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LETTER TO THE EDITOR

Dear Editor,

On the time spent below a random threshold by a system driven by a general counting process

Recently, Kirmani and Wesołowski [1] studied the time spent under a random threshold by a stochastic process driven by Poisson events. This is the process

$$S(t) = \int_0^t \mathbf{1}(\boldsymbol{Y}_{N(u)} \le X)\,\mathrm{d}u, \qquad t \ge 0,$$

where $\boldsymbol{Y} = (Y_i)_{i\ge 0}$ is a sequence of independent and identically distributed (i.i.d.) random variables, independent of a nonhomogeneous Poisson process $\boldsymbol{N} = (N(t))_{t\ge 0}$, X is a random variable independent of all else, and $\mathbf{1}(\cdot)$ is the indicator function. Let the Y_i have distribution function G and tail $\bar{G} = 1-G$. Kirmani and Wesołowski proved the following two propositions.

Proposition 1. ([1].) *In the model defined above, for any $t \ge 0$,*

$$\mathrm{E}(S(t)) = t\,\mathrm{E}(G(X)), \tag{1}$$

$$\operatorname{var}(S(t)) = \chi(t)\,\mathrm{E}(G(X)\bar{G}(X)) + t^2 \operatorname{var}(G(X)), \tag{2}$$

where

$$\chi(t) = 2 \iint_{0<u<v<t} \mathrm{P}(N(u) = N(v))\,\mathrm{d}u\,\mathrm{d}v.$$

Proposition 2. ([1].) *If $\boldsymbol{N}$ is a homogeneous Poisson process, then*

$$\frac{S(t)}{t} \xrightarrow{\mathrm{D}} G(X) \quad \textit{as } t \to \infty. \tag{3}$$

We show that all of these results hold in more general settings and that (3) holds in a more general sense. Extending (1) and (3) is trivial. Our main result is an extension of (2). Furthermore, our proof of (2) is simpler and more transparent than the proof in [1].

Our assumptions for (2) and initially for (1) are the same as those above except that now $\boldsymbol{N}$ is a general point process with points denoted by $0 \le T_1 \le T_2 \le \cdots$, where $T_i \to \infty$ as $i \to \infty$, and we define $T_0 = 0$. For any fixed t, let $\tau_i = T_{i+1} - T_i$, for $i \ge 0$ and $T_{i+1} \le t$, and $\tau_i = t - T_i$, for $T_i \le t$ and $T_{i+1} > t$. For any fixed x, let $\mathbf{1}_i$ be the indicator of the event $Y_i \le x$, $i \ge 0$. The $\mathbf{1}_i$ are i.i.d. with $\mathrm{E}(\mathbf{1}_i) = G(x)$ and $\operatorname{var}(\mathbf{1}_i) = G(x)\bar{G}(x)$. Let $S_x(t) = (S(t) \mid X = x)$. By definition,

$$S_x(t) = \sum_{i=0}^{N(t)} \tau_i\,\mathbf{1}_i \quad \text{and} \quad \sum_{i=0}^{N(t)} \tau_i = t.$$

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For any t, let A be the collection of random variables $\{N(t), \tau_0, \dots, \tau_{N(t)}\}$. We then obtain

$$\mathrm{E}(S_x(t) \mid A) = \mathrm{E}\biggl(\sum_{i=0}^{N(t)} \tau_i\, \mathbf{1}_i \biggm| A\biggr) = \biggl(\sum_{i=0}^{N(t)} \tau_i\, \mathrm{E}(\mathbf{1}_i) \biggm| A\biggr) = \biggl(\sum_{i=0}^{N(t)} \tau_i G(x) \biggm| A\biggr) = tG(x),$$

$$\mathrm{E}(S_x(t)) = tG(x), \tag{4}$$

$$\mathrm{var}(S_x(t) \mid A) = \biggl(\sum_{i=0}^{N(t)} \mathrm{var}(\tau_i\, \mathbf{1}_i) \biggm| A\biggr) = \biggl(\sum_{i=0}^{N(t)} \tau_i^2\, \mathrm{var}(\mathbf{1}_i) \biggm| A\biggr) = G(x)\bar{G}(x) \sum_{i=0}^{N(t)} \tau_i^2,$$

$$\mathrm{var}(S_x(t)) = \mathrm{E}[\mathrm{var}(S_x(t) \mid A)] + \mathrm{var}[\mathrm{E}(S_x(t) \mid A)] = G(x)\bar{G}(x)\, \mathrm{E}\biggl(\sum_{i=0}^{N(t)} \tau_i^2\biggr). \tag{5}$$

Now, (1) follows immediately from (4) and, as $\mathrm{E}(S_x(t) \mid A)$ is a constant, it has zero variance.

