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On Weak* Kadec–Klee Norms

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Abstract. We present partial positive results supporting a conjecture that admitting an equivalent Lipschitz (or uniformly) weak* Kadec–Klee norm is a three space property.

1 Introduction

A norm $\|\cdot\|$ on a Banach space X is called *Lipschitz weak* Kadec–Klee* (LKK*) if there exists $c \geq 0$ such that for every $\varepsilon > 0$, every $f \in X^*$ and every $f_n \in X^*$ such that $\|f_n\| \leq 1$, $f = w^*\text{-}\lim_n f_n$ and $\|f - f_n\| > \varepsilon$, one has $\|f\| \leq 1 - c\varepsilon$. To emphasize the role of c, we call a norm having the above property c-LKK*.

A norm $\|\cdot\|$ on a Banach space X is called *uniformly weak* Kadec–Klee*(UKK*) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $f \in X^*$ and every $f_n \in X^*$ satisfying $\|f_n\| \leq 1$, $f = w^*\text{-}\lim_n f_n$, and $\|f - f_n\| > \varepsilon$, one has $\|f\| \leq 1 - \delta$.

Lipschitz weak* Kadec–Klee norms are usefull tools when considering Lipschitz isomorphisms of Banach spaces, see [2]. We refer to [5, 6] for more information on UKK* norms.

In this note we try to answer the question whether admitting an equivalent LKK* norm is a three space property. A property P of a Banach space is called a three space property if X has P whenever there exists a closed subspace Z of X such that both Z and X/Z have P. We provide a positive answer in the case when $Z = c_0(\Gamma)$ and the dual unit ball of X/Z is an angelic space in its weak* topology. The same result is shown for UKK* norms. We also show that a Banach space X contains a copy of $c_0(\Gamma)$ where $|\Gamma| = w^*\text{-}\operatorname{dens} X^*$ provided X admits an equivalent LKK* norm and $w^*\text{-}\operatorname{dens} X^* \geq \omega_1$.

We note that a three space problem for either Lipschitz weak* or uniformly weak* Kadec–Klee norms is still an open problem.

Recall that a compact space K is called *angelic* if whenever $x \in \overline{A}$ for $A \subset K$, there is a sequence $x_n \in A$ such that $x = \lim_n x_n$. Let us note that Corson compacts are angelic and thus the dual unit ball of a weakly compactly generated space is angelic, see [3, Corollary 11.13, Theorem 12.50 and Exercise 12.55].

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2 Three Space Problem for LKK* Norms

Theorem 2.1 *Asume that X is a Banach space, $c_0(\Gamma) \subset X$, $X/c_0(\Gamma)$ admits an equivalent c-LKK* norm with $c \in (0,1)$, and that the dual unit ball of $(X/c_0(\Gamma))^*$ is an angelic space. Then X admits an equivalent c-LKK* norm.*

Proof We will proceed along the lines of the proof of Example 1 in [4] and Theorem 1 in [7]. Let Y stand for the space $X/c_0(\Gamma)$. First, we identify $X^* = \ell_1(\Gamma) \oplus Y^*$, with the duality given by

$$\langle (f,g), x\rangle = \langle l(f), x\rangle + \langle q^*(g), x\rangle, \quad f \in \ell_1(\Gamma),\, g \in Y^*,$$

where $q^*\colon Y^* \mapsto X^*$ is induced by the quotient map $q\colon X \mapsto Y = X/c_0(\Gamma)$ and $l\colon \ell_1(\Gamma) \mapsto X^*$ is a so-called lifting map satisfying $i^*(l(f)) = f$, $f \in \ell_1(\Gamma)$. (Here $i^*\colon X^* \mapsto \ell_1(\Gamma) = c_0(\Gamma)^*$ is the dual map to the inclusion $i\colon c_0(\Gamma) \mapsto X$.)

In the sequel, we are going to use the following lemma, which follows from the fact that Y^* can be identified with $c_0(\Gamma)^\perp = \{x^* \in X^* : x^*(x) = 0 \text{ for all } x \in c_0(\Gamma)\}$.

