TRANSVERSALLY AFFINE FOLIATIONS

by P. M. D. FURNESS and E. FÉDIDA

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1. Preliminaries. Let $\mathscr{F}$ be a smooth foliation of codimension p on a smooth manifold M^m. We can define $\mathscr{F}$ by an atlas of coordinate charts $(U, (x, y))$, called *leaf charts*, where $(x, y): U \to \mathbf{R}^{m-p} \times \mathbf{R}^p$ are coordinate functions for which the leaves of $\mathscr{F}$ are given by y^1 constant, ..., y^p constant, in U. Clearly, on the overlap of two such leaf charts $(U, (x, y))$ and $(U', (x', y'))$ we have a coordinate transformation of the form

$$x' = x'(x, y), \quad y' = y'(y).$$

If y' is always affine in y, i.e.

$$y'^i = A^i_j y^j + B^i, \tag{1}$$

where A^i_j and B^i are constants, we shall say that $\mathscr{F}$ is a *transversally affine foliation*. This notion is, in a sense, dual to that of *affine foliation*, see **[2]**, in which x' is affine in x and each leaf has an induced flat affine structure.

In this paper we establish some of the basic properties of transversally affine foliations of codimension one.

2. An equivalent definition. From now on we shall be concerned exclusively with foliations of codimension one.

LEMMA 1. *$\mathscr{F}$ is transversally affine and orientable if and only if some nowhere vanishing 1-form ω on M determining $\mathscr{F}$ has the following properties.*

(i) *$d\omega = \omega \wedge \theta$ for some 1-form θ.*
(ii) *$d\theta = 0$.*

Proof. Since $\mathscr{F}$ is determined by ω we have, by definition, $\omega \,|\, \mathscr{F} = 0$ and $\omega \wedge d\omega = 0$. This implies that there exists a 1-form θ such that $d\omega = \omega \wedge \theta$. Thus if $\mathscr{F}$ is transversally affine we must show that we can select θ to satisfy $d\theta = 0$.

Let $\mathscr{A}$ be an atlas of leaf charts satisfying (1). Suppose that $\omega = \omega_i\, dx^i + \alpha\, dy$ in the chart $(U, (x, y))$. The condition $\omega \,|\, \mathscr{F} = 0$ implies that $\omega_i = 0$. Thus on the overlap of the charts $(U, (x, y))$ and $(U', (x', y'))$ we have $\omega = \alpha'\, dy' = \alpha\, dy$. But $dy' = A\, dy$ for some constant A from (1), hence $\alpha' A = \alpha$. Also, since ω is nowhere vanishing, it follows that $\theta = -d\alpha/\alpha$ is a globally defined 1-form. Now $d\omega = d\alpha \wedge dy = d\alpha/\alpha \wedge \alpha\, dy = \omega \wedge \theta$ and $d\theta = 0$.

Conversely, suppose $\mathscr{F}$ is defined by a 1-form ω satisfying (i), (ii). Given an atlas $\mathscr{A}$ of leaf charts we want to modify it so as to satisfy (1). Since we can assume the domain U of each chart is topologically $\mathbf{R}^m$, the condition $d\theta = 0$ implies that there exists a real valued function f on U such that $\theta = df$. If we fix one such f for each chart, then on the overlap of two charts $\theta = df = df'$ and so $f' = f + B$, for some constant B. Now put $\alpha = e^{-f}$ and change to

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coordinates (X, Y) on U defined by

$$X^i = x^i, \quad Y = \int_0^y (\beta/\alpha)\, dy, \quad \text{where} \quad \omega = \beta\, dy$$

$$\left(\text{note that } \frac{\partial}{\partial x^i}(\beta/\alpha) = 0\right).$$

Clearly $(U, (X, Y))$ defines a leaf chart. However $\alpha\, dY = \beta\, dy = \omega$ and so

$$\alpha\, dY = \alpha'\, dY' = e^{-(f+B)} dY' = \alpha\, A dY',$$

where $A = e^{-B}$. Hence $A\, dY' = dY$ and we have constructed a leaf atlas for which $\mathscr{F}$ is transversally affine.

REMARK. If we put $\Omega = (\omega, \theta)$ then Ω can be regarded as a 1-form on M with values in the Lie algebra of the group of affine transformations of $\mathbf{R}$. (i) and (ii) imply $d\Omega + \frac{1}{2}[\Omega, \Omega] = 0$ which are the conditions for $(\mathscr{F}, \Omega)$ to be a *homogeneous foliation,* see **[1]**. As a consequence of Lemma 1 we have the following result.

