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LOWER BOUNDS OF OPERATORS ON WEIGHTED ℓ_p SPACES AND LORENTZ SEQUENCE SPACES

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Abstract. The problem considered is the determination of "lower bounds" of matrix operators on the spaces $\ell_p(w)$ or $d(w, p)$. Under fairly general conditions, the solution is the same for both spaces and is given by the infimum of a certain sequence. Specific cases are considered, with the weighting sequence defined by $w_n = 1/n^\alpha$. The exact solution is found for the Hilbert operator. For the averaging operator, two different upper bounds are given, and for certain values of p and α it is shown that the smaller of these two bounds is the exact solution.

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1. Introduction. The notion of "lower bounds" of matrix operators was introduced in [**11**], and has been intensively studied for ℓ_p spaces, e.g. [**2**,**4**,**5**]. If E is a Banach sequence space, we denote by $\delta(E)$ the set of decreasing, non-negative sequences in E. For a positive operator A mapping E into itself, the *lower bound* of A is

$$m_E(A) = \inf\{\|Ax\| : x \in \delta(E), \|x\| = 1\}.$$

In [**8**], lower bounds (as well as norms) were determined for certain classical operators on the Lorentz sequence space $d(w, 1)$, with the weighting sequence $w = (w_n)$ defined either by $w_n = 1/n^\alpha$ or by $W_n = n^{1-\alpha}$, where $W_n = w_1 + \cdots + w_n$. The present paper addresses the problem of finding analogous results for the case $p > 1$. Under a very mild condition, the lower bound problem is the same for the Lorentz sequence space $d(w, p)$ and the weighted ℓ_p space $\ell_p(w)$. The norms of these operators are considered in a companion paper [**9**].

For matrix operators satisfying fairly general conditions, the lower bound is given by an explicit formula in terms of $a_{i,j}$ and w_n. This reduces the problem to finding the infimum of a certain sequence, denoted in our notation by (V_n/W_n); (in the case $p = 1$, the norm of the operator is the supremum of the same sequence, but this is no longer true when $p > 1$). However, the problem of evaluating this infimum in particular cases can be far from trivial, and can lead to questions on inequalities of some interest in their own right.

The analogous problem for the continuous case concerns the space $L_p(w)$, where $w(x) = 1/x^\alpha$. This case is much easier, basically because the integral estimate for the tail of a series now becomes the exact quantity required instead of an approximation.

Indeed, for a wide class of operators, the function $V(x)/W(x)$ is actually constant, with its value expressed as a certain integral.

Our solutions need to reproduce the known results for ℓ_p when we take $w_n = 1$, and the results of [**8**] when we take $p = 1$. Some specific cases are as follows. For one version of the Hilbert operator, we can give an exact solution in the form of a certain infinite series. For the averaging (alias Cesàro) operator, the value of $m_{w,p}(A)^p$ is easily seen to be $p/(p+\alpha-1)$ in the continuous case. In the discrete case, we are unable to give a complete solution, but we show that under certain conditions it is the smaller of this quantity and $\zeta(p+\alpha)$.

2. General matrix operators on $\ell_p(w)$ and $d(w,p)$. Let (w_n) be a decreasing, non-negative sequence. We write $W_n = w_1 + \cdots + w_n$. Let $p \geq 1$. By $\ell_p(w)$ we mean the space of sequences $x = (x_n)$ having

$$S_p = \sum_{n=1}^{\infty} w_n |x_n|^p$$

convergent, with norm $\|x\|_{\ell_p(w)} = S_p^{1/p}$. When $w_n = 1$ for all n, this coincides with ℓ_p in the usual sense (the norm of which we denote by $\| \ \|_p$). However, we assume normally that $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n$ is divergent. Given a null sequence $x = (x_n)$, let (x_n^*) be the decreasing rearrangement of $|x_n|$. The Lorentz sequence space $d(w,p)$ is the space of null sequences x for which x^* is in $\ell_p(w)$, with norm $\|x\|_{w,p} = \|x^*\|_{\ell_p(w)}$.

Denote by $\delta_p(w)$ the set of decreasing, non-negative sequences in $\ell_p(w)$. Clearly, this is the same as the set of decreasing, non-negative sequences in $d(w,p)$, and the two norms coincide on it.

Let A be the operator defined by $Ax = y$, where $y_i = \sum_{j=1}^{\infty} a_{i,j} x_j$. We assume throughout that:

(1) $a_{i,j} \geq 0$ for all i, j,

(2) A maps $\ell_p(w)$ into itself.