Lemma 2.2 *If $w^*\text{-}\lim_\alpha (f_\alpha, g_\alpha) = (f,g)$ in X^*, then $w^*\text{-}\lim_\alpha f_\alpha = f$ in $c_0(\Gamma)^*$.*

Assume that Y admits an equivalent c-LKK* norm with $c \in (0,1)$, and let $\|\cdot\|_{Y^*}$ be the dual norm on Y^*. Let $\|\cdot\|_{X^*}$ be a dual norm on X^* that extends $\|\cdot\|_{Y^*}$, and let $\|\cdot\|_{\ell_1(\Gamma)}$ be the standard norm on $\ell_1(\Gamma)$. Let $A_0 \geq 1$ be such that

$$\frac{1}{\sqrt{A_0}}\|(f,g)\|_{X^*} \leq \|f\|_{\ell_1(\Gamma)} + \|g\|_{Y^*} \leq \sqrt{A_0}\|(f,g)\|_{X^*}, \quad (f,g) \in X^*. \tag{2.1}$$

For all $A > 0$ we define

$$|||(f,g)|||_A = A\|f\|_{\ell_1(\Gamma)} + \|g\|_{Y^*}, \quad (f,g) \in X^*. \tag{2.2}$$

Lemma 2.3 *For all $A \geq A_0$, the norm $|||\cdot|||_A$ defined by (2.2) is a dual norm on X^*.*

Proof We will follow the proof published in [4]. We need to show that the unit ball $B = \{(f,g) \in X^* : |||(f,g)|||_A \leq 1\}$ is weak* closed. Let $\{(f_\alpha, g_\alpha)\}_{\alpha\in I} \subset B$ be a net converging to (f,g) in the weak* topology of X^*. For every $\alpha \in I$ we write $f_\alpha = f^1_\alpha + f^2_\alpha$, where $f^1_\alpha, f^2_\alpha \in \ell_1(\Gamma)$ have disjoint supports and

$$\lim_{\alpha\in I} \|f^1_\alpha - f\|_{\ell_1(\Gamma)} = 0. \tag{2.3}$$

To get this decomposition, enumerate the support of f by $\{\gamma_n\}$ and find $\alpha_n \in I$, $n \in \mathbb{N}$, such that

$$|f_\alpha(\gamma_i) - f(\gamma_i)| < 2^{-n}, \quad \text{if } i = 1,\dots,n \text{ and } \alpha > \alpha_n.$$

Then set

$$f^1_\alpha = f_\alpha\restriction_{\{\gamma_1,\dots,\gamma_n\}}, \quad \text{if } \alpha > \alpha_n \text{ and } \alpha \not> \alpha_{n+1},$$
$$f^2_\alpha = f_\alpha - f^1_\alpha, \quad \alpha \in I.$$

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Since $\{(f^2_\alpha, 0)\}$ is a bounded net in X^*, we may assume without loss of generality that $\{(f^2_\alpha, 0)\}$ converges to (f', g') in the weak* topology of X^*. By Lemma 2.2 and the construction above, $f' = 0$. Thus $\{(0, g_\alpha)\}$ is weak* convergent to $(0, g'') = (0, g - g')$. Using weak* lower semicontinuity of the dual norms, by (2.3), (2.2) and disjoint supports of f^1_n, f^2_n, we get

