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ON THE DIVERGENCE OF HERMITE-FEJÉR TYPE INTERPOLATION WITH EQUIDISTANT NODES

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If $f(x)$ is defined on $[-1,1]$, let $\overline{H}_{1n}(f,x)$ denote the Lagrange interpolation polynomial of degree n (or less) for f which agrees with f at the $n+1$ equally spaced points $x_{k,n} = -1 + (2k)/n$ $(0 \leqslant k \leqslant n)$. A famous example due to S. Bernstein shows that even for the simple function $h(x) \equiv |x|$, the sequence $\overline{H}_{1n}(h,x)$ diverges as $n \to \infty$ for each x in $0 < |x| < 1$. A generalisation of Lagrange interpolation is the Hermite-Fejér interpolation polynomial $\overline{H}_{mn}(f,x)$, which is the unique polynomial of degree no greater than $m(n+1) - 1$ which satisfies $\overline{H}^{(p)}_{mn}(f,x_{k,n}) = \delta_{0,p} f(x_{k,n})$ $(0 \leqslant p \leqslant m-1, 0 \leqslant k \leqslant n)$. In general terms, if m is an even number, the polynomials $\overline{H}_{mn}(f,x)$ seem to possess better convergence properties than the $\overline{H}_{1n}(f,x)$. Nevertheless, D.L. Berman was able to show that for $g(x) \equiv x$, the sequence $\overline{H}_{2n}(g,x)$ diverges as $n \to \infty$ for each x in $0 < |x|$. In this paper we extend Berman's result by showing that for any even m, $\overline{H}_{mn}(g,x)$ diverges as $n \to \infty$ for each x in $0 < |x| < 1$. Further, we are able to obtain an estimate for the error $\left|\overline{H}_{mn}(g,x) - g(x)\right|$.

1. Introduction

Suppose $-1 \leqslant x_{0,n} < x_{1,n} < \ldots < x_{n,n} \leqslant 1$ is an arbitrary system of interpolation nodes. (We shall often write $x_{k,n}$ as x_k.) Let $m \geqslant 1$ be an integer, and suppose f is a real-valued function defined on $[-1,1]$. The $(0,1,\ldots,m-1)$ Hermite-Fejér (HF) interpolation polynomial $\overline{H}_{mn}(f,x)$ for f is the unique polynomial of degree at most $m(n+1)-1$ which satisfies the $m(n+1)$ conditions

$$\begin{cases} \overline{H}_{mn}(f,x_k) = f(x_k) & (k = 0,1,\ldots,n), \\ \overline{H}^{(p)}_{mn}(f,x_k) = 0 & (p = 1,2,\ldots,m-1;\ k = 0,1,\ldots,n). \end{cases}$$

Note that $\overline{H}_{1n}(f,x)$ is the well-known Lagrange interpolation polynomial for $f(x)$. In 1914, Faber [4] showed that for *any* system of nodes, there exists a function $f(x)$, continuous on $[-1,1]$, such that $\overline{H}_{1n}(f,x)$ does not converge uniformly to $f(x)$ on $[-1,1]$ as $n \to \infty$. On the other hand, Fejér [5] showed in 1916 that if

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$x_k = -\cos(((2k+1)\pi)/(2n+2))$ (so the x_k are the zeros of the Chebyshev polynomial $T_{n+1}(x) = \cos((n+1)\arccos x)$), and if f is continuous on $[-1,1]$, then

$$\lim_{n\to\infty} \overline{H}_{2n}(f,x) = f(x),$$

uniformly in $[-1,1]$. Thus $(0,1)$ HF interpolation seems to possess better convergence properties than Lagrange interpolation.

