an idempotent of S/ρ, where ϕ is the natural homomorphism corresponding to the congruence ρ, and so $a'\phi$ is an idempotent of S/ρ, by lemma 1.3 and lemma 2.2. Hence $a' \in A$, which completes the proof of the lemma.

Let ρ be a congruence on an orthodox semigroup S, and let $\mathscr{A} = \{A_i : i \in I\}$ be the kernel of ρ. Then the set $\mathscr{B} = \{B_i : i \in I\}$ of maximal regular subsemigroups of the elements of the kernel of ρ is called the *regular kernel* of ρ. We note that $\mathscr{B}$ is well-defined in the sense that for each element A_i of the kernel $\mathscr{A}$ of ρ there is a unique maximal regular subsemigroup B_i of A_i. In fact it is easily verified that

$$B_i = \{x \in A_i : V(x) \cap A_i \neq \square\} \tag{2}$$

is the unique maximal regular subsemigroup of A_i: for if x and y are elements of B_i, there exist elements $x' \in V(x) \cap A_i$, and $y' \in V(y) \cap A_i$, and so $y'x' \in V(xy) \cap A_i$, that is $xy \in B_i$. Hence B_i is a subsemigroup of A_i. That B_i is the unique maximal regular subsemigroup of A_i is now obvious. We shall make use of the characterization (2) of the B_i in the sequel.

THEOREM 2.4. *Let ρ and σ be congruences on an orthodox semigroup S having the same regular kernel $\mathscr{B} = \{B_i : i \in I\}$. Then $\rho = \sigma$.*

PROOF. Let $\mathscr{A} = \{A_i : i \in I\}$ be the kernel of ρ and let $\mathscr{A}' = \{A'_j : j \in J\}$ be the kernel of σ. Then by definition of the regular kernel $\mathscr{B}$, $|I| = |J|$, and (with a suitable indexing of the A_i and the A'_j) B_i is the maximal regular subsemigroup of A_i and of A'_i, for all $i \in I$. In view of the result of Preston it suffices to show that $A_i = A'_i$ for all $i \in I$, and to show this, we show that for all $i \in I$,

$$A_i(= A'_i) = \{a \in A : a' \in V(a) \text{ implies } aa'a'a \in B_i\},$$
$$\text{where } A = \{x \in S : xx' \in B_i,\ x' \in V(x), \text{ implies } x^2x' \in B_i\}.$$

Let

$$C_i = \{a \in A : a' \in V(a) \text{ implies } aa'a'a \in B_i\}.$$

Let a be an arbitrary element of C_i, and let a' be an arbitrary inverse of a. Then

$a \in A$ and so $aa', a^2a' \in B_j$, some $j \in I$. Hence $(aa', a^2a') \in \rho$, and it follows that $(aa'a, a^2a'a) = (a, a^2) \in \rho$. Thus $a \in \bigcup_{i \in I} A_i$, and so $a' \in \bigcup_{i \in I} A_i$, by lemma 2.3. It follows that $(a', (a')^2) \in \rho$ and hence that $(a, aa'a'a) \in \rho$, and since $aa'a'a \in B_i$ we deduce that $a \in A_i$. Hence $C_i \subseteq A_i$ for each $i \in I$.

Conversely, let a be an arbitrary element of A_i and let a' be an arbitrary inverse of a. Then $(a^2, a) \in \rho$, and so $(a^2a', aa') \in \rho$. Furthermore, $a' \in \bigcup_{i \in I} A_i$, by lemma 2.3, and so $((a')^2, a') \in \rho$. From this it follows that $(a(a')^2, aa') \in \rho$, and hence that $a(a')^2$, a^2a', and aa' are all contained in the same ρ-class, which must be an element of the kernel of ρ since aa' is an idempotent of S. Hence $a(a')^2, a^2a', aa' \in A_k$, for some $k \in I$. But $a(a')^2$ and a^2a' are mutually inverse elements of S and so $a(a')^2$, $a^2a' \in B_k$. Clearly, $aa' \in B_k$ since $aa' \in E_S \cap A_k$. Hence $aa' \in B_k$ implies $a^2a' \in B_k$, and it follows that $a \in A$. Furthermore, $((a')^2, a') \in \rho$ so $(a, aa'a'a) \in \rho$, and so $aa'a'a \in B_i$. It follows that $a \in C_i$ and hence that $A_i = C_i$ for all $i \in I$. Since we also have that $A_i' = C_i$ for all $i \in I$, the theorem is proved.

As an immediate corollary to the proof of this theorem we deduce the following result.

COROLLARY 2.5. *Let ρ be a congruence on an orthodox semigroup S with kernel $\mathscr{A} = \{A_i : i \in I\}$ and regular kernel $\mathscr{B} = \{B_i : i \in I\}$. Define $A = \{x \in S : xx' \in B_i$, $x' \in V(x)$, implies $x^2x' \in B_i\}$. Then $A = \bigcup_{i \in I} A_i$, and for each $i \in I$, $A_i = \{a \in A :$ $a' \in V(a)$ implies $aa'a'a \in B_i\}$.*

This result, of course, shows us how to obtain the kernel of a congruence on an orthodox semigroup when we are given the regular kernel. The following obvious corollary provides us with a necessary and sufficient condition (on the regular kernel of a congruence on an orthodox semigroup) for the kernel and the regular kernel to coincide.

COROLLARY 2.6. *A necessary and sufficient condition for the kernel $\mathscr{A} = \{A_i : i \in I\}$ and the regular kernel $\mathscr{B} = \{B_i : i \in I\}$ of a congruence ρ on an orthodox semigroup S to coincide is that $\bigcup_{i \in I} B_i = \{x \in S : xx' \in B_i,\ x' \in V(x)$, implies $x^2x' \in B_i\}$.*

3. Regular kernel normal systems

In § 2 we have shown that a congruence on an orthodox semigroup S is uniquely determined by its regular kernel $\mathscr{B}$. We proceed to obtain a characterization of such sets $\mathscr{B}$ and derive a construction for the associated congruences.

The set $\mathscr{B} = \{B_i : i \in I\}$ is defined to be a *regular kernel normal system* of the orthodox semigroup S if

(K1) each B_i is a regular subsemigroup of S;
(K2) $B_i \cap B_j = \square$ if $i \neq j$;

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(K3) each idempotent of S is contained in some B_i;
(K4) for each $a \in S$, $a' \in V(a)$, and $i \in I$, there is some $j = j(a, a', i) \in I$ such that $a'B_i a \subseteq B_j$;
(K5) for each $i, j \in I$, there is some $k \in I$ such that $B_i B_j B_i \subseteq B_k$;
(K6) if a, ab, bb', $b'b \in B_i$ for some $b' \in V(b)$, then $b \in B_i$;
(K7) for each $i \in I$ and for each $j \in I$, there is some $k \in I$ such that $E_i E_j \subseteq E_k$, where E_i is the set of idempotents of B_i.

LEMMA 3.1. *The regular kernel $\mathscr{B} = \{B_i : i \in I\}$ of a congruence ρ on an orthodox semigroup S is a regular kernel normal system of S.*

PROOF. Conditions K1, K2, K3, and K7 are trivial to verify. To prove that K4 is satisfied we first verify that for each $a \in S$, $a' \in V(a)$, and $i \in I$, there is some $j \in I$ such that $a'A_i a \subseteq A_j$, where $\mathscr{A} = \{A_i : i \in I\}$ is the kernel of ρ. Choose $e \in A_i \cap E_S$ and note that $a'ea \in E_S$, by lemma 1.2. Hence $a'ea \in A_j$ for some $j \in I$. But $(a'ea, a'xa) \in \rho$ for all $x \in A_i$, and so $a'A_i a \subseteq A_j$. From this we deduce immediately that $a'B_i a \subseteq a'A_i a \subseteq A_j$. Now let $a'ba$ be an arbitrary element of $a'B_i a$, where $b \in B_i$. Since B_i is regular, there is an inverse b' of b such that $b' \in B_i$. But then $a'b'a \in V(a'ba) \cap a'B_i a \subseteq V(a'ba) \cap A_j$, and so $a'ba \in B_j$. Thus, finally, $a'B_i a \subseteq B_j$.