THEOREM 1. *Let $\mathscr{F}$ be a transversally affine and orientable foliation of codimension one on a closed manifold M^m. Then*

(a) *The Godbillon-Vey invariant of $\mathscr{F}$ is trivial,*
(b) $H^1(M; \mathbf{R}) \neq 0$.

Proof. The Godbillon-Vey invariant of $\mathscr{F}$ is $[-\theta \wedge d\theta] \in H^3(M; \mathbf{R})$, see **[5]**. It is obviously trivial.

By Lemma 1 we have a closed form θ defined on M. Suppose $[\theta] = 0$, i.e. $\theta = df$ for some real valued function f, where $[\theta] \in H^1(M; \mathbf{R})$. Consider the nowhere vanishing 1-form $\phi = e^f \omega$. Then

$$d\phi = e^f \theta \wedge \omega + e^f d\omega = 0 \text{ by (i).}$$

Now, ϕ cannot be exact because a smooth real valued function on a closed manifold has at least two critical points. Hence $[\phi] \neq 0$. Thus $H^1(M; \mathbf{R}) \neq 0$. On the other hand if $[\theta] \neq 0$ then $H^1(M; \mathbf{R}) \neq 0$.

COROLLARY. *There do not exist transversally affine foliations of codimension one on spheres.*

REMARK. Part (b) of the theorem is a consequence of a more general result concerning transversally analytic foliations, see **[6]**, of which transversally affine foliations are clearly examples.

An important class of transversally affine foliations are those for which $\theta \equiv 0$, i.e. $d\omega \equiv 0$. These are the foliations without holonomy and have been studied in detail by Tischler **[11]** and Moussu **[9]**. The following construction yields another class of examples.

Consider $M^m = V^{m-1} \times S^1$. The trivial foliation is determined by a closed form α, the pull back of the standard volume form on S^1. Let $f: V^{m-1} \to \mathbf{R}$ be a smooth function for which zero is not a critical value. We can extend f to M in the obvious way. Define $\omega = df + f\alpha$. This is a nowhere vanishing 1-form and $d\omega = df \wedge \alpha = \omega \wedge \alpha$. Thus ω determines a transversally affine foliation on M^m. If we take $V^{m-1} = S^2$, with f as the standard height function, we obtain a transversally affine foliation of $S^2 \times S^1$, with one torus leaf T^2 and all other leaves diffeomorphic to $\mathbf{R}^2$. All the leaves are proper.

The next result gives a class of examples from pseudo-riemannian geometry.

THEOREM 2. *Let (M^m, g) be a smooth, pseudo-riemannian manifold which admits a flat parallel field of tangent lines. Then M^m admits a transversally affine foliation of codimension one.*

Proof. A parallel field of tangent lines is a tangent line bundle which is invariant with respect to parallel transport. Such a field is said to be flat if locally it is spanned by a parallel vector field. The parallel field of tangent $(m-1)$-planes $P^\perp$, which is the orthogonal conjugate of P, is integrable and hence tangent to a foliation $\mathscr{F}$ of codimension one, see [**4**]. Note that $P^\perp$ is complementary to P if and only if P is non-null. For the non-null case it is known, see [**4**], that $\mathscr{F}$ is determined by a closed 1-form and so is clearly transversally affine. In the null case, by considering a canonical form for the metric, see [**12**, **4**], one can obtain an atlas $\mathscr{A}$ of leaf charts for $\mathscr{F}$ with the following properties. Each chart has coordinates

$$(x, y, t) \in \mathbf{R} \times \mathbf{R}^{m-2} \times \mathbf{R}$$

such that the metric has the canonical form

$$ds^2 = 2\,dx\,dt + g_{ij}\,dy^i\,dy^j + 2H_i dy^i\,dt + K\,dt^2, \tag{2}$$

where g_{ij}, H_i, K are independent of x. The leaves of $\mathscr{F}$ are given locally by t constant, and P is spanned in each chart by the parallel vector field $\partial/\partial x$. Moreover, on the overlap of two charts $(U, (x, y, t))$ and $(U', (x', y', t'))$, we have, by virtue of (2), a coordinate transformation of the form

$$x' = (dt/dc)\,.\,x + h(y, t), \qquad y' = y'(y, t), \qquad t' = c(t).$$

Now, since $\partial/\partial x$ and $\partial/\partial x'$ are both parallel we must have dc/dt constant. Hence $c(t) = At + B$ where A and B are constants and so $\mathscr{F}$ is transversally affine.