Denote by e_n the sequence having 1 in place n and 0 elsewhere. Also, write

$$r_{i,n} = \sum_{j=1}^{n} a_{i,j}, \qquad c_{m,j} = \sum_{i=1}^{m} a_{i,j}.$$

Note that if $x = e_1 + \cdots + e_n$, then $y_i = r_{i,n}$.

We will not explore general conditions ensuring that (2) holds. For the particular operators considered later, this property is easily verified. (The problem of determining their norms is considered in [**9**].) However (2) implies in particular that, for each n, the element $A(e_1 + \cdots + e_n)$ is in $d(w,p)$ so that $\sum_{i=1}^{\infty} w_i r_{i,n}^p$ is convergent. (This in turn is equivalent to the statement that $\sum_{i=1}^{\infty} w_i a_{i,j}^p$ converges for each j.)

The next lemma is what we need to convert statements for $\ell_p(w)$ to ones for $d(w,p)$.

LEMMA 1. *The following condition is equivalent to the statement that $A(x)$ is decreasing for every decreasing, non-negative sequence in $d(w,p)$:*

(3) *$r_{i,n}$ decreases with i for each n.*

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Proof. Suppose that (x_j) is decreasing and $y = A(x)$. By Abel summation,

$$y_i = \sum_{j=1}^{\infty} a_{i,j} x_j = \sum_{j=1}^{\infty} r_{i,j}(x_j - x_{j+1}).$$

If (3) holds, it follows that $y_i \geq y_{i+1}$ for all i. The converse follows from the fact that $y_i = r_{i,n}$ when $x = e_1 + \cdots + e_n$.

Write

$$m_{w,p}(A) = \inf\{\|A(x)\|_{\ell_p(w)} : x \in \delta_p(w), \|x\|_{\ell_p(w)} = 1\}.$$

This is the "lower bound" of A as an operator on $\ell_p(w)$. In the presence of condition (3), Lemma 1 shows that it is equally the lower bound of A as an operator on $d(w,p)$, since then $A(x)$ is decreasing, so has the same norm in $\ell_p(w)$ and in $d(w,p)$. We also write $m_p(A)$ for the lower bound of A as an operator on ℓ_p.

We now give a characterization of $m_{w,p}(A)$ that simultaneously generalizes the known results for ℓ_p [**2**, Theorem 2] and for $d(w,1)$ [**8**, Proposition 1]. The following lemma is a variant of [**2**, Proposition 1]. The proof is short and so we include it.

Lemma 2. *Let $p \geq 1$. Suppose that (a_j), (x_j) are non-negative sequences, and that (x_j) is decreasing and tends to 0. Write $A_n = \sum_{j=1}^{n} a_j$ (with $A_0 = 0$), and $S_n = \sum_{j=1}^{n} a_j x_j$. Then*

(i) *$S_n^p - S_{n-1}^p \geq (A_n^p - A_{n-1}^p)x_n^p$, for all n;*

(ii) *if $\sum_{j=1}^{\infty} a_j x_j$ is convergent, then*

$$\left(\sum_{j=1}^{\infty} a_j x_j\right)^p \geq \sum_{n=1}^{\infty} A_n^p (x_n^p - x_{n+1}^p).$$

Proof. Differentiation shows that if $c > 0$, then $(x+c)^p - x^p$ is an increasing function of x for $x > 0$. Since $x_j \geq x_n$ for $j \leq n$, we have $S_{n-1} \geq A_{n-1}x_n$. Now

$$S_n = S_{n-1} + a_n x_n,$$
$$A_n x_n = A_{n-1} x_n + a_n x_n.$$

Statement (i) follows. Summing for $1 \leq n \leq N$, we obtain

$$S_N^p \geq \sum_{n=1}^{N} (A_n^p - A_{n-1}^p) x_n^p,$$

for each N. By letting $N \to \infty$ and applying Abel summation, we obtain statement (ii).

Corollary. *If (x_j) is decreasing and non-negative, and $X_n = x_1 + \cdots + x_n$, then for each n,*

$$X_n^p - X_{n-1}^p \geq [n^p - (n-1)^p] x_n^p.$$

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Proof. Take $a_j = 1$ in statement (i).