$$
\begin{aligned}
|||(f,g)|||_A &= A\|f\|_{\ell_1(\Gamma)} + \|g\|_{Y^*}\\
&\leq A \liminf_{\alpha\in I} \|f^1_\alpha\|_{\ell_1(\Gamma)} + \|g'\|_{Y^*} + \|g''\|_{Y^*}\\
&\leq A \liminf_{\alpha\in I} \|f^1_\alpha\|_{\ell_1(\Gamma)} + \sqrt{A_0}\|(0,g')\|_{X^*} + \liminf_{\alpha\in I}\|g_\alpha\|_{Y^*}\\
&\leq A \liminf_{\alpha\in I} \|f^1_\alpha\|_{\ell_1(\Gamma)} + \sqrt{A_0}\liminf_{\alpha\in I}\|(f^2_\alpha,0)\|_{X^*} + \liminf_{\alpha\in I}\|g_\alpha\|_{Y^*}\\
&\leq A \liminf_{\alpha\in I} \|f^1_\alpha\|_{\ell_1(\Gamma)} + A_0\liminf_{\alpha\in I}\|f^2_\alpha\|_{\ell_1(\Gamma)} + \liminf_{\alpha\in I}\|g_\alpha\|_{Y^*}\\
&\leq \liminf_{\alpha\in I}|||(f_\alpha,g_\alpha)|||_A \leq 1.
\end{aligned}
$$

This finishes the proof of Lemma 2.3. ∎

In order to prove Theorem 2.1, we select a number $A \geq A_0$ such that $\frac{A-A_0}{A+A_0} > c$. We must show that for a given $\varepsilon > 0$ and a sequence $(f_n, g_n) \in X^*$ satisfying $|||(f_n,g_n)|||_A \leq 1$, $w^*\text{-}\lim_n (f_n, g_n) = (f, g)$, and $|||(f_n, g_n) - (f, g)|||_A > \varepsilon$, one has $|||(f,g)|||_A \leq 1 - c\varepsilon$. Let $\varepsilon > 0$ and $\{(f_n, g_n)\}$ be as above. Enumerate the support of f by $\{\gamma_n\}$ and for each $n \in \mathbb{N}$ set $f^1_n = f_n \restriction_{\{\gamma_1,\dots,\gamma_n\}}$, $f^2_n = f_n - f^1_n$. It follows that

$$\lim_{n\to\infty} \|f^1_n - f\|_{\ell_1(\Gamma)} = 0. \tag{2.4}$$

Since $\{(f^2_n, 0)\}$ is a bounded net in X^*, it has a weak* accumulation point $(f', g') \in X^*$. By Lemma 2.2, f' is a weak* accumulation point of f^2_n in the weak* topology of $\ell_1(\Gamma)$. It follows that $f' = 0$, because

$$
f^2_n(\gamma) = \begin{cases} 0 & \text{if } \gamma \in \operatorname{spt} f \text{ and } n \text{ is large enough,}\\ f_n(\gamma) & \text{if } \gamma \notin \operatorname{spt} f. \end{cases}
$$

Thus $(0, g - g')$ is a weak* accumulation point of the sequence $\{(0, g_n)\}$. Hence, $g'' = g - g'$ is a weak* accumulation point of g_n in Y^*. Since the dual unit ball of Y^* is angelic, there is a sequence $\{g_{n_k}\}$ weak* converging to g''.

After passing to a subsequence if necessary, we may summarize the properties of $\{(f_n, g_n)\}$:

(i) $|||(f_n, g_n) - (f, g)|||_A > \varepsilon$ for all $n \in \mathbb{N}$,
(ii) $f_n = f^1_n + f^2_n$ and $\operatorname{spt} f^1_n \cap \operatorname{spt} f^2_n = \varnothing$ for each $n \in \mathbb{N}$,
(iii) $\lim_n \|f_n - f^1_n\|_{\ell_1(\Gamma)} = 0$,
(iv) $w^* - \lim_n (f^2_n, 0) = (0, g')$ in X^* and $w^* - \lim_n g_n = g''$ in Y^*,
(v) $\lim_n \|f^2_n\|_{\ell_1(\Gamma)} = \liminf_n \|f^2_n\|_{\ell_1(\Gamma)}$ and $\lim_n \|g_n\|_{Y^*} = \liminf_n \|g_n\|_{Y^*}$.

