BULL. AUSTRAL. MATH. SOC. 16P40, 16D40, 16D50, 16D70
VOL. 62 (2000) [159–164]

DECOMPOSITIONS OF MODULES INTO PROJECTIVE MODULES AND CS-MODULES

SOMYOT PLUBTIENG

Let M be a right R-module. It is shown that M is a locally Noetherian module if every finitely generated module in $\sigma[M]$ is a direct sum of a projective module and a CS-module. Moreover, if every module in $\sigma[M]$ is a direct sum of a projective module and a CS-module, then every module in $\sigma[M]$ is a direct sum of modules which are either indecomposable projective or uniform $\sum$-quasi-injective. In particular, if every module in $\sigma[M]$ is a direct sum of a projective module and a quasi-continuous module, then every module in $\sigma[M]$ is a direct sum of a projective module and a quasi-injective module.

1. INTRODUCTION

A module M is called a CS-module (or extending module [5]) if every submodule of M is essential in a direct summand of M. CS-modules provide a useful generalisation of (quasi-)injective modules and (quasi-)continuous modules (see [11]). The study of rings over which finitely generated right modules are CS was initiated by Dung and Smith [4]. It was shown further in Huynh, Rizvi and Yousif [9] and Vanaja [12] that such rings must be right Noetherian. Huynh and Rizvi [10] recently investigated rings over which every countably generated right module is a direct sum of a projective module and a CS-module, and they showed that these rings form a special class of right Artinian rings. They gave also several characterisations of rings over which every (countably generated) right R-module is a direct sum of a projective module and a quasi-continuous module.

In this paper, we use module-theoretic methods to consider the related properties in more general settings. First, we show that a module M is locally Noetherian if every finitely generated module in $\sigma[M]$ is a direct sum of a projective module and a CS-module. Further, we study the modules M satisfying the stronger property that every module in $\sigma[M]$ is a direct sum of a projective module and a CS-module. We show that such modules M turn out to be pure semisimple in the sense of Wisbauer [13, Section 53], and

Received October, 1999
I would like to thank Professor Nguyen Viet Dung for pointing out Proposition 3 and for his several helpful suggestions. Moreover I wish to express my thanks to Professor Sompong Dhompongsa for drawing my attention to the subject, and for many useful discussions.

https://doi.org/10.1017/S000497270001858X Published online by Cambridge University Press

every module in $\sigma[M]$ is a direct sum of indecomposable projective modules and uniform $\sum$-quasi-injective modules. As a consequence, we deduce that if every module in $\sigma[M]$ is a direct sum of a projective module and a quasi-continuous module, then every module in $\sigma[M]$ is a direct sum of a projective module and a quasi-injective module. Specialising to the special case when $M_R = R_R$, our results provide new additional information on certain classes of Artinian rings studied recently by Huynh and Rizvi [**10**].

2. The results

Throughout this paper we consider associative rings R with identity and unitary right R-modules. For a right R-module M, $\sigma[M]$ will denote the category of all right R-modules which are submodules of M-generated modules. For basic definitions and properties of rings, modules and categories we refer to Anderson and Fuller [**1**] and Wisbauer [**13**].

We shall consider the following two conditions on a right R-module M:

(∗) Every finitely generated module in $\sigma[M]$ is a direct sum of a projective module and a CS-module;

(∗∗) Every module in $\sigma[M]$ is a direct sum of a projective module and a CS-module.

We start our investigation by proving the following result.

Theorem 1. *Let M be a right R-module satisfying (∗). Then M is locally Noetherian.*

Proof: Let M be a right R-module satisfying (∗) and let N be a finitely generated submodule of M. We first aim to show that $N/\operatorname{Soc}(N)$ is Noetherian. Let E be an essential submodule of N, and set $K = N/E$. Then K is a singular module. Clearly every finitely generated module in $\sigma[K]$ can not contain nonzero projective submodules. Thus, by (∗), every finitely generated module in $\sigma[K]$ is CS. Then, by [**9**, Theorem 5], it follows that K is Noetherian. Therefore, N has ACC on essential submodules, hence $N/\operatorname{Soc}(N)$ is Noetherian by [**5**, Theorem 5.15 (1)].