Fejér's result has prompted many authors to study $(0,1,\dots,m-1)$ HF interpolation, particularly when the nodes of interpolation are the zeros of some orthogonal polynomials (such as the Chebyshev polynomials). Much less popular has been a study of $(0,1,\dots,m-1)$ HF interpolation based on the equidistant nodes

$$(1.1) \qquad x_{k,n} = x_k = -1 + \frac{2k}{n} \qquad (k = 0,1,\dots,n).$$

One reason for the lack of attention paid to equidistant nodes is a result of Bernstein [**2**], who showed in 1918 that if $h(x) \equiv |x|$, and the x_k are given by (1.1), then the sequence $\overline{H}_{1n}(h,x)$ diverges as $n \to \infty$ for each x in $0 < |x| < 1$. Thus Lagrange interpolation on equidistant nodes diverges for a simple function such as $h(x)$. A quantitative version of Bernstein's result was developed by Byrne, Mills and Smith [**3**], who showed that if $0 < |x| < 1$, then

$$(1.2) \quad \limsup_{n\to\infty} \frac{1}{n} \log\left|\overline{H}_{1n}(h,x) - h(x)\right| = \frac{1}{2}\left[(1+x)\log(1+x) + (1-x)\log(1-x)\right].$$

(See also Li and Mohapatra [**6**].)

For $(0,1)$ HF interpolation on the equidistant nodes (1.1), Berman [**1**] showed in 1958 that even for $g(x) \equiv x$, the sequence $\overline{H}_{2n}(g,x)$ diverges as $n \to \infty$ for each x in $0 < |x| < 1$. The only results of this type for $(0,1,\dots,m-1)$ HF interpolation $(m \geqslant 3)$ that we have been able to locate in the literature are due to Mendelevič [**7**]. In particular, Mendelevič showed that if m is even, the $(0,1,\dots,m-1)$ HF interpolation process based on the equidistant nodes $y_{k,n} = k/n$ $(k = 1,2,\dots,n)$ diverges for the function

$$\Psi(x) = \begin{cases} 0 & (0 \leqslant x < 1/2), \\ x - 1/2 & (1/2 \leqslant x \leqslant 1), \end{cases}$$

on a set $E \subset [0,1]$, where E has measure greater than 0.26.

In this paper we shall prove the following theorem that both generalises and quantifies Berman's divergence result, and also provides a simpler example of divergence of $(0,1,\dots,m-1)$ HF interpolation on equidistant nodes for even m than Mendelevič's example.

Theorem 1. *Suppose $m \geqslant 2$ is even, and $g(x) \equiv x$. Then for each x in $0 < |x| < 1$, the $(0,1,\ldots,m-1)$ Hermite-Fejér interpolation polynomials $\overline{H}_{mn}(g,x)$ based on the equidistant nodes*

$$x_{k,n} = x_k = -1 + \frac{2k}{n} \qquad (k = 0,1,\ldots,n)$$

satisfy

(1.3)
$$\limsup_{n\to\infty} \frac{1}{n} \log\left|\overline{H}_{mn}(g,x) - g(x)\right| = \frac{m}{2}\left[(1+x)\log(1+x) + (1-x)\log(1-x)\right].$$

The proof of Theorem 1 will be presented in Section 3. We note that since ± 1 are interpolation nodes for all n, then $g(-1) = \overline{H}_{mn}(g,-1)$ and $g(1) = \overline{H}_{mn}(g,1)$ for all n. Furthermore, since the equidistant nodes are distributed symmetrically about 0, then $\overline{H}_{mn}(f,x)$ is an odd function whenever $f(x)$ is odd. Hence $g(0) = \overline{H}_{mn}(g,0) = 0$ for all n. Thus Theorem 1 settles the convergence behaviour of $\overline{H}_{mn}(g,x)$ for all x in $[-1,1]$. We also point out that our proof of Theorem 1 does not readily adapt to the case when $m(\geqslant 3)$ is an odd number. However, we conjecture that Theorem 1 remains true for all such values of m. (For $m = 1$, Theorem 1 is false, since $\overline{H}_{mn}(g,x) \equiv g(x)$, although (by (1.2)) it does hold true if g is replaced by $h(x) \equiv |x|$.)