For an example to illustrate this theorem, in which every leaf of $\mathscr{F}$ is dense and $\mathscr{F}$ has non-trivial holonomy, see [**2**, §4].

3. Properties of the leaves. In this section we shall prove that a transversally affine foliation of codimension one cannot have exceptional leaves. This is not true for analytic foliations in general, see [**7**].

THEOREM 3. *If $\mathscr{F}$ is a transversally affine foliation of codimension one on a smooth manifold M^m, then each leaf of $\mathscr{F}$ is either proper or locally dense. In particular there are no exceptional leaves.*

COROLLARY. *If M^m is closed then*:

(i) *if every leaf of $\mathscr{F}$ is non-compact then every leaf is dense*;

(ii) *if the compact leaves are not isolated then every leaf is compact and if in addition $\mathscr{F}$ is transversally orientable then $\mathscr{F}$ is a fibring.*

Proof. (i) follows from the known fact, see **[10]**, that if $\mathscr{F}$ has no exceptional leaves then every non-dense leaf has a compact leaf in its closure.

(ii) is a consequence of the fact that $\mathscr{F}$ is transversally analytic, see **[6]**.

Proof of Theorem 3. Recall that a leaf L is *proper* (resp. *locally dense*) if it intersects a transverse arc in a non-empty discrete set (resp. dense set). Note that a leaf can be locally dense without being dense in M^m.

If every leaf is compact then every leaf is proper and we are finished. Let L be a non-proper leaf. Then it is not difficult to show that there is a smoothly embedded circle S intersecting L which is transversal to $\mathscr{F}$. The union of all leaves through S is an open submanifold $U \subset M^m$. Thus if we can show that each leaf of $\mathscr{F} \mid U$ is either proper or locally dense then the proof is complete.

There is a flat affine structure on S, induced by $\mathscr{F}$, which we shall denote by (S, Γ), where Γ is a connection on S having zero curvature and torsion. This is a consequence of $\mathscr{F}$ being transversally affine. Also, $\mathscr{F}$ induces a pseudogroup $\mathscr{G}$ of local affine diffeomorphisms of (S, Γ). The orbit of $x \in S$ under the action of $\mathscr{G}$ is precisely $L_x \cap S$ where L_x denotes the leaf through x. By Theorem 3 of **[3]** there is a covering map

$$f:(X, \Gamma^*) \to (S, \Gamma),$$

where (X, Γ^*) is one of the following spaces.

(I) $X = \mathbf{R}$ and Γ^* is the standard flat euclidean connection. The group of deck transformations of f is generated by the affine diffeomorphism ϕ where $\phi(x) = x+1$, $x \in \mathbf{R}$.

(II) $X = \mathbf{R}^+ = \{x \in \mathbf{R} : x > 0\}$. Γ^* is the (incomplete) flat connection induced by the euclidean connection. Here $\phi(x) = \alpha x$, $x \in \mathbf{R}^+$ and $\alpha > 1$ is constant.

Each $\xi \in \mathscr{G}$ can be lifted (not uniquely) to an element $\xi' \in \mathrm{Aff}(\mathbf{R}, \Gamma^*)$ (the Lie group of affine diffeomorphisms of $(\mathbf{R}, \Gamma^*)$). To do this one uses analytic continuation, see **[8**, Chapter VI**]**. In case (II) it is clear that $\xi'(0) = 0$.

If $\phi \in \mathrm{Aff}(X, \Gamma^*)$ is a generator of the group of deck transformations of f then the group G generated by $\{\xi' : \xi \in \mathscr{G}\} \cup \{\phi\}$ is independent of the particular choice of lifts.

LEMMA 2. *Let* $x \in X$. *Then* $f(G(x)) = S \cap L_{f(x)}$.

NOTE. This shows that $L_{f(x)}$ is proper or locally dense if and only if $G(x) \subset X (= \mathbf{R}$ or $\mathbf{R}^+)$ is proper or locally dense.