We show now that $\operatorname{Soc}(N)$ is finitely generated, which would imply that N is Noetherian. Assume on the contrary that $\operatorname{Soc}(N)$ is infinitely generated. Then we may write $\operatorname{Soc}(N) = H_1 \oplus H_2$, where H_1 and H_2 are infinite direct sums of simple modules.

By hypothesis, we have $N/H_1 = \overline{P_1} \oplus \overline{Q_1}$ where $\overline{P_1}$ is a projective module and $\overline{Q_1}$ is a CS-module. Let Q_1 be the inverse image of $\overline{Q_1}$ in N. Then clearly $\overline{P_1} \simeq N/Q_1$, and Q_1/H_1 (being isomorphic to $\overline{Q_1}$) is a CS-module. Since $\overline{P_1}$ is projective, $N = Q_1 \oplus Q_2$ for some submodule Q_2 of N. Then $\operatorname{Soc}(N) = \operatorname{Soc}(Q_1) \oplus \operatorname{Soc}(Q_2)$.

Observe that, because $\overline{Q_1}/\operatorname{Soc}\left(\overline{Q_1}\right)$ is Noetherian by the above argument, and $\overline{Q_1}$ is a finitely generated CS-module, it follows from [**5**, Lemma 9.1] that $\operatorname{Soc}\left(\overline{Q_1}\right)$ is finitely generated. Hence $\overline{Q_1}$, and so Q_1/H_1, has finite uniform dimension. Therefore, this clearly implies that $\operatorname{Soc}(Q_2)$ is infinitely generated.

Note that

$$N/\operatorname{Soc}(N) \simeq \big(Q_1/\operatorname{Soc}(Q_1)\big) \oplus \big(Q_2/\operatorname{Soc}(Q_2)\big),$$

where $Q_1 \neq \operatorname{Soc}(Q_1)$ and $Q_2 \neq \operatorname{Soc}(Q_2)$. Hence, $N/\operatorname{Soc}(N)$ has uniform dimension at least 2. Applying the same arguments to the module Q_2, and continuing the process in a similar manner, an obvious induction shows that $N/\operatorname{Soc}(N)$ has infinite uniform dimension, which is a contradiction to the fact that $N/\operatorname{Soc}(N)$ is Noetherian. This shows that $\operatorname{Soc}(N)$ is finitely generated, and therefore N is Noetherian, completing our proof. ∎

From Theorem 1 we obtain immediately the following consequence.

Corollary 2. *Let R be a ring such that every finitely generated right R-module is a direct sum of a projective module and a CS-module. Then R is right Noetherian.*

We now prove the following fact which will be crucial for the proof of our main result. Recall that a module N in $\sigma[M]$ is called $\sum$-pure-injective in $\sigma[M]$ if every direct sum of copies of M is pure-injective in $\sigma[M]$. A module M is called pure semisimple if every module in the category $\sigma[M]$ is pure-injective. In this case, $\sigma[M]$ is called a pure semisimple category (see, for example, [**13**]).

Proposition 3. *Let M be a module and suppose that there is a cardinal number* $\mathbf{c}$ *such that every module in $\sigma[M]$ is a direct sum of* $\mathbf{c}$*-generated modules. Then every module in $\sigma[M]$ is a direct sum of modules with local endomorphism rings.*

Proof: It follows from Garcia and Martinez Hernandez [**8**] (see Garcia and Dung [**7**, Theorem 2.4]) that a pure-injective module N in $\sigma[M]$ is $\sum$-pure-injective if and only if there is an infinite cardinal number $\mathbf{m}$ such that the pure-injective envelope in $\sigma[M]$ of any direct sum of copies of N is a direct sum of $\mathbf{m}$-generated modules. Hence, our hypothesis combined with this result implies that every pure-injective module in $\sigma[M]$ is $\sum$-pure-injective, hence is a direct sum of indecomposable modules with local endomorphism rings. This implies that $\sigma[M]$ is a pure semisimple category, so every module in $\sigma[M]$ is a direct sum of modules with local endomorphism rings (see, for example, [**8**]). ∎

We are now in a position to prove the main result.