THEOREM 1. *Suppose that the operator A satisfies conditions* (1) *and* (2). *Write $r_{i,n} = \sum_{j=1}^{n} a_{i,j}$ and*

$$V_n = \sum_{i=1}^{\infty} w_i r_{i,n}^p.$$

Then

$$m_{w,p}(A)^p = \inf_{n\geq 1} \frac{V_n}{W_n},$$

and $m_{w,p}(A)$ is determined by elements of the form $e_1 + \cdots + e_n$.

Proof. Denote the stated infimum by c. Take $x \in \delta_p(w)$, and let $y = A(x)$. By Lemma 2(ii), we have

$$y_i^p \geq \sum_{n=1}^{\infty} r_{i,n}^p (x_n^p - x_{n+1}^p).$$

Hence

$$\begin{aligned}
\sum_{i=1}^{\infty} w_i y_i^p &\geq \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} r_{i,n}^p (x_n^p - x_{n+1}^p) \\
&= \sum_{n=1}^{\infty} (x_n^p - x_{n+1}^p) \sum_{i=1}^{\infty} w_i r_{i,n}^p \\
&= \sum_{n=1}^{\infty} V_n (x_n^p - x_{n+1}^p) \\
&\geq c \sum_{n=1}^{\infty} W_n (x_n^p - x_{n+1}^p) \\
&= c \sum_{j=1}^{\infty} w_j x_j^p \qquad \text{by Abel summation.}
\end{aligned}$$

This shows that $m_{w.p}(A)^p \geq c$. If $x = e_1 + \cdots + e_n$, then $\sum_{j=1}^{\infty} w_j x_j^p = W_n$, and

$$\sum_{i=1}^{\infty} w_i y_i^p = \sum_{i=1}^{\infty} w_i r_{i,n}^p = V_n.$$

This shows equality, and the sufficiency of the elements $e_1 + \cdots + e_n$.

NOTE 1. In the same way, one can show that if A is regarded as an operator from $\ell_p(w)$ to $\ell_q(w)$, where $p \geq q \geq 1$, then its lower bound is $\inf_{n\geq 1}(V_n^{1/q}/W_n^{1/p})$. See [**10**, Chapter 3].

NOTE 2. In the case $p = 1$, the sequence (V_n/W_n) also determines the norm: in fact, $\|A\|_{w,1} = \sup_{n\geq 1}(V_n/W_n)$ by [**8**]. For $p > 1$, the last part of the proof of Theorem 1 shows that $\|A\|_{w,p}^p \geq \sup_{n\geq 1}(V_n/W_n)$, but equality does not hold. See [**9**].

Write

$$u_n = \sum_{i=1}^{\infty} w_i a_{i,n}^p,$$

$$v_n = V_n - V_{n-1} = \sum_{i=1}^{\infty} w_i (r_{i,n}^p - r_{i,n-1}^p),$$

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so that $V_n = v_1 + \cdots + v_n$ (in accordance with our convention). When $p = 1$, we have $u_n = v_n$. It is elementary that $\inf_{n\geq 1}(V_n/W_n) \geq \inf_{n\geq 1}(v_n/w_n)$, and that equality will hold if (together with the other assumptions about (w_n)) the infimum of (v_n/w_n) is either its first term or its limit. By a further application of Lemma 2, we have also the following result.

PROPOSITION 1. *If A satisfies the conditions of Theorem* 1*, and also* $(a_{i,j})$ *decreases with j for each i, then*

$$m_{w,p}(A)^p \geq \inf_{n\geq 1}[n^p - (n-1)^p]\frac{u_n}{w_n}.$$

Proof. By the Corollary of Lemma 2,

$$r_{i,n}^p - r_{i,n-1}^p \geq [n^p - (n-1)^p]a_{i,n}^p.$$

Hence

$$v_n \geq [n^p - (n-1)^p]\sum_{i=1}^{\infty} w_i a_{i,n}^p = [n^p - (n-1)^p]u_n,$$

and so the stated expression is not greater than $\inf_{n\geq 1}(v_n/w_n)$.

Next, we identify a class of operators for which the lower bound problem is very easy.

PROPOSITION 2. *Suppose that A satisfies conditions* (1), (2) *and also:* (4) *A is upper triangular, i.e.* $a_{i,j} = 0$ *for* $i > j$*;* (5) $c_{j,j} = 1$ *for all j (in other words, A is a quasi-summability matrix). Then* $m_{w,p}(A) = 1$ *for any w and any* $p \geq 1$.