To verify that K5 is satisfied, we prove first that for each $i \in I$ and for each $j \in I$, there is some $k \in I$ such that $A_i A_j A_i \subseteq A_k$. Indeed this follows easily since in fact the A_i satisfy the stronger condition $A_i A_j \subseteq A_l$ for some $l \in I$. Now let $b_1 b_2 b_3$ be an arbitrary element of $B_i B_j B_i \subseteq A_i A_j A_i \subseteq A_k$, and choose $b_1' \in V(b_1) \cap B_i$, $b_2' \in V(b_2) \cap B_j$, and $b_3' \in V(b_3) \cap B_i$. Then $b_3' b_2' b_1' \in V(b_1 b_2 b_3) \cap B_i B_j B_i \subseteq V(b_1 b_2 b_3) \cap A_k$, and it follows that $b_1 b_2 b_3 \in B_k$.

To prove that K6 is satisfied, we first note that if a, ab, $bb' \in A_i$ for some $b' \in V(b)$, then $b \in A_i$. This is easy to prove, since if $(a, bb') \in \rho$, then $(ab, b) \in \rho$, and so $b \in A_i$. Now suppose that a, ab, bb', $b'b \in B_i$. Then in particular, $a, ab, bb' \in A_i$, so $b \in A_i$. But then $b, bb', b'b \in A_i$, and so $b' \in A_i$. Hence $b \in B_i$. This completes the proof of the lemma.

We now introduce the following notation: if $\mathscr{B} = \{B_i : i \in I\}$ is a regular kernel normal system of the orthodox semigroup S, then *we define $a \sim b$ if and only if there is some $i \in I$ such that $a \in B_i$ and $b \in B_i$*. Note that $\sim$ is a partial equivalence on S. Let $\mathscr{B} = \{B_i : i \in I\}$ be a regular kernel normal system of the orthodox semigroup S and consider the relation

(3) $$\rho_{\mathscr{B}} = \{(a, b) \in S \times S\text{: there exists } a' \in V(a) \text{ and } b' \in V(b) \text{ such that } aa', bb', ab' \in B_i,\ a'a, b'b, a'b \in B_j \text{ for some } i, j \in I\}.$$

In terms of the notation just introduced we have

(3′) $$\rho_{\mathscr{B}} = \{(a, b) \in S \times S : aa' \sim bb' \sim ab',\ a'a \sim b'b \sim a'b, \text{ for some } a' \in V(a), b' \in V(b)\}.$$

We now prove the following lemma.

LEMMA 3.2. *Let* $\mathscr{B} = \{B_i : i \in I\}$ *be a regular kernel normal system of the orthodox semigroup* S *and let* $\rho_{\mathscr{B}}$ *be defined by equation* (3) (*or equivalently by equation* (3′)). *Then the transitive closure* $\rho^t_{\mathscr{B}}$ *of the relation* $\rho_{\mathscr{B}}$ *is a congruence on* S.

PROOF. It suffices to prove that $\rho_{\mathscr{B}}$ is a reflexive, symmetric and compatible relation on S. The fact that $\rho_{\mathscr{B}}$ is reflexive follows immediately from K3.

Suppose now that $(a, b) \in \rho_{\mathscr{B}}$. Then there are inverses a' and b' of a and b respectively such that

$$aa' \sim bb' \sim ab' \text{ and } a'a \sim b'b \sim a'b.$$

To prove that $(b, a) \in \rho_{\mathscr{B}}$, it clearly suffices to prove that

$$(4) \qquad bb' \sim aa' \sim ba' \text{ and } b'b \sim a'a \sim b'a.$$

Now

$$(ab')(ba') = a(b'b)a' \sim a(a'a)a' = aa', \qquad \text{by K4,}$$

and

$$(ba')(ab') = b(a'a)b' \sim b(b'b)b' = bb', \qquad \text{by K4.}$$

Hence $(ab') \sim (ab')(ba') \sim (ba')(ab')$. From this it follows immediately from K6 that $ba' \sim ab' \sim aa' \sim bb'$, since $ba' \in V(ab')$. The condition $b'b \sim a'a \sim b'a$ follows from the above proof by interchanging a with a' and b with b' throughout. Hence $(b, a) \in \rho_{\mathscr{B}}$, and thus $\rho_{\mathscr{B}}$ is symmetric. *We remark that in fact we have proved that if* $aa', bb', ab' \in B_i$, *and* $a'a, b'b, a'b \in B_j$, *where* $a' \in V(a)$ *and* $b' \in V(b)$, *then it follows that* $ba' \in B_i$ *and* $b'a \in B_j$. *We make use of this remark in the sequel without comment.*

We now prove that $\rho_{\mathscr{B}}$ is left compatible. Suppose $(a, b) \in \rho_{\mathscr{B}}$, and let c be an arbitrary element of S. We aim to prove that $(ca, cb) \in \rho_{\mathscr{B}}$. Since $(a, b) \in \rho_{\mathscr{B}}$, there are inverses a' of a and b' of b respectively such that $aa' \sim bb' \sim ab'$ and $a'a \sim b'b \sim a'b$. Let c' be an arbitrary inverse of c. Since $a'c' \in V(ca)$, and $b'c' \in V(cb)$, it clearly suffices to prove that

$$(5) \qquad (ca)(a'c') \sim (cb)(b'c') \sim (ca)(b'c'),$$

and that

$$(6) \qquad (a'c')(ca) \sim (b'c')(cb) \sim (a'c')(cb).$$

Now $(ca)(a'c') = c(aa')c' \sim c(bb')c'$, by K4, so $(ca)(a'c') \sim (cb)(b'c)$. Also, $(ca)(b'c') = c(ab')c' \sim c(aa')c'$, by K4, so $(ca)(b'c') \sim (ca)(a'c')$, and (5) is verified. To prove (6) we proceed as follows. Note that $(a'c')(ca) = a'(aa')(c'c)(aa')a$. But $aa' \sim ab'$ and $aa' \sim ba'$, so $(aa')(c'c)(aa') \sim (ab')(c'c)(ba')$, by K5, since $c'c \in B_k$ for some $k \in I$. Hence $a'(aa')(c'c)(aa')a \sim a'(ab')(c'c)(ba')a$, by K4, i.e.

$$\begin{aligned}(a'c')(ca) &\sim (a'a)[(b'c')(cb)](a'a)\\ &\sim (b'b)[(b'c')(cb)](b'b) \qquad \text{(by K5 or K7)}\\ &= (b'c')(cb).\end{aligned}$$

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Thus $(a'c')(ca) \sim (b'c')(cb)$.

Now let $x = (b'c')(cb)$, $y = (a'c')(cb)$, and $y' = (b'c')(ca)$. Then $y' \in V(y)$, and

$$\begin{aligned} xy &= (b'c'c)(ba')(c'cb) \\ &\sim (b'c'c)(bb')(c'cb) \qquad \text{(by K4)} \\ &= (b'c'cb)(b'c'cb) = (b'c')(cb), \text{ since } (b'c')(cb) \in E_S. \end{aligned}$$

Further,

$$\begin{aligned} yy' &= (a'c'c)(bb')(c'ca) \\ &\sim (a'c'c)(aa')(c'ca) \ \text{(by K4)} \\ &= (a'c'ca)(a'c'ca) = (a'c')(ca) \sim (b'c')(cb) = x, \end{aligned}$$

by what was proved earlier, and

$$\begin{aligned} y'y &= (b'c'c)(aa')(c'cb) \\ &\sim (b'c'c)(bb')(c'cb) \\ &= (b'c'cb)(b'c'cb) = (b'c')(cb). \end{aligned}$$

Hence $x \sim xy \sim yy' \sim y'y$, and so $x \sim y$, by K6. Thus $(a'c')(ca) \sim (b'c')(cb) \sim (a'c')(cb)$. Hence (6) is verified, and the left compatibility of $\rho_{\mathscr{B}}$ is established. The right compatibility of $\rho_{\mathscr{B}}$ follows similarly by the dual argument to the above. This completes the proof that $\rho^t_{\mathscr{B}}$ is a congruence.