2. Preliminary results

In this section we introduce further notation and some preliminary results that will be needed for the proof of Theorem 1.

Suppose

$$-1 \leqslant x_{0,n} < x_{1,n} < \ldots < x_{n,n} \leqslant 1 \tag{2.1}$$

is a system of interpolatory nodes, and let $m \geqslant 1$ be an integer. If $f(x)$ is $m-1$ times differentiable on $[-1,1]$, the $(0,1,\ldots,m-1)$ Hermite interpolation polynomial for f is the unique polynomial $H_{mn}(f,x)$ of degree $m(n+1)-1$ or less which satisfies

$$H_{mn}^{(p)}(f,\, x_k) = f^{(p)}(x_k) \qquad (p = 0,\, 1,\, \ldots,\, m-1;\; k = 0,\, 1,\, \ldots,\, n).$$

$H_{mn}(f,x)$ can be written in the form

$$H_{mn}(f,x) = \sum_{k=0}^{n}\sum_{j=0}^{m-1} f^{(j)}(x_k) A_{jk}(x), \tag{2.2}$$

where the polynomials $A_{jk}(x)$ (more precisely, $A_{jkmn}(x)$) are the unique polynomials of degree $m(n+1)-1$ or less which satisfy

$$A_{jk}^{(p)}(x_q) = \delta_{jp}\delta_{kq} \qquad (j,p = 0,1,\ldots,m-1;\; k,q = 0,1,\ldots,n).$$

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Note that the $(0,1,\ldots,m-1)$ HF interpolation polynomial for f can be written as

$$\overline{H}_{mn}(f,x)=\sum_{k=0}^{n} f(x_k)A_{0k}(x). \tag{2.3}$$

Now, if $g(x)\equiv x$, then $H_{mn}(g,x)\equiv x$. Hence, by (2.2),

$$\sum_{k=0}^{n} x_k A_{0k}(x)+\sum_{k=0}^{n} A_{1k}(x)=x$$

for all x in $[-1,1]$, and so by (2.3),

$$x-\overline{H}_{mn}(g,x)=\sum_{k=0}^{n} A_{1k}(x). \tag{2.4}$$

Thus to prove (1.3) it will suffice to consider $\sum\limits_{k=0}^{n} A_{1k}(x)$.

The following formula for the $A_{jk}(x)$ was developed by Szabados [**8**, Lemma 1]. Define

$$\omega_n(x)=\prod_{k=0}^{n}(x-x_k), \tag{2.5}$$

and put

$$\ell_k(x)=\ell_{kn}(x)=\frac{\omega_n(x)}{\omega_n'(x_k)(x-x_k)} \qquad (k=0,1,\ldots,n). \tag{2.6}$$

Then

$$A_{jk}(x)=\frac{\ell_k(x)^m}{j!}(x-x_k)^j B_{jk}(x) \qquad (j=0,1,\ldots,m-1;\ k=0,1,\ldots,n), \tag{2.7}$$

where

(2.8)

$$B_{jk}(x)=B_{jkmn}(x)=\sum_{i=0}^{m-j-1} b_{ik}(x-x_k)^i \qquad (j=0,1,\ldots,m-1;\ k=0,1,\ldots,n),$$

and

$$b_{ik}=b_{ikmn}=\frac{\left[\ell_k(x)^{-m}\right]^{(i)}_{x=x_k}}{i!} \qquad (i=0,1,\ldots;\ k=0,1,\ldots,n). \tag{2.9}$$

We shall need the following lemma which is also due to Szabados [**8**, Lemmas 2 and 3].