Proof. Take a decreasing, non-negative element x. Then

$$\begin{aligned}\sum_{i=1}^{n} y_i &\geq \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}x_j\\ &= \sum_{j=1}^{n} x_j \sum_{i=1}^{n} a_{i,j}\\ &= \sum_{j=1}^{n} x_j,\end{aligned}$$

since $\sum_{i=1}^{n} a_{i,j} = \sum_{i=1}^{j} a_{i,j} = 1$ for $j \leq n$. Hence we have $Y_n \geq X_n$ for all n. By the majorization principle (also known as Karamata's inequality) [**1**, Section 1.30] this implies that $\sum_{j=1}^{n} y_j^p \geq \sum_{j=1}^{n} x_j^p$ for all n. By Abel summation, it follows that $\|y\|_{\ell_p(w)} \geq \|x\|_{\ell_p(w)}$. Further, $a_{1,1} = 1$, and so $A(e_1) = e_1$. The statement follows.

In particular, this applies to the "Copson" operator C, defined by $y_i = \sum_{j=1}^{\infty}(x_j/j)$, which is given by the transpose of the Cesàro matrix:

$$a_{i,j} = \begin{cases} 1/j & \text{for } i \leq j,\\ 0 & \text{for } i > j.\end{cases}$$

3. Integral operators on $L_p(w)$. Let $w(x)$ be a decreasing, non-negative function on $(0, \infty)$. We assume that $W(x) = \int_0^x w(t)\, dt$ is finite for each x; (hence $1/x^\alpha$ is permitted for $0 < \alpha < 1$, but not for $\alpha = 1$). Then $L_p(w)$ is the space of functions f having

$$I_p = \int_0^\infty w(x)|f(x)|^p\, dx$$

convergent, with norm $\|f\|_{L_p(w)} = I_p^{1/p}$. Let A be the operator defined by

$$(Af)(x) = \int_0^\infty a(x, y)f(y)\, dy.$$

We assume that $a(x, y)$ is non-negative and that A maps $L_p(w)$ into itself. Again we denote the lower bound of A on $L_p(w)$ by $m_{w,p}(A)$, and define

$$r(x, y) = \int_0^y a(x, t)\, dt,$$

$$V(y) = \int_0^\infty w(x)r(x, y)^p\, dx.$$

Note that $r(x, y) = (Af)(x)$ when f is the characteristic function of $[0, y]$.

Corresponding to Theorem 1, we have the following result.

Theorem 2. *With the above notation,*

$$m_{w,p}(A)^p = \inf_{y>0} \frac{V(y)}{W(y)}.$$

We omit the proof, which is a routine adaptation of the proof of [**2**,Theorem 7], inserting $w(x)$ where appropriate. It is also essentially a smoother version of the above proof for the discrete case.

However, at this point the similarity between the discrete and continuous cases breaks down. When we take $w(x) = 1/x^\alpha$, the next result shows that for a wide class of operators (including those considered below), the function $V(y)/W(y)$ is actually constant and so the problem is already solved.

Proposition 3. *Suppose that $w(x) = 1/x^\alpha$ (where $0 < \alpha < 1$) and that $a(x, y)$ satisfies*

$$a(\lambda x, \lambda y) = \frac{1}{\lambda} a(x, y),$$

for all $x, y, \lambda > 0$. Then, with the above notation, $V(y)/W(y)$ is constant, and $m_{w,p}(A)^p = (1 - \alpha)V(1)$.

Proof. We have $W(y) = y^{1-\alpha}/(1 - \alpha)$. Also,

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$$\begin{aligned} r(\lambda x, \lambda y) &= \int_0^{\lambda y} a(\lambda x, t)\, dt \\ &= \int_0^{y} a(\lambda x, \lambda u)\lambda\, du \\ &= \int_0^{y} a(x, u)\, du \\ &= r(x, y). \end{aligned}$$

Hence

$$\begin{aligned} V(y) &= \int_0^{\infty} \frac{1}{x^{\alpha}} r(x, y)^p\, dx \\ &= \int_0^{\infty} \frac{1}{y^{\alpha} t^{\alpha}} r(yt, y)^p y\, dt \\ &= y^{1-\alpha} \int_0^{\infty} \frac{1}{t^{\alpha}} r(t, 1)^p\, dt, \end{aligned}$$

so that $V(y)/W(y) = (1-\alpha)V(1)$.

4. The Hilbert operator. In the continuous case, the Hilbert operator H is given by the kernel $a(x, y) = 1/(x+y)$. This kernel satisfies the condition of Proposition 3 and so we obtain the following result.