We now prove that, with the above notation, $\mathscr{B}$ is the regular kernel of the congruence $\rho^t_{\mathscr{B}}$. The following 'inductive lemma' is used in the proof.

LEMMA 3.3. *Let $s_1, s_2, \cdots, s_{n-1}$ be elements of the orthodox semigroup S, and let s'_i, s''_i be inverses of s_i for $i = 1, \cdots n-1$ such that relative to some regular kernel normal system $\mathscr{B}$ we have*

$$s_r s''_r \sim s_{r+1} s'_{r+1}, \ s''_r s_r \sim s'_{r+1} s_{r+1}, \text{ for } r = 1, \cdots n-2.$$

Then the following formulae hold:

$$s_1 s'_1 \sim (s_{n-1} s'_{n-1})(s_{n-2} s'_{n-2}) \cdots (s_1 s'_1); \tag{7}$$

$$s'_1 s_1 \sim (s'_1 s_1) \cdots (s'_{n-2} s_{n-2})(s'_{n-1} s_{n-1}); \tag{7'}$$

$$s''_{n-1} s_{n-1} \sim (s''_{n-1} s_{n-1}) \cdots (s''_2 s_2)(s''_1 s_1); \tag{8}$$

$$s_{n-1} s''_{n-1} \sim (s_1 s''_1)(s_2 s''_2) \cdots (s_{n-1} s''_{n-1}). \tag{8'}$$

PROOF. To prove (7) we first prove by induction that for $r = 1, \cdots n-1$,

$$s_1 s'_1 \sim (s_r s'_r)(s_{r-1} s'_{r-1}) \cdots (s_1 s'_1). \tag{9}$$

Evidently, (9) holds true for $r = 1$, so suppose that (9) holds for $r = k$. Then

$$\begin{aligned} s_1 s'_1 &\sim (s_k s'_k)(s_{k-1} s'_{k-1}) \cdots (s_1 s'_1) \\ &= (s_k s''_k)(s_k s'_k)(s_{k-1} s'_{k-1}) \cdots (s_1 s'_1) \\ &\sim (s_{k+1} s'_{k+1})(s_k s'_k)(s_{k-1} s'_{k-1}) \cdots (s_1 s'_1), \qquad \text{by K7,} \end{aligned}$$

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and so the result (9) is proved by induction, and (7) follows from (9) by applying (9) with $r = n-1$. The result (7′) follows by the dual of the argument used to prove (7).

To prove (8) we first prove by induction that for $r = 1, \cdots n-1$,

$$(10) \qquad s''_{n-1}s_{n-1} \sim (s''_{n-1}s_{n-1}) \cdots (s''_{n-r+1}s_{n-r+1})(s''_{n-r}s_{n-r}).$$

Clearly (10) holds for $r = 1$, so suppose that (10) holds for $r = k$. Then

$$\begin{aligned} s''_{n-1}s_{n-1} &\sim (s''_{n-1}s_{n-1}) \cdots (s''_{n-k+1}s_{n-k+1})(s''_{n-k}s_{n-k}) \\ &= (s''_{n-1}s_{n-1}) \cdots (s''_{n-k+1}s_{n-k+1})(s''_{n-k}s_{n-k})(s'_{n-k}s_{n-k}) \\ &\sim (s''_{n-1}s_{n-1}) \cdots (s''_{n-k+1}s_{n-k+1})(s''_{n-k}s_{n-k}) \\ &\qquad (s''_{n-k-1}s_{n-k-1}), \qquad \text{by K7.} \end{aligned}$$

Thus the result (10) follows by induction, and (8) follows immediately from (10). As before, the result (8′) is proved by the dual of the argument used to prove (8).

We now proceed to the proof of the statement that $\mathscr{B}$ is the regular kernel of the congruence $\rho^t_{\mathscr{B}}$. The proof of this is contained in the following two lemmas.

LEMMA 3.4. *Let $\mathscr{B} = \{B_i : i \in I\}$ be a regular kernel normal system of the orthodox semigroup S and let $\rho_{\mathscr{B}}$ be defined by equation* (3). *Let $\{A_j : j \in J\}$ be the kernel of the congruence $\rho^t_{\mathscr{B}}$. Then $|I| = |J|$, and it is possible to index the A_i so that for all $i \in I$, B_i is a regular subsemigroup of A_i.*

PROOF. First note that if $a \sim b$, then $(a, b) \in \rho^t_{\mathscr{B}}$: for if $a, b \in B_i$, then there are inverses a' of a and b' of b such that $a', b' \in B_i$. But then aa', bb', ab', $a'a$, $b'b$, $a'b \in B_i$, and hence $(a, b) \in \rho_{\mathscr{B}} \subseteq \rho^t_{\mathscr{B}}$. Thus B_i is a subsemigroup of some $\rho^t_{\mathscr{B}}$-class $a\rho^t_{\mathscr{B}}$. But since B_i contains an idempotent (being regular), we see that $a\rho^t_{\mathscr{B}}$ contains an idempotent, and so $a\rho^t_{\mathscr{B}} = A_j$ for some $j \in J$. Thus each element B_i of $\mathscr{B}$ is a subsemigroup of some element A_j of the kernel of $\rho^t_{\mathscr{B}}$.

It remains to verify that distinct sets B_i and B_j are contained in distinct elements of the kernel of $\rho^t_{\mathscr{B}}$ and that every element of the kernel of $\rho^t_{\mathscr{B}}$ contains some set $B_i \in \mathscr{B}$. The latter assertion follows because every element of the kernel of $\rho^t_{\mathscr{B}}$ contains at least one idempotent of S and every idempotent of S is contained in some element B_i of $\mathscr{B}$ by K3. To verify the former assertion it clearly suffices to prove that if two idempotents of S lie in the same element A_j of the kernel of $\rho^t_{\mathscr{B}}$, then they lie in the same set $B_j \in \mathscr{B}$.

Let $e, f \in E_S$ and suppose that $(e, f) \in \rho^t_{\mathscr{B}}$. Now $\rho^t_{\mathscr{B}} = \bigcup_{n=1}^{\infty} \rho^n_{\mathscr{B}}$, where $\rho^n_{\mathscr{B}}$ is the n-fold composition of $\rho_{\mathscr{B}}$ with itself, and so $(e, f) \in \rho^n_{\mathscr{B}}$ for some $n \geqq 1$. We consider the cases $n = 1$ and $n > 1$ separately. Suppose first that $(e, f) \in \rho_{\mathscr{B}}$. Then there are inverses e' of e and f' of f such that $ee' \sim ff' \sim ef'(\sim fe')$ and $e'e \sim f'f \sim e'f(\sim f'e)$. Then $e = ee'e = e(e'e')e = (ee')(e'e) \sim (ff')(f'f)$ by K7, so $e \sim ff'f = f$, as required.