LEMMA 1. *Define*

$$a_{ik}=a_{ikmn}=m\sum_{\substack{\nu=0\\ \nu\neq k}}^{n}\frac{1}{(x_\nu-x_k)^i} \qquad (k=0,1,\ldots,n;\ i=1,2,\ldots), \tag{2.10}$$

and let $B_{jk}(x)$ *and* b_{ik} *be given by (2.8) and (2.9). Then*

$$b_{ik} = \frac{1}{i}\sum_{\nu=1}^{i} a_{\nu k} b_{i-\nu,k} \qquad (k = 0,1,\dots,n;\ i = 1,2,\dots). \tag{2.11}$$

Also, there exists a positive number c *(depending only on* j *and* m*) so that*

$$B_{jk}(x) \geqslant c\left(\frac{x - x_k}{x_k - x_{k\pm 1}}\right)^{m-j-1} \tag{2.12}$$

$$(-\infty < x < \infty,\ m - j \text{ odd},\ 0 \leqslant j \leqslant m-1,\ 0 \leqslant k \leqslant n),$$

with one of the signs in $x_{k\pm1}$.

The formulas and results of this section so far are valid for an arbitrary system (2.1) of nodes. Henceforth we shall assume that the interpolation nodes are the equidistant nodes

$$x_k = -1 + \frac{2k}{n} \qquad (k = 0,1,\dots,n). \tag{2.13}$$

We now develop an upper bound for $|B_{1k}(x)|$ which, by (2.4) and (2.7), will be useful later when obtaining bounds for $\left|\overline{H}_{mn}(g,x) - x\right|$.

LEMMA 2. *There exist constants* c_{im} $(i = 0,1,\dots;\ m = 1,2,\dots)$ *so that for* $n \geqslant 2$,

$$|b_{ik}| \leqslant c_{im}(n\log n)^i \qquad (i = 0,1,\dots;\ k = 0,1,\dots,n). \tag{2.14}$$

PROOF: From (2.10) and (2.13) we have

$$a_{ik} = m\left(\frac{n}{2}\right)^i \sum_{\substack{\nu=0\\ \nu\neq k}}^{n} \frac{1}{(\nu-k)^i} \qquad (k = 0,1,\dots,n;\ i = 1,2,\dots),$$

and so

$$|a_{ik}| \leqslant m\left(\frac{n}{2}\right)^i \times 2\sum_{r=1}^{n}\frac{1}{r^i} \leqslant m\,n^i \sum_{r=1}^{n}\frac{1}{r^i}.$$

Thus there exist constants c_m (independent of n) so that

$$|a_{ik}| \leqslant \begin{cases} c_m n\log n & (i = 1;\, k = 0,1,\dots,n), \\ c_m n^i & (i = 2,3,\dots;\ k = 0,1,\dots,n). \end{cases} \tag{2.15}$$

We now prove (2.14) by induction on i. Since $\ell_k(x_k) = 1$ for all k, (2.9) yields $b_{0k} = 1$ for all k, and so (2.14) holds true for $i = 0$ if we define $c_{0m} = 1$. If (2.14) holds true for $i = 0,1,\dots,r-1$, then by (2.11) and (2.15) we have

$$\begin{aligned}|b_{rk}| &\leqslant \frac{1}{r}\sum_{\nu=1}^{r}|a_{\nu k}|\,|b_{r-\nu,k}| \\ &\leqslant \frac{c_m}{r}\left(n\log n \times c_{r-1,m}(n\log n)^{r-1} + \sum_{\nu=2}^{r} n^{\nu}\times c_{r-\nu,m}(n\log n)^{r-\nu}\right) \\ &= \frac{c_m}{r}(n\log n)^r\left(c_{r-1,m} + \sum_{\nu=2}^{r}\frac{c_{r-\nu,m}}{(\log n)^{\nu}}\right) \\ &\leqslant \frac{c_m}{r}(n\log n)^r\left(c_{r-1,m} + \sum_{\nu=2}^{r}\frac{c_{r-\nu,m}}{(\log 2)^{\nu}}\right).\end{aligned}$$