PROPOSITION 4. *Let $w(x) = 1/x^{\alpha}$, where $0 \leq \alpha < 1$. Then*

$$m_{w,p}(H)^p = (1-\alpha) \int_0^{\infty} \frac{1}{x^{\alpha}} \left[\log\left(1 + \frac{1}{x}\right)\right]^p dx.$$

Proof. We need only note that

$$r(x, 1) = \int_0^1 \frac{1}{x+y}\, dy = \log\left(1 + \frac{1}{x}\right).$$

When $\alpha = 0$, this integral equates to $\Gamma(p+1)\zeta(p)$ by [**2**, p. 97].

In the discrete case, two versions of the Hilbert operator, which we denote by H_1 and H_0 respectively, are given by the matrices

$$a_{i,j} = \frac{1}{i+j}, \qquad a_{i,j} = \frac{1}{i+j-1}.$$

We consider the lower bound of H_1. In the case $p = 1$, this was shown in [**8**, Theorem 13] to be $\sum_{i=1}^{\infty} 1/[i^{\alpha}(i+1)]$. We generalize this, using our Proposition 1.

THEOREM 3. *Let $w_n = 1/n^{\alpha}$, where $0 < \alpha < 1$, and let $p \geq 1$. Then*

$$m_{w,p}(H_1)^p = \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}(i+1)^p}.$$

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Proof. Firstly, $\|e_1\|_{w,p} = 1$, while if $y = H_1(e_1)$, then $y_i = 1/(i+1)$; hence

$$\|y\|_{w,p}^p = \sum_{i=1}^{\infty} \frac{1}{i^\alpha (i+1)^p}.$$

Therefore $m_{w,p}(H_1)$ is no greater than the quantity stated. To prove the reverse inequality, we use Proposition 1. With the notation used there,

$$u_n = \sum_{i=1}^{\infty} \frac{1}{i^\alpha (i+n)^p},$$

and

$$m_{w,p}(H_1)^p \geq \inf_{n \geq 1} [n^p - (n-1)^p] n^\alpha u_n.$$

Now $n^p - n^{p-1} = n^{p-1}(n-1) \geq (n-1)^p$; hence $n^p - (n-1)^p \geq n^{p-1}$, and $m_{w,p}(H_1)^p \geq \inf_{n\geq 1} C_n$, where $C_n = n^{p+\alpha-1} u_n$. A small adaptation to the proof of [**8**, Theorem 13] shows that $C_n \geq C_1$ for all n; hence $\inf_{n\geq 1} C_n = C_1 = u_1$, as required. (With rather more work, it is shown in [**6**] that C_n increases with n.)

Corollary. *We have* $m_p(H_1)^p = \zeta(p-1)$.

Proof. This is the case $w_n = 1$ for all n.

We conclude this section with some brief remarks about H_0. It was shown in [**2**] that $m_p(H_0)^p = \zeta(p)$. The discussion in [**8**] shows that when $w_n = 1/n^\alpha$, there are no easy candidates for $m_{w,p}(H_0)$, even when $p = 1$ (though a solution is found for the w defined by $W_n = n^{1-\alpha}$). However, it is not hard to give a lower estimate.

Proposition 5. *If* $w_n = 1/n^\alpha$, *then*

$$m_{w,p}(H_0)^p \geq \frac{1}{p+\alpha-1}.$$

Proof. We now have

$$u_n = \sum_{i=1}^{\infty} \frac{1}{i^\alpha (i+n-1)^p} = \sum_{j=n}^{\infty} \frac{1}{j^p (j-n+1)^\alpha)} \geq \sum_{j=n}^{\infty} \frac{1}{j^{p+\alpha}}.$$

By integral estimation, this is not less than $1/(p+\alpha-1)n^{p+\alpha-1}$. The statement follows, by Proposition 1.

5. The averaging operator. In this section, A will mean the averaging operator, defined in the discrete case by $y = A(x)$, where $y_n = \frac{1}{n}(x_1 + \cdots + x_n)$. It is given by the Cesàro matrix

$$a_{i,j} = \begin{cases} 1/i & \text{for } j \leq i, \\ 0 & \text{for } j > i. \end{cases}$$

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In the continuous case, the operator is given by $(Af)(x) = \frac{1}{x}\int_0^x f$, so that

$$a(x, y) = \begin{cases} 1/x & \text{for } y \leq x, \\ 0 & \text{for } y > x. \end{cases}$$

This function satisfies the condition of Proposition 3 and so we have the following result.