Now suppose that $(e, f) \in \rho^n_{\mathscr{B}}$ for some $n > 1$. Then there exist $s_1, s_2, \cdots$

$s_{n-1} \in S$ such that $(e, s_1) \in \rho_{\mathscr{B}}$, $(s_1, s_2) \in \rho_{\mathscr{B}}$, $\cdots (s_{n-1}, f) \in \rho_{\mathscr{B}}$, and thus there exist $e' \in V(e), f' \in V(f)$, and $s_i', s_i'' \in V(s_i)$, for $i = 1, \cdots n-1$, such that

$$(11) \quad \begin{cases} ee' \sim s_1 s_1' \sim es_1', & e'e \sim s_1' s_1 \sim e's_1 \\ s_i s_i'' \sim s_{i+1} s_{i+1}' \sim s_i s_{i+1}', & s_i'' s_i \sim s_{i+1}' s_{i+1} \sim s_i'' s_{i+1} \quad \text{for } i = 1, \cdots n-2, \\ s_{n-1} s_{n-1}'' \sim ff' \sim s_{n-1} f', & s_{n-1}'' s_{n-1} \sim f'f \sim s_{n-1}'' f. \end{cases}$$

Now $e = (ee')(e'e) \sim (s_1 s_1')(s_1' s_1)$, by K7. Thus by (7) and (7′) of the inductive lemma, and by K7, we have,

$$\begin{aligned} e &\sim (s_{n-1} s_{n-1}') \cdots (s_2 s_2')(s_1 s_1')(s_1' s_1)(s_2' s_2) \cdots (s_{n-1}' s_{n-1}) \\ &= (s_{n-1} s_{n-1}'')(s_{n-1} s_{n-1}') \cdots (s_2 s_2')(s_1 s_1')(s_1' s_1)(s_2' s_2) \cdots (s_{n-1}' s_{n-1})(s_{n-1}'' s_{n-1}) \\ &\sim (ff')(s_{n-1} s_{n-1}') \cdots (s_2 s_2')(s_1 s_1')(s_1' s_1)(s_2' s_2) \cdots (s_{n-1}' s_{n-1})(f'f) = fsf, \end{aligned}$$

where

$$s = f'(s_{n-1} s_{n-1}') \cdots (s_2 s_2')(s_1 s_1')(s_1' s_1) \cdots (s_{n-1}' s_{n-1})f'.$$

Now $f' = (f'f)(ff') \sim (s_{n-1}'' s_{n-1})(s_{n-1} s_{n-1}'')$, by K7, so

$$\begin{aligned} s &\sim (s_{n-1}'' s_{n-1})(s_{n-1} s_{n-1}'')(s_{n-1} s_{n-1}') \cdots (s_2 s_2')(s_1 s_1')(s_1' s_1)(s_2' s_2) \\ &\qquad \cdots (s_{n-1}' s_{n-1})(s_{n-1}'' s_{n-1})(s_{n-1} s_{n-1}'') \\ &= (s_{n-1}'' s_{n-1})(s_{n-1} s_{n-1}') \cdots (s_1 s_1')(s_1' s_1) \cdots (s_{n-1}' s_{n-1})(s_{n-1} s_{n-1}''). \end{aligned}$$

Thus by (7), (7′), (8) and (8′) of the inductive lemma, and by K7, we have

$$\begin{aligned} s &\sim (s_{n-1}'' s_{n-1}) \cdots (s_2'' s_2)(s_1'' s_1)(s_1 s_1')(s_1' s_1)(s_1 s_1'')(s_2 s_2'') \cdots (s_{n-1} s_{n-1}'') \\ &= (s_{n-1}'' s_{n-1}) \cdots (s_2'' s_2) s_1''[s_1 s_1 s_1' s_1' s_1 s_1] s_1''(s_2 s_2'') \cdots (s_{n-1} s_{n-1}'') \\ &= (s_{n-1}'' s_{n-1}) \cdots (s_2'' s_2)(s_1'' s_1)(s_1 s_1'')(s_2 s_2'') \cdots (s_{n-1} s_{n-1}''), \end{aligned}$$

since $s_1' s_1' \in V(s_1 s_1)$. Hence by (7) and (7′) of the inductive lemma and by K7, we finally obtain

$$s \sim (s_{n-1}'' s_{n-1})(s_{n-1} s_{n-1}'') \sim (f'f)(ff') = f', \quad \text{and so by K7}$$

$e \sim fsf \sim ff'f = f$, as requied. This completes the proof of the lemma.

LEMMA 3.5. *Let $\mathscr{B} = \{B_i : i \in I\}$ be a regular kernel normal system of the orthodox semigroup S and let $\rho_{\mathscr{B}}$ be defined by equation* (3). *Let $\{A_i : i \in I\}$ be the kernel of the congruence $\rho_{\mathscr{B}}^t$, indexed in accordance with lemma* 3.4. *Then for all $i \in I$, B_i is the maximal regular subsemigroup of A_i.*

PROOF. Let a be any element of A_i for which $V(a) \cap A_i \neq \square$, and let $a^* \in V(a) \cap A_i$. We show that in fact $a \in B_i$, from which the result follows by virtue of the characterization (2) of the maximal regular subsemigroups of the A_i.

Let e be any idempotent of B_i. (Such an idempotent exists since B_i is a regular subsemigroup of S.) Now A_i is the $\rho_{\mathscr{B}}^t$-class containing B_i, so $(a, e) \in \rho_{\mathscr{B}}^t$, and hence

$(a, e) \in \rho^n_{\mathscr{B}}$ for some natural number $n \geqq 1$. We consider the cases $n = 1$ and $n > 1$ separately. Note first that $e, aa^*, a^*a \in B_i$.

Suppose first that $(a, e) \in \rho_{\mathscr{B}}$. The there are inverses a' of a and e' of e respectively such that

$$aa' \sim ee' \sim ae' \ (\sim ea') \text{ and } a'a \sim e'e \sim a'e \ (\sim e'a).$$

Let $w = ee'aa^*$. Then $w \sim aa'aa^* = aa^* \in B_i$, by K7. Furthermore, $wa = e(e'a)(a^*a) \in B_j$ for some $j \in I$, by K5. But $w \in B_i$ and $a \in A_i$, so $wa \in B_i$. Hence $w, wa, aa^*, a^*a \in B_i$, and it follows by K6 that $a \in B_i$.

Suppose now that $(a, e) \in \rho^n_{\mathscr{B}}$ for some $n > 1$. Then there are elements $c_1, c_2, \cdots c_{n-1} \in S$ such that $(e, c_1) \in \rho_{\mathscr{B}}, (c_1, c_2) \in \rho_{\mathscr{B}}, \cdots (c_{n-1}, a) \in \rho_{\mathscr{B}}$, and hence there exist $e' \in V(e)$, $a' \in V(a)$, and $c'_i, c''_i \in V(c_i)$, for $i = 1, \ldots n-1$ such that

$$(12)\begin{cases} ee' \sim c_1c'_1 \sim ec'_1, & e'e \sim c'_1c_1 \sim e'e_1 \\ c_rc''_r \sim c_{r+1}c'_{r+1} \sim c_rc'_{r+1}, & c''_rc_r \sim c'_{r+1}c_{r+1} \sim c''_rc_{r+1}, \quad \text{for } r = 1, \cdots n-2, \\ c_{n-1}c''_{n-1} \sim aa' \sim c_{n-1}a', & c''_{n-1}c_{n-1} \sim a'a \sim c''_{n-1}a. \end{cases}$$

Put $w_n = e(c''_{n-1}c_{n-1}) \cdots (c''_1c_1)e'(c_1c''_1) \cdots (c_{n-1}c''_{n-1})(aa^*)$. Then

$$\begin{aligned} w_n &\sim e(c''_{n-1}c_{n-1}) \cdots (c''_1c_1)e'(c_1c''_1) \cdots (c_{n-1}c''_{n-1})e \\ &= e(e'e)(c''_{n-1}c_{n-1}) \cdots (c''_1c_1)e'(c_1c''_1) \cdots (c_{n-1}c''_{n-1})(ee')e \\ &\sim e(c'_1c_1)(c''_{n-1}c_{n-1})e'(c_{n-1}c''_{n-1})(c_1c'_1)e \\ &\sim e(c'_1c_1) \cdots (c'_{n-1}c_{n-1})(c''_{n-1}c_{n-1})e'(c_{n-1}c''_{n-1})(c_{n-1}c'_{n-1}) \cdots (c_1c'_1)e \\ &= e(c'_1c_1) \cdots (c'_{n-1}c_{n-1})e'(c_{n-1}c'_{n-1}) \cdots (c_1c'_1)e \\ &\sim e(c'_1c_1)e'(c_1c'_1)e \\ &\sim e(e'e)e'(ee')e = e. \end{aligned}$$