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By (i) above and the triangle inequality we have

$$\begin{aligned}\varepsilon &< |||(f_n, g_n) - (f, g)|||_A \\ &\le |||(f_n^1, 0) - (f, 0)|||_A + |||(f_n^2, 0) - (0, g')|||_A + |||(0, g_n) - (0, g'')|||_A.\end{aligned}$$

As $\lim_n \|f_n - f^1\|_{\ell_1(\Gamma)} = 0$, we may assume by omitting finitely many n's that

$$\begin{aligned}\varepsilon &< |||(f_n^2, 0) - (0, g')|||_A + |||(0, g_n) - (0, g'')|||_A \\ &= |||(f_n^2, 0) - (0, g')|||_A + \|g_n - g''\|_{Y^*}\end{aligned} \tag{2.5}$$

for all $n \in \mathbb{N}$.

The next step of the proof is based on the following elementary lemma whose proof is omitted.

Lemma 2.4 *Let $\{a_n\}$, $\{b_n\}$ be bounded sequences of nonnegative real numbers such that $a_n + b_n > \varepsilon$ for all $n \in \mathbb{N}$. Then for every $\eta > 0$ there exist $\varepsilon_1, \varepsilon_2 \ge 0$ and an infinite set $M \subset \mathbb{N}$ such that $\varepsilon_1 + \varepsilon_2 > \varepsilon - \eta$ and $a_n > \varepsilon_1$, $b_n > \varepsilon_2$ for each $n \in M$.*

Fix $\eta > 0$. By Lemma 2.4, there exist $\varepsilon_1, \varepsilon_2 \ge 0$ and infinite set M of natural numbers such that for $n \in M$,

$$\begin{aligned}\varepsilon_1 &< |||(f_n^2, 0) - (0, g')|||_A, \\ \varepsilon_2 &< \|g_n - g''\|_{Y^*}, \\ \varepsilon_1 + \varepsilon_2 &> \varepsilon - \eta.\end{aligned}$$

Again we will assume that the above is true for all $n \in \mathbb{N}$. Because $\|\cdot\|_{Y^*}$ is a dual norm to a c-LKK* norm, one has

$$\|g''\|_{Y^*} \le \limsup_{n\to\infty} \|g_n\|_{Y^*} - c\varepsilon_2. \tag{2.6}$$

Since $\varepsilon_1 < |||(f_n^2, 0) - (0, g')|||_A = A\|f_n^2\|_{\ell_1(\Gamma)} + \|g'\|_{Y^*}$, property (iv) and Lemma 2.3 imply

$$\begin{aligned}\varepsilon_1 &\le A \liminf_{n\to\infty} \|f_n^2\|_{\ell_1(\Gamma)} + A_0 \liminf_{n\to\infty} \|f_n^2\|_{\ell_1(\Gamma)} \\ &= \frac{A + A_0}{A} \liminf_{n\to\infty} |||(f_n^2, 0)|||_A.\end{aligned}$$

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Using Lemma 2.3 again, we estimate

$$
\begin{aligned}
(2.7)\qquad |||(0,g')|||_A &= \|g'\|_{Y^*} = |||(0,g')|||_{A_0} \\
&\leq \liminf_{n\to\infty} |||(f_n^2,0)|||_{A_0} \\
&= A_0 \liminf_{n\to\infty} \|f_n^2\|_{\ell_1(\Gamma)} \\
&= \frac{A_0}{A} \liminf_{n\to\infty} |||(f_n^2,0)|||_A \\
&= \liminf_{n\to\infty} \Big(1 - \frac{A-A_0}{A}\Big) |||(f_n^2,0)|||_A \\
&\leq \limsup_{n\to\infty} |||(f_n^2,0)|||_A - \frac{A-A_0}{A} \liminf_{n\to\infty} |||(f_n^2,0)|||_A \\
&\leq \limsup_{n\to\infty} |||(f_n^2,0)|||_A - \frac{A-A_0}{A} \varepsilon_1 \frac{A}{A+A_0}.
\end{aligned}
$$