Proposition 6. *Let $w(x) = 1/x^\alpha$, where $0 \leq \alpha < 1$, and let A be the averaging operator on $L_p(w)$. Then*

$$m_{w,p}(A)^p = \frac{p}{p+\alpha-1}.$$

Proof. With our previous notation, $r(x, 1)$ equals 1 for $x \leq 1$ and $1/x$ for $x > 1$, so that

$$\begin{aligned} V(1) = \int_0^\infty \frac{1}{x^\alpha} r(x,1)^p \, dx &= \int_0^1 \frac{1}{x^\alpha} \, dx + \int_1^\infty \frac{1}{x^{p+\alpha}} \, dx \\ &= \frac{1}{1-\alpha} + \frac{1}{p+\alpha-1}. \end{aligned}$$

Hence

$$m_{w,p}(A)^p = (1-\alpha)V(1) = 1 + \frac{1-\alpha}{p+\alpha-1} = \frac{p}{p+\alpha-1}.$$

Note. The norm of A as an operator on $L_p(w)$ is $p/(p+\alpha-1)$ by [**12**, Chapter 1, Theorem 9.16].

As usual, the problem is much harder in the discrete case, and we can only give a partial solution. It is shown in [**2**] that $m_p(A) = \zeta(p)^{1/p}$ for $p > 1$, and in [**8**] that $m_{w,1}(A) = 1/\alpha$ when $w_n = 1/n^\alpha$. With our usual notation,

$$r_{i,n} = \begin{cases} 1 & \text{if } i \leq n, \\ n/i & \text{if } i > n, \end{cases}$$

so that

$$V_n = \sum_{i=1}^\infty w_i r_{i,n}^p = W_n + n^p \sum_{i=n+1}^\infty \frac{w_i}{i^p}.$$

Now let $w_n = 1/n^\alpha$, where $0 < \alpha \leq 1$. Write

$$R_n = \sum_{i=n}^\infty \frac{1}{i^{p+\alpha}}.$$

By Theorem 1, $m_{w,p}(A)^p = \inf_{n\geq 1} B_n$, where

$$B_n = \frac{V_n}{W_n} = 1 + \frac{n^p R_{n+1}}{W_n}.$$

Clearly, $m_{w,p}(A) \geq 1$ in all cases. Note that $B_1 = 1 + R_2 = \zeta(p+\alpha)$.

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LEMMA 3. *Let* $S_n = \sum_{i=n}^{\infty} 1/i^{1+\alpha}$, *where* $\alpha > 0$. *Then* $n^\alpha S_n \to 1/\alpha$ *as* $n \to \infty$. *Also,*

$$n^\alpha S_n \geq \frac{1}{\alpha} + \frac{1}{2n}.$$

In particular, $\zeta(1+\alpha) \geq 1/\alpha + \frac{1}{2}$.

Proof. By comparison with the integral of $1/x^{1+\alpha}$, we have

$$\frac{1}{\alpha n^\alpha} \leq S_n \leq \frac{1}{\alpha(n-1)^\alpha},$$

which proves the stated limit. The function $1/x^{1+\alpha}$ is convex and so the trapezium rule overestimates its integral on each interval $[k, k+1]$. The second statement follows on combining such intervals.

LEMMA 4. *Let* B_n *be as above, where* $0 < \alpha \leq 1$. *Then*

$$B_n \to \frac{p}{p+\alpha-1} \qquad \text{as } n \to \infty.$$

Proof. By Lemma 3, $n^{p+\alpha-1} R_{n+1} \to 1/(p+\alpha-1)$ as $n \to \infty$. Also, for $0 < \alpha < 1$, integral estimation gives

$$\frac{1}{1-\alpha}(n^{1-\alpha} - 1) \;\leq\; W_n \;\leq\; \frac{n^{1-\alpha}}{1-\alpha};$$

hence

$$\frac{n^{1-\alpha}}{W_n} \to 1-\alpha \qquad \text{as } n \to \infty.$$

It follows that

$$B_n = 1 + \frac{n^{1-\alpha}}{W_n}\, n^{p+\alpha-1} R_{n+1} \to 1 + \frac{1-\alpha}{p+\alpha-1} = \frac{p}{p+\alpha-1} \qquad \text{as } n \to \infty.$$

If $\alpha = 1$, then $n^p R_{n+1} \to 1/p$ and $W_n \to \infty$ as $n \to \infty$, so that $B_n \to 1$.
Hence we have the following result.