Theorem 4. *Let M be a right R-module satisfying $(**)$. Then every module N in $\sigma[M]$ has a decomposition $N = \bigoplus_{i\in I} N_i$, where for each $i \in I$, either N_i is indecomposable projective or N_i is uniform Σ-quasi-injective.*

Proof: First we show that there exists a cardinal number $\mathbf{c}$ such that each module $N \in \sigma[M]$ is a direct sum of $\mathbf{c}$-generated modules. It follows from Theorem 1 that M is a locally Noetherian module. Let N be any module in $\sigma[M]$. By the condition $(**)$, we have that $N = P \oplus K$, where P is a projective module and K is a CS-module. By Kaplansky's Theorem (see, for example, [**1**, Corollary 26.2]), P is a direct sum of countably generated modules.

Note that K is a locally Noetherian CS-module. Hence by [**5**, Corollary 8.3], K has a decomposition $K = \bigoplus_{j \in J} K_j$, where each K_j is an uniform module. For each K_j, we consider the M-injective envelope $E(K_j)$ of K_j (that is, the injective envelope of K_j in $\sigma[M]$). Since the category $\sigma[M]$ has a generating set consisting of finitely generated modules, clearly the collection of all isomorphism classes of uniform M-injective modules forms a set, implying that the collection of all isomorphism classes of uniform modules in $\sigma[M]$ is also a set. Hence there exists an infinite cardinal number $\mathbf{c}$ such that every uniform module in $\sigma[M]$ is $\mathbf{c}$-generated. Therefore, the module N in $\sigma[M]$ has a decomposition $N = \bigoplus_{i \in I} N_i$, where each N_i is a $\mathbf{c}$-generated module. By Proposition 3, we get that $\sigma[M]$ is a pure semisimple category, and so every module N in $\sigma[M]$ is a direct sum of modules with local endomorphism rings.

Finally we show that every indecomposable direct summand of N is projective or Σ-quasi-injective. Let U be any indecomposable direct summand of N, and assume that U is not projective. Consider the module $U^{(I)}$, where I is any index set. By the condition $(**)$, we know that $U^{(I)} = Q \oplus Y$ where Q is projective and Y is CS. If $Q \neq 0$, then by Azumaya's Theorem (see [**1**, Theorem 12.6]) Q must contain a direct summand isomorphic to U. Hence U is projective, which is a contradiction. This implies that $Q = 0$, and so $U^{(I)} = Y$ is a CS-module. Hence, $U^{(I)}$ is a CS-module for each index set I, that is, U is Σ-CS in the sense of [**3**] (see Clark and Wisbauer [**2**]). Now we shall use an argument in [**4**, Theorem 7, p.279] to show that U is Σ-quasi-injective.

Let $V = \bigoplus_{i=1}^{\infty} U_i$, with $U_i \simeq U$ for all i. Because V is a CS-module and $\operatorname{End}(U_i)$ is local for each i, the family $\{U_i \mid i \geqslant 1\}$ is locally semi-T-nilpotent (see [**3**, Theorem 2.4]). Let $\theta : U \to U$ be any monomorphism, and suppose that θ is not an isomorphism. By the locally semi-T-nilpotency of $\{U_i \mid i \geqslant 1\}$, it follows that, for any $x \in U$, there is a positive integer n such that $\theta^n(x) = 0$, which implies that $x = 0$, a contradiction. Thus any monomorphism $\theta : U \to U$ is an isomorphism. Since $U \oplus U$ is CS, by [**4**, Lemma 3(b)], it follows that U is U-injective, that is, U is quasi-injective. It follows now from [**5**, Corollary 8.10] that U is Σ -quasi-injective since U is Σ-CS. This completes the proof. ▯

The next result can also be derived from [**10**, Theorem 5] which was proved by different techniques.

Proposition 5. *If every right R-module is a direct sum of a projective module and a CS-module then R is a right Artinian ring.*

Proof: Under our hypothesis, it follows from the proof of Theorem 4 (for the case $M_R = R_R$) that there is a cardinal number $\mathbf{c}$ such that every right R-module is a direct sum of $\mathbf{c}$-generated modules. Hence, by [**6**, Theorem 20.23], R is a right Artinian ring. ▯

Finally, we consider the modules M satisfying the property that every module $N \in \sigma[M]$ is a direct sum of a projective module and a quasi-continuous module.