This estimate together with (2.6) and properties (ii), (iii) and (v) give

$$
\begin{aligned}
(2.8)\quad |||(f,g)|||_A &= A\|f\|_{\ell_1(\Gamma)} + \|g\|_{Y^*} \\
&\leq A \limsup_{n\to\infty} \|f_n^1\|_{\ell_1(\Gamma)} + \|g'\|_{Y^*} + \|g''\|_{Y^*} \\
&\leq A \limsup_{n\to\infty} \|f_n^1\|_{\ell_1(\Gamma)} + \limsup_{n\to\infty} |||(f_n^2,0)|||_A - \frac{A-A_0}{A+A_0}\varepsilon_1 \\
&\quad + \limsup_{n\to\infty} \|g_n\|_{Y^*} - c\varepsilon_2 \\
&\leq \limsup_{n\to\infty} \Big(A\big(\|f_n^1\|_{\ell_1(\Gamma)} + \|f_n^2\|_{\ell_1(\Gamma)}\big) + \|g_n\|_{Y^*}\Big) - \frac{A-A_0}{A+A_0}\varepsilon_1 - c\varepsilon_2 \\
&\leq \limsup_{n\to\infty} |||(f_n,g_n)|||_A - c(\varepsilon-\eta) \leq 1 - c(\varepsilon-\eta).
\end{aligned}
$$

As η is arbitrary, $|||(f,g)|||_A \leq 1 - c\varepsilon$, which concludes the proof. ∎

3 Three-Space Problem for UKK* Norms

Theorem 3.1 *Assume X is a Banach space, $c_0(\Gamma) \subset X$, $X/c_0(\Gamma)$ admits an equivalent UKK* norm, and the dual unit ball of $(X/c_0(\Gamma))^*$ is an angelic space. Then X admits an equivalent UKK* norm.*

Proof We will modify the method used in the proof of Theorem 2.1. Roughly speaking, in the proof of Theorem 2.1 we had to split and track down the decrement of the norm in both parts f_n^2 and g_n to get exactly the same c as before. Here the procedure can be simplified, namely it is enough to take care of just the part of the decomposition that is further from its limit point.

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Exactly as in the proof of Theorem 2.1, we denote by Y the space $X/c_0(\Gamma)$ and write $X^* = \ell_1(\Gamma) \oplus Y^*$ when the norm on Y is UKK*. We find A_0 satisfying (2.1) and define norms

$$|||(f,g)|||_A = \|f\|_{\ell_1(\Gamma)} + \|g\|_{Y^*} \quad (f,g) \in X^*,$$

on X^* for all $A > 0$. By Lemma 2.3, $|||\cdot|||_A$ is a dual norm for each $A \geq A_0$. Select $A \geq A_0$ such that $\frac{A-A_0}{A+A_0} > \frac{2}{3}$.

Let $\varepsilon > 0$ be given. Since the norm on Y is UKK*, we can choose $\delta > 0$ from the definition UKK* for $\varepsilon/3$. We will show that the final estimate for the norm $|||\cdot|||_A$ in the definition is satisfied for $\min(\frac{\varepsilon\delta}{6}, \frac{2\varepsilon}{9})$. Suppose that $\{(f_n, g_n)\}$ is a sequence in $(X^*, |||\cdot|||_A)$ converging in the weak* topology to (f,g) such that $|||(f_n,g_n)|||_A \leq 1$ and $|||(f,g) - (f_n,g_n)|||_A > \varepsilon$.

We follow word for word the proof of Theorem 2.1 up to Lemma 2.4. According to that lemma and (2.5), we can find $\varepsilon_1, \varepsilon_2 \geq 0$ such that $\varepsilon_1 + \varepsilon_2 > \frac{2\varepsilon}{3}$, and

$$\begin{aligned} \varepsilon_1 &< |||(f_n^2, 0) - (0, g')|||_A, \\ \varepsilon_2 &< \|g_n - g''\|_{Y^*}, \end{aligned}$$

for infinitely many n's. We will assume that these inequalities hold for every $n \in \mathbb{N}$.