Proof. Let $h: M^* \to M^m$ be a simply connected covering and let $\mathscr{F}^* = h^{-1}\mathscr{F}$. Let $U^* \subset M^*$ be a connected open set such that $(h \mid U^*): U^* \to U$ is a covering. Now, $S \subset M^m$ represents an element of infinite order in $\pi_1(M^m)$. This follows because $\mathscr{F}$ is transversally analytic, see **[6]**. Thus each lift K of S in U^* is an embedded $\mathbf{R}$. Moreover, it is not difficult to prove that K intersects each leaf of $\mathscr{F}^* \mid U^*$ precisely once. We can identify K with the space X, where K has the flat affine structure induced by $\mathscr{F}^*$.

Let P be the group of deck transformations of the covering $h: M^* \to M^m$. Clearly, each element of $\mathscr{G}$ is induced by an element of P (in particular one keeping U^* setwise fixed).

Define $k: U^* \to K$ by $k(y) = z$ when $y \in L_z$. Let $p \in P$. Then $k \circ p: K \to K$ is an affine diffeomorphism (with respect to the structure induced by $\mathscr{F}^*$) because p is $\mathscr{F}^*$ preserving.

The following lemma completes the proof of the theorem.

LEMMA 3. *In both cases* (I) *and* (II) *above, if* $x \in X$ *then* $G(x)$ *is either proper or dense.*

Proof. Case (I). $X = \mathbf{R}$ and ϕ is the translation $\phi(x) = x+1$.

(i) Suppose G consists only of translations. If G contains the translations $x \mapsto x+\alpha$ and $x \mapsto x+\beta$ such that $(\beta-\alpha)$ is irrational then $G(x)$ will be dense for all $x \in \mathbf{R}$. If this is not the case then G consists of elements of the form

$$\psi : x \mapsto x+p/q, \quad \text{where} \quad p, q \in \mathbf{Z}.$$

Clearly $\phi^{-p} \circ \psi^{q} = 1_{\mathbf{R}}$. To obtain a set of generators we can assume $0 < p/q \leqq 1$. If the set of generators is finite then all orbits are proper. If the set of generators is infinite then the set $\{p/q\}$ must have a limit point in $[0, 1]$. In this case there must be "arbitrarily small" translations in G and so all orbits must be dense.

(ii) Suppose G contains at least one non-translation. By suitably changing coordinates we can assume this element, say ψ, has the form $\psi(x) = \lambda x$, $\lambda > 1$. Now,

$$\psi^{p} \circ \phi \circ \psi^{-p}(x) = x+\lambda^{p}, \; p \in \mathbf{Z}.$$

Thus by choosing p negative, we may again get arbitrarily small translations in G and so every orbit is dense.

Case (II). $X = \mathbf{R}^{+}$ and ϕ has the form $\phi(x) = \alpha x$, $\alpha > 1$. G must preserve $\mathbf{R}^{+}$, hence $G(0) = 0$. Thus every $\psi \in G$ has the form $\psi(x) = \beta x$, $\beta > 0$. If there is a $\psi \in G$ such that $(\log \beta / \log \alpha)$ is irrational then every orbit is dense. Otherwise every orbit is proper.

The following elementary examples of foliations on the torus T^2 show that both the cases (I) and (II) can occur.

(I) Take $\mathbf{R}^2$ with the standard flat euclidean connection. If G is the group of affine transformations generated by $(x, y) \mapsto (x+1, y)$ and $(x, y) \mapsto (x, y+1)$ then $\mathbf{R}^2/G$ is the standard euclidean torus. The foliation of $\mathbf{R}^2$ defined by $dx + \alpha\, dy = 0$ for $\alpha \in \mathbf{R}$ induces a transversally affine foliation on T^2. If $L = \{(x, 0) : x \in \mathbf{R}\}$ then we can take $S = L/G$. Every leaf is compact if α is rational and every leaf is dense if α is irrational.

(II) Put $X = \mathbf{R}^2 - \{(0, 0)\}$. Let G_λ be the group generated by $(x, y) \mapsto (\lambda x, \lambda y)$ $\lambda > 1$. Then X/G_λ is a flat affine torus. The foliation of X defined by $dx = 0$ induces a foliation of T^2 with every leaf proper and precisely two compact leaves. If $L = \{(x, 0) : x > 0\}$ then we can take $S = L/G$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTHAMPTON
SOUTHAMPTON
SO9 5NH

INSTITUT DE RECHERCHE MATHEMATIQUE AVANCEE
UNIVERSITÉ LOUIS PASTEUR
7, RUE RENE DESCARTES
67084 STRASBOURG CEDEX