Hence $w_n \in B_i$. Also,

$$\begin{aligned} w_na &= e(c''_{n-1}c_{n-1}) \cdots (c''_1c_1)e'(c_1c''_1) \cdots (c_{n-1}c''_{n-1})(aa)^*a \\ &= e(c''_{n-1}c_{n-1}) \cdots (c''_1c_1)(e'c_1)(c''_1c_2) \cdots (c''_{n-1}a)(a^*a). \end{aligned}$$

Hence $w_na \in B_k$ for some $k \in I$ by repeated application of K5. But $w_n \in B_i$ and $a \in A_i$, so $w_na \in A_i$. Hence $B_k = B_i$, and so $w_na \in B_i$. Then, since $w_n, w_na, aa^*, a^*a \in B_i$, we deduce that $a \in B_i$, by K6.

We may summarize the results obtained so far in the following theorem.

THEOREM 3.6. *If ρ is a congruence on an orthodox semigroup S then the regular kernel $\mathscr{B}$ of ρ is a regular kernel normal system of S, and $\rho = \rho^t_{\mathscr{B}}$, the transitive closure of the relation $\rho_{\mathscr{B}}$ defined by* (3). *Conversely, if $\mathscr{B}$ is a regular kernel normal system of S, then there is precisely one congruence ρ on S such that $\mathscr{B}$ is the regular kernel of ρ and $\rho = \rho^t_{\mathscr{B}}$.*

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REMARK. In the previous theorem we have been concerned with the transitive closure of the relation $\rho_{\mathscr{B}}$. In fact this complication is forced on us because the relation $\rho_{\mathscr{B}}$ is not necessarily transitive for an arbitrarily prescribed regular kernel normal system $\mathscr{B}$ of an arbitrary orthodox semigroup, as the following example readily shows.

EXAMPLE 3.7. Let S be the semigroup of example 1.4 and consider the relation ρ on S which partitions S into the two classes $S_1 = \{a_{11}, a_{21}, b_{11}, b_{21}\}$ and $S_2 = \{a_{12}, a_{22}, b_{12}, b_{22}\}$. Clearly ρ is a congruence on S, and

$$\mathscr{B} = \{\{a_{11}, b_{11}, b_{21}\}, \{a_{22}, b_{12}, b_{22}\}\}$$

is the regular kernel of ρ. By virtue of theorem 3.6, ρ is the transitive closure of the relation $\rho_{\mathscr{B}}$. But it is easy to see that $\rho_{\mathscr{B}} \neq \rho$, since a_{11} and a_{12} are elements of S which are equivalent under ρ but not under $\rho_{\mathscr{B}}$. (This follows since the only inverse of a_{11} is a_{11} and the only inverse of a_{12} is a_{21}, and $a_{11} = a_{11}a_{11}$ lies in a different element of $\mathscr{B}$ than $a_{22} = a_{21}a_{12}$). Hence for this choice of $\mathscr{B}$ and this choice of S, $\rho_{\mathscr{B}} \neq \rho^t_{\mathscr{B}}$, and so $\rho_{\mathscr{B}}$ is not transitive.

4. Idempotent-separating congruences

A congruence ρ on a semigroup S is called an idempotent-separating congruence if each congruence class contains at most one idempotent of S. Lallement [5] has proved that any idempotent-separating congruence on a regular semigroup is contained in Green's equivalence $\mathscr{H}$. We make use of this result to investigate idempotent-separating congruences on orthodox semigroups.

In theorem 4.2 we obtain a simplification of theorem 3.6 in the case where the congruence considered is an idempotent-separating congruence, and in theorem 4.3 we obtain a necessary and sufficient condition for Green's equivalence $\mathscr{H}$ to be a congruence on an orthodox semigroup. These results may also be deduced from théorème 3.11 (and the ensuing remarks) in [5].

Note first that it ρ is an idempotent-separating congruence on a regular semigroup S, then the kernel of ρ is a set $\mathscr{N} = \{N_e : e \in E_S\}$ of normal subgroups of the set $\{H_e : e \in E_S\}$ of maximal subgroups of S. This is obvious since the restriction to H_e of the natural homomorphism determined by ρ is a group homomorphism of H_e with kernel N_e. In particular, the kernel of ρ is composed of regular subsemigroups of S, and so the kernel of ρ and the regular kernel of ρ coincide.

Now let $\mathscr{N} = \{N_e : e \in E_S\}$ be the kernel of an idempotent-separating congruence ρ on the orthodox semigroup S, and consider the relation

(13) $$\rho_{\mathscr{N}} = \{(a, b) \in S \times S\text{: there are inverses } a' \text{ of } a \text{ and } b' \text{ of } b \text{ such that } aa' = bb' = e,\ ab' \in N_e,\ a'a = b'b = f,\ a'b \in N_f, \text{ for some } e, f \in E_S\}.$$

(Evidently, $\rho_{\mathscr{N}}$ is just a special example of the relation $\rho_{\mathscr{B}}$ defined by (3) corresponding to the idempotent-separating (regular) kernel normal system $\mathscr{N}$). We show that $\rho_{\mathscr{N}}$ is in fact a transitive relation, and hence that $\rho_{\mathscr{N}} = \rho_{\mathscr{N}}^t = \rho$. To prove this, suppose that $(a, b) \in \rho_{\mathscr{N}}$ and $(b, c) \in \rho_{\mathscr{N}}$. Than there are inverses a' of a, b' and b^* of b, and c^* of c, and idempotents e, f, g, h of S such that

$$aa' = bb' = e,\ ab' \in N_e;\ a'a = b'b = f,\ a'b \in N_f,$$

and

$$bb^* = cc^* = g,\ bc^* \in N_g;\ b^*b = c^*c = h,\ b^*c \in N_h.$$

Now $(a, b) \in \rho_{\mathscr{N}} \subseteq \rho_{\mathscr{N}}^t = \rho$, and $(b, c) \in \rho$, so $(a, b) \in \mathscr{H}$ and $(b, c) \in \mathscr{H}$, since $\rho \subseteq \mathscr{H}$. Hence a, b, and c are $\mathscr{H}$-equivalent elements of S, and so there are inverses a^* of a and c' of c such that $aa^* = bb^* = cc^* = g$, $a^*a = b^*b = c^*c = h$, $aa' = bb' = cc' = e$, and $a'a = b'b = c'c = f$. (See for example [1], § 2.3) Now,

$$(ac')(ca') = a(c'c)a' = a(a'a)a' = aa' = e,$$

and similarly $(ca')(ac') = e$. Also

$$(ac')e = (ac')(cc') = ac' = (aa')(ac') = e(ac').$$

Hence $ac' \in H_e$, and by the dual argument $a'c \in H_f$. But

$$\begin{aligned} ac' &= (aa'a)c' = a(b'b)c' = (ab')(bb^*bc') \\ &= (ab')(bc^*cc') = (ab')(bc^*)(cc'), \end{aligned}$$

and

$$ab', bc^*, cc' \in N = \bigcup \{N_e : e \in E_s\}.$$

Hence $ac' \in N$, since N is clearly a subsemigroup of S, being the inverse image under the natural homomorphism corresponding to the congruence $\rho = \rho_{\mathscr{N}}^t$ of the set of idempotents of S/p. Thus $ac' \in N \cap H_e = N_e$. Also, $a'c = a'cc^*c = (a'b)(b^*c) \in N$, so $a'c \in N \cap H_f = N_f$. Thus $aa' = cc' = e$, $ac' \in N_e$ and $a'a = c'c = f$, $a'c \in N_f$. Hence $(a, c) \in \rho_{\mathscr{N}}$ and this completes the proof of the statement that $\rho_{\mathscr{N}}$ is transitive, and hence that $\rho_{\mathscr{N}} = \rho_{\mathscr{N}}^t = \rho$.