THEOREM 4. *Let* $w_n = 1/n^\alpha$. *If* $0 < \alpha < 1$, *then*

$$m_{w,p}(A)^p \;\leq\; \min\left(\zeta(p+\alpha),\; \frac{p}{p+\alpha-1}\right).$$

If $\alpha = 1$, *then* $m_{w,p}(A) = 1$.

Either of $\zeta(p+\alpha)$ and $p/(p+\alpha-1)$ can be smaller. Indeed, for a fixed value c of $p+\alpha$, the latter is smaller when $p \leq p_0 = (c-1)\zeta(c)$, so that $\alpha \geq \alpha_0 = c - p_0$. (In this, α_0 increases towards 1 as c increases.) The next lemma gives a simple sufficient condition for $p/(p+\alpha-1)$ to be smaller.

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LEMMA 5. *If* $p \le 1+\alpha$, *then*

$$\zeta(p+\alpha) \ \ge\ \frac{p}{p+\alpha-1}.$$

Proof. By Lemma 3,

$$\zeta(p+\alpha) \ \ge\ \frac{1}{p+\alpha-1}+\frac{1}{2}.$$

Hence the required inequality holds provided that

$$\frac{p-1}{p+\alpha-1} \ \le\ \frac{1}{2},$$

or $2(p-1) \le p+\alpha-1$, which is equivalent to $p \le 1+\alpha$.

Numerical examples suggest that (B_n) either decreases for all n or increases for a certain number of terms and then decreases. This would imply that $m_{w,p}(A)^p$ is the smaller of $\zeta(p+\alpha)$ and $p/(p+\alpha-1)$. We shall use Proposition 1 to prove that this is correct when $p \le 2$ and α exceeds a certain number $f(p)$. With the notation of Proposition 1,

$$u_n = \sum_{i=1}^{\infty} w_i a_{i,n}^p = \sum_{i=n}^{\infty} \frac{1}{i^{p+\alpha}} = R_n,$$

and hence $m_{w,p}(A)^p \ge \inf_{n\ge 1} D_n$, where

$$D_n = n^{\alpha}[n^p-(n-1)^p]R_n.$$

(Lemma 2 is not used, since for the averaging operator, it is easily seen that $r_{i,n}^p - r_{i,n-1}^p = [n^p-(n-1)^p]a_{i,n}^p$.) Note that $D_1 = R_1 = \zeta(p+\alpha)$.

LEMMA 6. *For* $1 \le p \le 2$, *we have*

$$n^p-(n-1)^p \ \ge\ \frac{p}{2}\left[n^{p-1}+(n-1)^{p-1}\right].$$

Proof. The function $f(t) = t^{p-1}$ is concave and hence

$$\begin{aligned} n^p-(n-1)^p &= p\int_{n-1}^{n} t^{p-1}\,dt \\ &\ge \frac{p}{2}[f(n)+f(n-1)]. \end{aligned}$$

LEMMA 7. *For* $1 \le p \le 2$ *and* $n \ge 2$, *we have*

$$\frac{n^p-(n-1)^p}{n^{p-1}} \ge \frac{p}{2}\left(1+\frac{n-1}{p+n-2}\right).$$

Proof. By Lemma 6,

$$\frac{n^p-(n-1)^p}{n^{p-1}} \ge \frac{p}{2}\left[1+\left(\frac{n-1}{n}\right)^{p-1}\right].$$

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By the elementary inequality $(1+x)^{p-1} \le 1+(p-1)x$, we have

$$\left(\frac{n}{n-1}\right)^{p-1} \le 1+\frac{p-1}{n-1} = \frac{p+n-2}{n-1}.$$

The statement follows.

Note. Equality holds in Lemma 7 when p is 1 or 2. This rather strange looking inequality is a companion to the more elementary fact that the left-hand side is not greater than p.