Then either ε_1 or ε_2 is greater than $\varepsilon/3$. Assume first that $\varepsilon_1 > \varepsilon/3$. Then, as in (2.7), we have

$$\begin{aligned} |||(0,g')|||_A &\leq \limsup_{n\to\infty} |||(f_n^2, 0)|||_A - \varepsilon_1 \frac{A-A_0}{A+A_0} \\ &\leq \limsup_{n\to\infty} |||(f_n^2, 0)|||_A - \frac{2\varepsilon}{9}. \end{aligned} \tag{3.1}$$

Using (3.1) we proceed as in (2.8) to get $|||(f,g)|||_A \leq 1 - \frac{2\varepsilon}{9}$.

Suppose now that $\varepsilon_2 > \varepsilon/3$. Set $s = \limsup_n \|g_n\|_{Y^*}$. Then

$$\|(1/s)(g_n - g'')\|_{Y^*} > \frac{\varepsilon_2}{s} > \frac{\varepsilon}{3}.$$

Since $|||\cdot|||_A$ is UKK*, $\|g''/s\|_{Y^*} \leq 1 - \delta$, *i.e.*,

$$\|g''\|_{Y^*} \leq s(1-\delta). \tag{3.2}$$

Since $\varepsilon_2 < \|g_n - g''\|_{Y^*} \leq \|g_n\|_{Y^*} + \|g''\|_{Y^*} \leq 2\limsup_{n\to\infty} \|g_n\|_{Y^*}$, we get from (3.2)

$$\begin{aligned} \|g''\|_{Y^*} \leq s - s\delta &\leq \limsup_{n\to\infty} \|g_n\|_{Y^*} - \frac{\varepsilon_2\delta}{2} \\ &\leq \limsup_{n\to\infty} \|g_n\|_{Y^*} - \frac{\varepsilon\delta}{6}. \end{aligned}$$

We again imitate estimates (2.8) to conclude that $|||(f,g)|||_A \leq 1 - \frac{\varepsilon\delta}{6}$. This finishes the proof. ■

4 Containing $c_0(\Gamma)$

Theorem 4.1 *Let X admit a c-Lipschitz weak* Kadec–Klee norm and weak* density of X^* is κ, $\kappa \geq \omega_1$. Then X contains an isomorphic copy of $c_0(\kappa)$.*

Proof We will use the following lemma that is formulated and proved in [2, Lemma 4.3].

Lemma 4.2 *Let $\|\cdot\|$ be a c-LKK* norm on X. Then for every $x \in X$ there exists a separable $E \subset X^*$ such that for every $y \in E_\perp$ one has*

$$\max\Big(\|x\|, \frac{\|y\|}{2-c}\Big) \leq \|x+y\| \leq \max\Big(\|x\|, \frac{\|y\|}{c}\Big). \tag{4.1}$$

We remark that $\operatorname{dens} X \geq \kappa$, as w^*-$\operatorname{dens} X^* \geq \kappa$. For every $\alpha \in [0,\kappa)$, we will construct by transfinite induction a point $x_\alpha \in X$, a subspace $Y_\alpha \subset X$, and a subspace $E_\alpha \subset X^*$ such that

(i) $Y_\alpha = \overline{\operatorname{span}}\{x_\beta : \beta \leq \alpha\}$;
(ii) for every $x \in Y_\alpha$ and $y \in (E_\alpha)_\perp$ holds (4.1);
(iii) $E_\beta \subset E_\alpha$ if $\beta \leq \alpha$;
(iv) $\|x_\alpha\| = 1$;
(v) $\|y\| = \sup\{|f(y)| : f \in E_\alpha, \|f\| \leq 1\}$ for each $y \in Y_\alpha$;
(vi) if $\alpha < \beta$, then $x_\beta \in (E_\alpha)_\perp$;
(vii) both $\operatorname{dens} Y_\alpha$ and w^*-$\operatorname{dens} E_\alpha$ are smaller than κ.

To start the construction, pick an arbitrary $x_0 \in X$ with norm 1. Using Lemma 4.2 we find a separable space $E_0 \subset X^*$ such that for x_0 and each $y \in (E_0)_\perp$, condition (4.1) holds true.