We now show that in the idempotent-separating case there is a simple characterization of regular kernel normal systems. Following Preston [7], we define a set $\mathscr{N} = \{N_e : e \in E_S\}$ of normal subgroups of the maximal subgroups $\{H_e : e \in E_S\}$ of the orthodox semigroup S to be a *group kernel normal system* of S if the N_e satisfy the conditions:

(i) $a'N_e a \subseteq N_{a'ea}$ for all $a \in S$, $a' \in V(a)$, and $e \in E_S$;

(ii) $N_e N_f \subseteq N_{ef}$ for all $e, f \in E_S$.

LEMMA 4.1. *A set $\mathscr{N} = \{N_e : e \in E_S\}$ of normal subgroups of the set $\{H_e : e \in E_S\}$ of maximal subgroups of the orthodox semigroup S is a (regular) kernel normal system of S if and only if $\mathscr{N}$ is a group kernel normal system of S.*

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PROOF. If $\mathcal{N} = \{N_e : e \in E_S\}$ is a regular kernel normal system of S, then condition (i) is clearly satisfied. Further, $\rho_{\mathcal{N}}$ is a congruence on S and $\mathcal{N}$ is the kernel of $\rho_{\mathcal{N}}$, so condition (ii) is satisfied. Conversely, suppose that $\mathcal{N}$ satisfies condition (i) and (ii). Then conditions (K4), (K5), and (K7) for regular kernel normal systems are trivially satisfied, and conditons (K1), (K2), and (K3) are automatically satisfied by the definition of $\mathcal{N}$. It remains to verify that (K6) is satisfied. Let a, ab, bb', $b'b \in N_e$ for some $b' \in V(b)$. Then there exists an element $a^* \in V(a) \cap N_e$, and we have $e = a^*a = bb'$. Hence $b = (bb')b = (a^*a)b = a^*(ab)$, the product of two elements of N_e, and so $b \in N_e$. Thus K6 is verified and the lemma is proved.

We may summarize the results of this section in the following theorem.

THEOREM 4.2. *If ρ is an idempotent-separating congruence on an orthodox semigroup S then the kernel $\mathcal{N}$ of ρ is a group kernel normal system of S, and $\rho = \rho_{\mathcal{N}}$, the relation defined by* (13). *Conversely, if $\mathcal{N}$ is a group kernel normal system of S, then there is precisely one congruence ρ on S such that $\mathcal{N}$ is the kernel of ρ. This congruence ρ is an idempotent-separating congruence on S and $\rho = \rho_{\mathcal{N}}$.*

We now determine a necessary and sufficient condition for Green's equivalence $\mathscr{H}$ to be a congruence on an orthodox semigroup. Note first that on a regular semigroup S, $\mathscr{H}$ is given by

$$\mathscr{H} = \{(a, b) \in S \times S : aa' = bb', a'a = b'b \text{ for some } a' \in V(a),\ b' \in V(b)\}. \tag{14}$$

(This is proved in [1], § 2.3). Note also that if $aa' = bb' = e$ and $a'a = b'b = f$, then $ab' \in H_e$ and $a'b \in H_f$. For $(ab')e = (ab')(bb') = ab'$, and $e(ab') = (aa')(ab') = ab'$, while $(ab')(ba') = a(b'b)a' = a(a'a)a' = aa' = e$, and $(ba')(ab') = b(a'a)b' = b(b'b)b' = bb' = e$. Hence $ab' \in H_e$, and by a similar argument $a'b \in H_f$. Thus if $\mathcal{N} = \{H_e : e \in E_S\}$, we see that in fact $\mathscr{H} = \rho_{\mathcal{N}}$, where $\rho_{\mathcal{N}}$ is defined by (13). By virtue of this remark, we see that $\mathscr{H}$ is a congruence on S if and only if $\{H_e : e \in E_S\}$ is a group kernel normal system of S. We are now in a position to prove the following theorem.

THEOREM 4.3. *A necessary and sufficient condition for $\mathscr{H}$ to be a congruence on an orthodox semigroup S is that the set $\{H_e : e \in E_S\}$ of maximal subgroups of S satisfies the condition $H_e H_f \subseteq H_{ef}$, for all $e, f \in E_S$.*

PROOF. Clearly this condition is satisfied if $\mathscr{H}$ is a congruence on S, for then $\{H_e : e \in E_S\}$ is a group kernel normal system of S. Conversely, suppose that $\{H_e : e \in H_S\}$ satisfies the condition $H_e H_f \subseteq H_{ef}$, for all $e, f \in E_S$. To prove that $\mathscr{H}$ is a congruence on S it clearly suffices to prove that H_e satisfy the condition $a'H_e a \subseteq H_{a'ea}$, for all $a \in S$, $a' \in V(a)$, and $e \in E_S$.

Let $a'ha$ be an arbitrary element of $a'H_e a$, and let h' be the inverse of h which is in H_e. Then

$$\begin{aligned}(a'ha)(a'ea) &= (a'hea)(a'ea)\\ &= a'h(eaa')(eaa')a = a'h(eaa')a\\ &= a'hea = a'ha,\end{aligned}$$

and

$$\begin{aligned}(a'ea)(a'ha) &= (a'ea)(a'eha)\\ &= a'(aa'e)(aa'e)ha = a'(aa'e)ha = a'eha = a'ha.\end{aligned}$$

Also, $(a'ha)(a'h'a) = a'(haa'h')a$. But h, h', and e are all in the same class H_e, so by the hypothesis of the theorem, $h(aa')h'$ is in the same $\mathscr{H}$-class as $e(aa')e$. Since both $h(aa')h'$ and $e(aa')e$ are idempotents, we have that $h(aa')h' = e(aa')e$. Hence

$$(a'ha)(a'h'a) = a'(eaa'e)a = (a'ea)(a'ea) = a'ea.$$

A similar argument to the above shows that $h'aa'h = eaa'e$, and hence $(a'h'a)(a'ha) = a'(h'aa'h)a = a'(eaa'e)a = a'ea$. Thus we have proved that $(a'ha)(a'ea) = (a'ea)(a'ha) = a'ha$, and that $(a'ha)(a'h'a) = (a'h'a)(a'ha) = a'ea$, from which it follows that $a'ha \in H_{a'ea}$, for all $h \in H_e$. Hence $a'H_ea \subseteq H_{a'ea}$, and the proof of the theorem is complete.

Finally, we determine the maximal idempotent-separating congruence on an orthodox semigroup, thus generalizing the result of J. M. Howie [4] from inverse semigroups to orthodox semigroups.

THEOREM 4.4. *The maximal idempotent-separating congruence on an orthodox semigroup S is*

$$\mu = \{(a, b) \in S \times S : \text{there are inverses } a' \text{ of } a \text{ and } b' \text{ of } b \text{ such that } a'ea = b'eb \text{ and } aea' = beb' \text{ for all } e \in E_S\}.$$

PROOF. That μ is reflexive and symmetric is obvious. To prove that μ is transitive note first that if $(a, b) \in \mu$, then $a'(aa'bb')a = b'(aa'bb')b$, since $aa'bb' \in E_S$, where a' and b' are the inverses of a and b respectively which appear in the definition of μ. Hence $a'(bb')a = b'(aa')b$. But $bb' \in E_S$, so $a'(bb')a = b'(bb')b = b'b$, and similarly $b'(aa')b = a'a$. Hence $a'a = b'b$. In a similar fashion, it is not difficult to see that $aa' = bb'$. From these two results we deduce that, in particular $\mu \subseteq \mathscr{H}$. We now proceed to the proof of the transitivity of μ.