Theorem 6. *The following conditions are equivalent for a right R-module M:*

(1) *Every module $N \in \sigma[M]$ is a direct sum of a projective module and a quasi-continuous module;*

(2) *Every module $N \in \sigma[M]$ is a direct sum of a projective module and a quasi-injective module.*

Proof: (2)⇒ (1) is clear.

(1)⇒(2) Assume that (1) holds. Let N be any module in $\sigma[M]$. Then $N = P \oplus K$, where P is projective and K is quasi-continuous. By Theorem 4, it follows that $K = \bigoplus_{i \in I} K_i$, where each K_i is indecomposable, hence uniform. Without loss of generally we clearly may assume that each K_i is non-projective. Thus, by Theorem 4, each K_i is quasi-injective. Because K is quasi-continuous, it follows by [**11**, Theorem 2.13] that for each $j \in I$, $\left(\bigoplus_{i \neq j} K_i\right)$ is K_j-injective. Hence, by [**11**, Proposition 1.18], this implies that $K = \bigoplus_{i \in I} K_i$ is quasi-injective. Therefore, N is a direct sum of a projective module and a quasi-injective module. □

We conclude the paper with some remarks.

Remarks.

(a) The results in this paper remain true (with similar arguments) if the conditions (*) and (**) are replaced by the weaker ones that every (finitely generated) module in $\sigma[M]$ is a direct sum of a module which is projective in $\sigma[M]$ and a CS-module.

(b) Rings satisfying the property that every right R-module is a direct sum of a projective module and a quasi-injective module have recently been studied by Huynh and Rizvi [**10**]. We refer to this work for several characterisations and ideal-theoretic descriptions of these rings.

References

[1] F.W. Anderson and K.R. Fuller, *Rings and categories of modules* (Springer-Verlag, Berlin, Heidelberg, New York, 1974).

[2] J. Clark and R. Wisbauer, '$\sum$-extending modules', *J. Pure Appl. Algebra* **104** (1995), 19–32.

[3] N.V. Dung, 'On indecomposable decompositions of CS-modules II', *J. Pure Appl. Algebra* **119** (1997), 139–153.

[4] N.V. Dung and P.F. Smith, 'Rings for which certain modules are CS', *J. Pure Appl. Algebra* **102** (1995), 273–287.

[5] N.V. Dung, D.V. Huynh, P.F. Smith and R. Wisbauer, *Extending modules*, Pitman Research Notes in Mathematics Series **313** (Longman, Harlow, 1994).

[6] C. Faith, *Algebra II: Ring theory* (Springer-Verlag, Berlin, Heidelberg, New York, 1976).

[7] J.L. Garcia and N.V. Dung, 'Some decomposition properties of injective and pure-injective modules', *Osaka J. Math.* **31** (1994), 95–108.

[8] J.L. Garcia and J. Martinez Hernandez, 'Purity through Gabriel's functor rings', *Bull. Soc. Math. Belgique* **31** (1994), 95–108.

[9] D.V. Huynh, S.T. Rizvi and M.F. Yousif, 'Rings whose finitely generated modules are extending', *J. Pure Appl. Algebra* **111** (1996), 325–328.

[10] D.V. Huynh and S.T. Rizvi, 'On some classes of artinian rings', *J. Algebra* (to appear).

[11] S.H. Mohamed and B.J. Müller, *Continuous and discrete modules*, London Math. Soc. Lecture Notes **147** (Cambridge Univ. Press., Cambridge, 1990).

[12] N. Vanaja, 'All finitely generated M-subgenerated modules are extending', *Comm. Algebra* **24** (1996), 543–572.

[13] R. Wisbauer, *Foundations of module and ring theory* (Gordon and Breach, Philadelphia, P.A., 1991).

Department of Mathematics
Naresuan Unviersity
Phitsanulok 65000
Thailand
e-mail: somyotp@nu.ac.th