Theorem 5. *Suppose that* $1 \le p \le 2$ *and* $\alpha \ge f(p)$, *where*

$$f(p) = \frac{(3-p)(p-1)}{p+1}.$$

Define D_n *as above. Then* $D_n \ge p/(p+\alpha-1)$, *for all* $n \ge 2$. *Hence if* $w_n = 1/n^\alpha$, *then*

$$m_{w,p}(A)^p \ = \ \min\left(\zeta(p+\alpha),\ \frac{p}{p+\alpha-1}\right).$$

Proof. Note that

$$D_n = \frac{n^p-(n-1)^p}{n^{p-1}}\ n^{p+\alpha-1}R_n.$$

By Lemmas 3 and 7,

$$D_n \ge \frac{p}{2}\left(1+\frac{n-1}{p+n-2}\right)\left(\frac{1}{p+\alpha-1}+\frac{1}{2n}\right),$$

so that

$$D_n - \frac{p}{p+\alpha-1} \ge \frac{p}{4n}\left(1+\frac{n-1}{p+n-2}\right) - \frac{p}{2(p+\alpha-1)}\left(1-\frac{n-1}{p+n-2}\right).$$

Call this E_n. Then

$$\begin{aligned}\frac{2}{p}(p+n-2)E_n &= \frac{1}{2n}(p+2n-3) - \frac{p-1}{p+\alpha-1}\\ &= \frac{\alpha}{p+\alpha-1} - \frac{3-p}{2n}.\end{aligned}$$

Hence, to ensure that $E_n \ge 0$ for all $n \ge 2$, it is sufficient if the right-hand side is non-negative when $n=2$. This equates to $4\alpha \ge (p+\alpha-1)(3-p)$ and hence to $(p+1)\alpha \ge (p-1)(3-p)$.

Corollary. *If* $p \le 1+\alpha$, *then* $m_{w,p}(A)^p = p/(p+\alpha-1)$.

Proof. Since $\alpha \le 1$, we have $p \le 2$. Also, $3-p \le p+1$, since $p \ge 1$. Hence $\alpha \ge p-1 \ge f(p)$, and Theorem 5 applies. By Lemma 5, $p/(p+\alpha-1) \le \zeta(p+\alpha)$.

Further Remarks. Some values of f are: $f(1) = 0$, $f(\frac{3}{2}) = \frac{3}{10}$, $f(2) = \frac{1}{3}$. For p in $[1, 2]$, the greatest value of $f(p)$ is $6 - 4\sqrt{2} \approx 0.343$, occurring when $p = 2\sqrt{2} - 1$.

There are cases where $\alpha \geq f(p)$ and $\zeta(p + \alpha)$ is smaller than $p/(p + \alpha - 1)$. For example, let $p = 2$ and $\alpha = 1/3$. Then $\zeta(p + \alpha) \approx 1.415$, while $p/(p + \alpha - 1) = 3/2$.

When $2 \leq p < 5$, similar methods show that the conclusion of Theorem 5 holds when $\alpha \geq g(p)$, where $g(p) = (p - 1)^2/(5 - p)$. Then again $g(2) = \frac{1}{3}$, but $g(p) > 1$ (so that no conclusion follows) when $p > 2.56$.

The conclusion of Theorem 5 holds more widely than these results suggest, but there are values of p and α for which $D_2 < p/(p + \alpha - 1)$; e.g. $p = 2$, $\alpha \leq \frac{1}{10}$. Of course, this does not disprove our conjecture concerning $m_{w,p}(A)$.

REFERENCES

1. E. F. Beckenbach and R. Bellman, *Inequalities* (Springer-Verlag, 1961).

2. G. Bennett, Lower bounds for matrices, *Linear Algebra and Appl.* **82** (1986), 81–98.

3. G. Bennett, Some elementary inequalities III, *Quart. J. Math. Oxford Ser. (2)* **42** (1991), 149–174.

4. G. Bennett, Lower bounds for matrices II, *Canadian J. Math.* **44** (1992), 54–74.

5. G. Bennett, *Factorizing the classical inequalities, Mem. Amer. Math. Soc.* **576** (1996).

6. G. Bennett and G. J. O. Jameson, Monotonic averages of convex functions, preprint.

7. G. H. Hardy, J. Littlewood and G. Polya, *Inequalities* (Cambridge University Press, 1934).

8. G. J. O. Jameson, Norms and lower bounds of operators on the Lorentz sequence space $d(w, 1)$, *Illinois J. Math.* **43** (1999), 79–99.

9. G. J. O. Jameson and R. Lashkaripour, Norms of operators on the Lorentz sequence spaces $d(w, p)$, preprint.

10. R. Lashkaripour, *Lower bounds and norms of operators on Lorentz sequence spaces*, Doctoral dissertation (Lancaster, 1997).

11. R. Lyons, A lower bound for the Cesàro operator, *Proc. Amer. Math. Soc.* **86** (1982), 694.

12. A. Zygmund, *Trigonometric series* (Cambridge University Press, 1959).