Suppose that $(a, b) \in \mu$ and $(b, c) \in \mu$. Then there are inverses a' of a, b' and b^* of b, and c^* of c such that

$$a'ea = b'eb,\ aea' = beb',\ b^*eb = c^*ec,\ beb^* = cec^*,\ \textit{for all}\ e \in E_S.$$

In particular, we have seen that this implies that $aa' = bb'$, $a'a = b'b$, $cc^* = bb^*$, and $c^*c = b^*b$, and hence that a, b, and c are $\mathscr{H}$-equivalent elements of S. Hence there are inverses a^* of a and c' of c such that

$$aa' = bb' = cc', \; a'a = b'b = c'c,$$

and

$$aa^* = bb^* = cc^*, \; a^*a = b^*b = c^*c.$$

Now $a^*aa' \in V(a)$, and $c^*cc' \in V(c)$, and for all $e \in E_S$,

$$\begin{aligned}(a^*aa')ea &= (a^*a)(a'ea) = (a^*a)(b'eb)\\ &= (b^*b)(b'eb) = b^*(bb'e)b = c^*(bb'e)c\\ &= c^*(cc'e)c = (c^*cc')ec,\end{aligned}$$

while

$$\begin{aligned}ae(a^*aa') &= a(ea^*a)a' = b(ea^*a)b' = b(eb^*b)b'\\ &= (beb^*)(bb') = (cec^*)(bb') = (cec^*)(cc')\\ &= ce(c^*cc').\end{aligned}$$

Hence $(a, c) \in \mu$, and μ is transitive.

Now suppose that $(a, b) \in \mu$, and let $c \in S$. Then there are inverses a' of a and b' of b such that $a'ea = b'eb$ and $aea' = beb'$, for all $e \in E_S$. Let c' be an arbitrary inverse of c. Then

$$(c'a')e(ac) = c'(a'ea)c = c'(b'eb)c = (c'b')e(bc), \text{ for all } e \in E_S,$$

and

$$(ac)e(c'a) = a(cec')a' = b(cec')b' = (bc)e(c'b'), \text{ for all } e \in E_S.$$

Hence $(ac, bc) \in \mu$, since $c'a' \in V(ac)$ and $c'b' \in V(bc)$. Thus μ is right compatible. Now

$$(ca)e(a'c') = c(aea')c' = c(beb')c' = (cb)e(b'c), \text{ for all } e \in E_S,$$

and

$$(a'c')e(ca) = a'(c'ec)a = b'(c'ec)b = (b'c')e(cb) \text{ for all } e \in E_S,$$

so μ is left compatible. Hence μ is a congruence.

That μ separates idempotents is obvious since we have already proved that $\mu \subseteq \mathscr{H}$.

Finally, let ρ be any idempotent-separating congruence of S. Then if $(a, b) \in \rho$, we have that $(a, b) \in \mathscr{H}$, and hence there are inverses a' of a and b' of b such that $aa' = bb'$ and $a'a = b'b$. Then, since $(a, b) \in \rho$, we have $(aa', ba') \in \rho$, i.e. $(bb', ba') \in \rho$, and hence $(b'bb', b'ba') \in \rho$, i.e. $(b', a') \in \rho$. Hence, for all $e \in E_S$ we have $(aea', beb') \in \rho$, and so $aea' = beb'$ since both aea' and beb' are idempotents, and ρ separates idempotents. Also $(b'eb, a'ea) \in \rho$, and so $a'ea = b'eb$. Thus, finally, $(a, b) \in \mu$, and consequently $\rho \subseteq \mu$. This completes the proof that μ is the maximal idempotent-separating congruence on S.

We remark that *if $(x, y) \in \mu$, and if x^* is an arbitrary inverse of x, then there exists an inverse y^* of y such that $xex^* = yey^*$ and $x^*ex = y^*ey$ for all $e \in E_S$.* For let $(x, y) \in \mu$, and let x^* be an arbitrary inverse of x. Then there are inverses x' of x and y' of y such that $xex' = yey'$ and $x'ex = y'ey$ for all $e \in E_S$. Also, since

$(x, y) \in \mathscr{H}$, there is an inverse y^* of y sich that $xx^* = yy^*$ and $x^*x = y^*y$. Then for all $e \in E_S$,

$$xex^* = xe(x^*xx'xx^*) = x(ex^*x)x'(xx^*)$$
$$= y(ex^*x)y'(xx^*) = y(ey^*y)y'(yy^*) = yey^*,$$

and

$$x^*ex = (x^*xx'xx^*)ex = (x^*x)x'(xx^*e)x$$
$$= (x^*x)y'(xx^*e)y = (y^*y)y'(yy^*e)y = y^*ey.$$

5. Inverse semigroup congruences

A congruence ρ on a semigroup S is called an *inverse semigroup congruence* if S/ρ is an inverse semigroup. In this section we examine inverse semigroup congruences on orthodox semigroups from the point of view of the regular kernel normal systems of the congruences. We also provide an alternative proof of the result of T. E. Hall [3] that Yamada's equivalence

$$(15) \qquad \mathscr{Y} = \{(a, b) \in S \times S : V(a) = V(b)\}$$

(M. Yamada [9]) is a congruence on an orthodox semigroup S, and is the finest inverse semigroup congruence on S.

Let ρ be an inverse semigroup congruence on the orthodox semigroup S and let $\{A_i : i \in I\}$ be the kernel of ρ. Choose $a \in A_i$ and $e \in A_i \cap E_S$. Then $a\phi_\rho = e\phi_\rho \in E_{S/\rho}$, where ϕ_ρ is the natural homomorphism corresponding to the congruence ρ. Let a' be an arbitrary inverse of a. Then $a'\phi_\rho \in V(e\phi_\rho)$, so $a'\phi_\rho = e\phi_\rho$, and so $a' \in A_i$. Thus if $a \in A_i$, we have $V(a) \subseteq A_i$. In particular, each element A_i of the kernel of ρ is regular, and so *the kernel and the regular kernel of an inverse semigroup congruence on an orthodox semigroup coincide.*

Now let s be an arbitrary element of S and let s', $s'' \in V(s)$. Then $s'\phi_\rho$, $s''\phi_\rho \in V(s\phi_\rho)$, and so $s'\phi_\rho = s''\phi_\rho$. Thus ρ identifies all inverses of any given element of S.

Let $\mathscr{A} = \{A_i : i \in I\}$ be the kernel of the inverse semigroup congruence ρ on the orthodox semigroup S. Then we know that $\rho = \rho^t_{\mathscr{A}}$, the transitive closure of the relation $\rho_{\mathscr{A}}$ defined by (3). *We now show that in fact $\rho_{\mathscr{A}}$ is transitive, i.e. that* $\rho_{\mathscr{A}} = \rho^t_{\mathscr{A}} = \rho$. Let $(a, b) \in \rho_{\mathscr{A}}$ and let $(b, c) \in \rho_{\mathscr{A}}$. Then there are inverses a' of a, b' and b^* of b, and c^* of c such that $aa', bb', ab' \in A_i$, $a'a, b'b, a'b \in A_j$, bb^*, $cc^*, bc^* \in A_k$, and $b^*b, c^*c, b^*c \in A_l$ for some $i, j, k, l \in I$. Now $(b', b^*) \in \rho^t_{\mathscr{A}} = \rho$, so $(bb', bb^*) \in \rho$. But $bb' \in A_i$ and $bb^* \in A_k$. Hence $A_i = A_k$, and since $(b'b, b^*b) \in \rho$, we also have that $A_j = A_l$. Hence $aa', cc^* \in A_i$ and $a'a, c^*c \in A_j$. Further, $ab', bc^* \in A_i$, and so $(ab')(bc^*) = a(b'b)c^* \in A_i$. But $(b'b, a'a) \in \rho$, so $(a(b'b)c^*, a(a'a)c^*) \in \rho$, i.e. $ac^* \in A_i$. Also, $a'b, b^*c^* \in A_j$, so $a'(bb^*)c \in A_j$. But $(bb^*, cc^*) \in \rho$, so $(a'(bb^*)c, a'(cc^*)c) \in \rho$, i.e. $a'c \in A_j$. Hence $aa', cc^*, ac^* \in A_i$ and $a'a, c^*c, a'c \in A_j$, and consequently $(a, c) \in \rho_{\mathscr{A}}$. Hence $\rho_{\mathscr{A}}$ is transitive, and $\rho_{\mathscr{A}} = \rho^t_{\mathscr{A}} = \rho$.