Assume that the objects have been constructed for every $\beta < \alpha$, where $\alpha < \kappa$. Set

$$Z_1 = \overline{\operatorname{span}}\{Y_\beta : \beta < \alpha\} \quad \text{and} \quad E_1 = \overline{\operatorname{span}}\{E_\beta : \beta < \alpha\}.$$

We find a point $x_\alpha \in (E_1)_\perp$ of norm 1 (this is possible due to weak* density of X^* and condition (vii)). Let C be a dense subset of Z_1 of cardinality less than κ. For every $p, q \in \mathbb{Q}$ and $z \in C$ we apply Lemma 4.2 on $pz + qx_\alpha$ and get the appropriate separable subspace $E_{p,q,z}$. Set

$$E_2 = E_1 \cup \overline{\operatorname{span}}\{E_{p,q,c} : p, q \in \mathbb{Q}, z \in C\} \quad \text{and} \quad Y_\alpha = \overline{\operatorname{span}}\{x_\beta : \beta \leq \alpha\}.$$

As $\operatorname{dens} Y_\alpha < \kappa$, we can enlarge E_2 to get a space E_α such that w^*-$\operatorname{dens} E_\alpha < \kappa$ and E_α is norming for Y_α, *i.e.*, E_α satisfies (v).

It follows that given $x \in Y_\alpha$ and $y \in (E_\alpha)_\perp$, the inequalities (4.1) hold for them as well as the other properties listed above. This finishes the inductive construction.

Set $Y = \overline{\operatorname{span}}\{x_\alpha : \alpha < \kappa\}$. Notice that $x_\alpha \neq x_\beta$ whenever $\alpha < \beta < \kappa$, because of conditions (iv), (v), (vi). We claim that Y is isomorphic to $c_0(\kappa)$.

Let $x = \sum_{i=1}^{n} c_i x_{\alpha_i}$ where $\alpha_1 < \cdots < \alpha_n$. Inductive use of condition (vi) and (4.1) gives

$$\begin{aligned}\Big\|\sum_{i=1}^{n} c_i x_{\alpha_i}\Big\| &\le \max\Big(\Big\|\sum_{i=1}^{n-1} c_i x_{\alpha_i}\Big\|, \frac{\|c_n x_{\alpha_n}\|}{c}\Big)\\ &\le \max\Big(\max\Big(\Big\|\sum_{i=1}^{n-2} c_i x_{\alpha_i}\Big\|, \frac{\|c_{n-1} x_{\alpha_{n-1}}\|}{c}\Big), \frac{\|c_n x_{\alpha_n}\|}{c}\Big)\\ &\le \cdots \le \frac{1}{c}\max(\|c_1 x_{\alpha_1}\|, \dots, \|c_n x_{\alpha_n}\|)\\ &= \frac{1}{c}\max(|c_1|, \dots, |c_n|) .\end{aligned}$$

Similarly we obtain the inequality

$$\Big\|\sum_{i=1}^{n} c_i x_{\alpha_i}\Big\| \ge \frac{1}{2-c}\max(|c_1|, \dots, |c_n|) .$$

Thus by defining

$$\begin{gathered} T\colon\ \operatorname{span}\{x_\alpha : \alpha < \kappa\} \to c_0(\kappa) ,\\ \sum_{i=1}^{n} c_i x_{\alpha_i} \mapsto \sum_{i=1}^{n} c_i e_{\alpha_i} ,\end{gathered}$$

we obtain a linear map from $\operatorname{span}\{x_\alpha : \alpha < \kappa\}$ into $c_0(\kappa)$ such that

$$\frac{1}{2-c}\|Tx\|_\infty \le \|x\| \le \frac{1}{c}\|Tx\|_\infty \tag{4.2}$$

for every $x \in \operatorname{span}\{x_\alpha : \alpha < \kappa\}$.

Now let x be in Y. Then $x = \lim_n x_n$ where $x_n \in \operatorname{span}\{x_\alpha : \alpha < \kappa\}$. According to (4.2), $\{Tx_n\}$ is a Cauchy sequence in $c_0(\kappa)$. Thus we can define $Tx = \lim_n Tx_n$. Then T is a isomorphism between Y and $c_0(\kappa)$. ∎

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