We now show that

$$\mathrm{E}\biggl(\sum_{i=0}^{N(t)} \tau_i^2\biggr) = \chi(t).$$

For each (u, v) on a two-dimensional uv-plane, let $\mathbf{1}(u, v)$ be the indicator of the event $\{N(u) = N(v)\}$. We first claim that

$$\iint_{0 \le u, v \le t} \mathbf{1}(u, v)\, \mathrm{d}u\, \mathrm{d}v = \sum_{i=0}^{N(t)} \tau_i^2. \tag{6}$$

Consider a realization of $N(t) = 2$. The integration above is carried out over the square $(0, t) \times (0, t)$ on the uv-plane. In Figure 1, we show where the jumps in $\boldsymbol{N}$ occur for *the same* realization, where the jump locations for $\{N(u)\}$ are represented horizontally and the jump locations for $\{N(v)\}$ are represented vertically. Note that $(0, t)$ is cut into three segments, of lengths τ_0, τ_1, and τ_2, in each dimension. We have $\mathbf{1}(u, v) = 1$ if and only if (u, v) is in one of the three squares along the diagonal, with sides of length τ_0, τ_1, and τ_2, respectively. Hence, $\iint_{0 \le u, v \le t} \mathbf{1}(u, v)\, \mathrm{d}u\, \mathrm{d}v = \tau_0^2 + \tau_1^2 + \tau_2^2$ for this realization. The same idea holds in general, and (6) holds on every sample path.

Taking the expected value, we have

$$\begin{aligned}
\mathrm{E}\biggl(\sum_{i=0}^{N(t)} \tau_i^2\biggr) &= \mathrm{E}\biggl(\iint_{0 \le u, v \le t} \mathbf{1}(u, v)\, \mathrm{d}u\, \mathrm{d}v\biggr) \\
&= \iint_{0 \le u, v \le t} \mathrm{E}(\mathbf{1}(u, v))\, \mathrm{d}u\, \mathrm{d}v \\
&= \iint_{0 \le u, v \le t} \mathrm{P}(N(u) = N(v))\, \mathrm{d}u\, \mathrm{d}v \\
&= \chi(t).
\end{aligned}$$

Interchanging expectation and integration above is valid because the $\mathbf{1}(u, v)$ is nonnegative (Fubini's theorem). Combining this expression with (5), we have $\mathrm{var}(S_x(t)) = \chi(t)G(x)\bar{G}(x)$ and, as $\mathrm{var}(S(t)) = \mathrm{E}[\mathrm{var}(S(t) \mid X)] + \mathrm{var}[\mathrm{E}(S(t) \mid X)]$, we obtain (2).

In the derivation of (4), we see that independence of the Y_i is not needed. Hence, for (1), we require only that the Y_i all have the same distribution.

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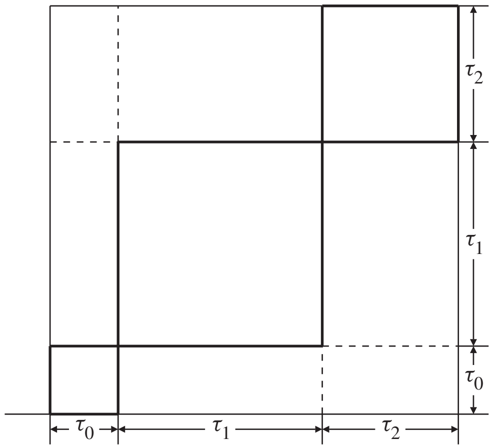

FIGURE 1: Illustration of $\iint_{0\le u,v\le t} \mathbf{1}(u,v)\,\mathrm{d}u\,\mathrm{d}v = \tau_0^2+\tau_1^2+\tau_2^2$.

Regarding limiting behavior as $t\to\infty$, we write

$$\frac{N(t)}{t}\frac{\sum_{i=0}^{N(t)-1}\tau_i\,\mathbf{1}_i}{N(t)} \le \frac{S_x(t)}{t} = \frac{\sum_{i=0}^{N(t)}\tau_i\,\mathbf{1}_i}{t} \le \frac{N(t)+1}{t}\frac{\sum_{i=0}^{N(t)-1}\tau_i\,\mathbf{1}_i + K}{N(t)+1},$$

where $K=(T_{N(t)+1}-T_{N(t)})\,\mathbf{1}_{N(t)}$. Thus, $S_x(t)$ converges with probability 1 to $G(x)$ provided that $N(t)/t$ converges to a finite constant with probability 1. This will occur when $\boldsymbol{N}$ is a stationary and ergodic point process and, when it does, $S(t)/t$ converges to the random variable $G(X)$ with probability 1, a stronger mode of convergence than in (3), under weaker conditions.

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References

[1] KIRMANI, S. N. U. A. AND WESOŁOWSKI, J. (2003). Time spent below a random threshold by a Poisson driven sequence of observations. *J. Appl. Prob.* **40,** 807–814.

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