We remark further that since $\rho_{\mathscr{A}} = \rho$, and since ρ identifies all inverses of an arbitrary element of S, we clearly have that

$$(16)\quad \begin{array}{l}\rho_{\mathscr{A}} = \{(a,b) \in S \times S : \text{there exist } i \text{ and } j \text{ in } I \text{ such that} \\ aa', bb', ab' \in A_i, a'a, b'b, a'b \in A_j \text{ for all } a' \in V(a), \text{ and all } b' \in V(b)\}.\end{array}$$

We shall make use of the characterization (16) *of inverse semigroup congruences on orthodox semigroups in the sequel.*

Before proceeding to the determination of the finest inverse semigroup congruence on an orthodox semigroup, we investigate some of the properties of the set $V(e)$ of inverses of an idempotent e of the orthodox semigroup S. We already know that $V(e) \subseteq E_S$. Suppose now that $e_1 \in V(e)$. *We prove that under these circumstances,* $V(e) = V(e_1)$. To prove this, let e_2 be an arbitrary element of $V(e)$. Then e_1e_2 and e_2e_1 are both in $V(e)$, and

$$(e_1e_2)e_1 = (e_1e_2ee_1e_2)e_1 = e_1e_2(ee_1)(e_1e_2e_1)(ee_1).$$

But $e_1 \in V(e_1)$ and $e_2e_1 \in V(e)$, so $e_1e_2e_1 \in V(ee_1)$. Hence

$$e_1e_2e_1 = e_1e_2ee_1 = e_1e(e_1e_2)ee_1 = e_1ee_1 = e_1.$$

The result $e_2e_1e_2 = e_2$ follows by interchanging e_1 and e_2 throughout. Hence $e_2 \in V(e_1)$ and so $V(e) \subseteq V(e_1)$. The converse result, $V(e_1) \subseteq V(e)$, follows by symmetry.

From this result we deduce that *if e_1 and e_2 are idempotents of S for which $V(e_1) \cap V(e_2) \neq \square$, then $V(e_1) = V(e_2)$. We also deduce that $V(e)$ is a subsemigroup of mutually inverse idempotents of S.* (Indeed, one can prove that $V(e)$ is a rectangular band). We make use of these results in the proof of the following theorem, due to T. E. Hall [3].

THEOREM 5.1. *The finest inverse semigroup congruence on an orthodox semigroup S is Yamada's equivalence $\mathscr{Y}$, defined by* (15).

PROOF. We first prove that $\mathscr{V} = \{V(e) : e \in E_S\}$ is a regular kernel normal system of S. That $V(e)$ is a regular subsemigroup of S is obvious, since if $a, b \in V(e)$ then $ab \in V(ee) = V(e)$. We have already proved that $V(e) \cap V(f) = \square$ if $V(e) \neq V(f)$, and it is obvious that $E_S \subseteq \cup \{V(e) : e \in E_S\}$. Hence $\mathscr{V}$ satisfies conditions K1, K2, and K3 of regular kernel normal systems. That $a'V(e)a \subseteq V(a'ea)$ for all $a \in S$, $a' \in V(a)$, and $e \in E_S$ is also obvious. To verify K5 and K7, note that if $e_1 \in V(e)$ and $f_1 \in V(f)$, then $e_1f_1 \in V(fe) = V(ef)$. Hence $V(e)V(f) \subseteq V(ef)$. Finally to verify that K6 is satisfied, suppose that $a, ab, bb', b'b \in V(e)$. Then $a \in V(bb')$, $bb' = bb'abb'$, $b = bb'b = bb'abb'b = (bb')(ab) \in V(e)$, and K6 is verified. Thus we have established that $\mathscr{V}$ forms the regular kernel of some congruence $\rho = \rho^t_{\mathscr{V}}$ on S. We note that each idempotent of S/ρ has a unique inverse in S/ρ, and hence the idempotents of S/ρ commute. Thus S/ρ is an inverse semigroup.

It follows that ρ is an inverse semigroup congruence on S and that

$$\rho = \rho_{\mathscr{V}} = \{(a, b) \in S \times S : \text{there are elements } e, f \in E_s \text{ such that } aa', bb', ab', ba' \in V(e), a'a, b'b, a'b, b'a \in V(f), \text{ for all } a' \in V(a), b' \in V(b)\},$$

and that $\mathscr{V}$ is the kernel of $\rho_{\mathscr{V}}$. Clearly, any inverse semigroup congruence on S must identify all inverses of an arbitrary idempotent, and hence it follows that $\rho_{\mathscr{V}}$ is the finest inverse semigroup congruence on S.

It remains to be proved that $\rho_{\mathscr{V}} = \mathscr{Y}$. Let $(a, b) \in \mathscr{Y}$, and let a' be an arbitrary inverse of a (and hence of b). Then $aa'ba'aa' = a(a'ba') = aa'$, since $b \in V(a')$, and $ba'aa'ba' = b(a'ba') = ba'$. Hence $aa' \in V(ba')$, and so $V(aa') = V(ba')$. Also, $a'aa'ba'a = (a'ba')a = a'a$, and $a'ba'aa'b = a'(ba'b) = a'b$, so $V(a'a) = V(a'b)$. Hence there is an inverse a' of a (and of b) such that $aa', ba' \in V(aa')$ and $a'a, a'b \in V(a'a)$ and it follows immediately that $(a, b) \in \rho_{\mathscr{V}}$, and hence that $\mathscr{Y} \subseteq \rho_{\mathscr{V}}$.

Suppose now that $(a, b) \in \rho_{\mathscr{V}}$. Then there are idempotents e and f of S such that $aa', bb', ab', ba' \in V(e)$ and $a'a, b'b, a'b, b'a \in V(f)$ for all $a' \in V(a), b' \in V(b)$. Let b' be any inverse of b. Then $ab'a = (ab'ba'ab')a$, since $ab' \in V(ba')$. Hence

$$\begin{aligned} ab'a &= ab'ba'(aa'ab'aa')a = ab'ba'(aa')a \\ &= ab'ba'a = a(a'ab'ba'a) = aa'a = a, \end{aligned}$$

while

$$b'ab' = (b'aa'bb'a)b', \quad \text{since } a'b \in V(b'a),$$

and so

$$\begin{aligned} b'ab' &= b'aa'b(b'bb'ab'b)b' = b'aa'b(b'b)b' \\ &= b'aa'bb' = b'(bb'aa'bb') = b'bb' = b', \text{ so } b' \in V(a). \end{aligned}$$

Hence $V(b) \subseteq V(a)$, and by symmetry, $V(a) \subseteq V(b)$. It follows that $(a, b) \in \mathscr{Y}$, and hence that $\rho_{\mathscr{V}} \subset \mathscr{Y}$. Hence $\rho_{\mathscr{V}} = \mathscr{Y}$, as stated.

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