On defining

$$c_{rm} = \frac{c_m}{r}\left(c_{r-1,m} + \sum_{\nu=2}^{r}\frac{c_{r-\nu,m}}{(\log 2)^{\nu}}\right),$$

the lemma is established. □

Corollary. *There exist constants d_m $(m = 1, 2, \ldots)$ so that for $n = 2, 3, \ldots,$*

(2.16) $$|B_{1k}(x)| \leqslant d_m(n\log n)^{m-2} \qquad (k = 0, 1, \ldots, n;\ -1 \leqslant x \leqslant 1).$$

Proof: Since $|x - x_k| \leqslant 2$, (2.8) gives

$$|B_{1k}(x)| \leqslant \sum_{i=0}^{m-2} 2^i\,|b_{ik}|.$$

By Lemma 2 we then have

$$\begin{aligned}|B_{1k}(x)| &\leqslant \sum_{i=0}^{m-2} 2^i c_{im}(n\log n)^i \\ &\leqslant d_m(n\log n)^{m-2},\end{aligned}$$

where $d_m = \sum_{i=0}^{m-2} 2^i c_{im}$. □

We consider next the polynomials $\ell_k(x)$ as defined by (2.5) and (2.6). Upon writing

(2.17) $$x = x_j + \frac{2\theta}{n} = -1 + \frac{2}{n}(j + \theta),$$

where $0 \leqslant \theta < 1$, and using (2.13), we obtain

(2.18) $$\ell_k(x) = \frac{(-1)^{k+j}(\theta)_{j+1}(1-\theta)_{n-j}}{(j + \theta - k)k!(n-k)!},$$

where

$$(a)_k = \begin{cases} 1 & (k = 0), \\ a(a+1)\ldots(a+k-1) & (k = 1, 2, \ldots). \end{cases}$$

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The following lemma will be useful in determining the behaviour of the $\ell_k(x)$ for large n.

Lemma 3. *If $|x| < 1$, and x is given by (2.17) with $0 \leqslant \theta < 1$, define*

$$q(x) = q_n(x) = (1+\theta)_j (2-\theta)_{n-j-1}. \tag{2.19}$$

Then

(2.20)

$$\lim_{n\to\infty} \left(\frac{1}{n}\log q(x) - \log n\right) = -1 - \log 2 + \frac{1}{2}\left[(1+x)\log(1+x) + (1-x)\log(1-x)\right].$$

Proof: Firstly note from (2.17) that

$$\lim_{n\to\infty} \frac{j}{n} = \frac{1+x}{2}, \tag{2.21}$$

and so j, $n-j \to \infty$ as $n \to \infty$. Now, (2.19) can be written as

$$q(x) = \frac{\Gamma(j+1+\theta)\,\Gamma(n+1-j-\theta)}{\Gamma(1+\theta)\,\Gamma(2-\theta)},$$

and hence, upon using the asymptotic expansion [9, page 252]

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + O\left(z^{-1}\right) \tag{2.22}$$

as $z \to \infty$, we obtain

$$\begin{aligned}
&\frac{1}{n}\log q(x) \\
&= \left(\frac{j+1/2+\theta}{n}\right)\log(j+1+\theta) + \left(\frac{n+1/2-j-\theta}{n}\right)\log(n+1-j-\theta) - 1 + O\left(n^{-1}\right) \\
&= \left(\frac{j+1/2+\theta}{n}\right)\log\left(\frac{j+1+\theta}{n}\right) + \left(\frac{n+1/2-j-\theta}{n}\right)\log\left(\frac{n+1-j-\theta}{n}\right) \\
&\quad + \log n - 1 + O\left(\frac{\log n}{n}\right).
\end{aligned}$$

The lemma is then established by letting $n \to \infty$ and using (2.21). ∎

3. Proof of the Theorem

We now prove Theorem 1. Since $\overline{H}_{mn}(g,x)$ is odd, we can assume without loss of generality that $x < 0$. Write $x = -1 + 2(j+\theta)/n$, where $0 \leqslant \theta < 1$. By (2.4) and (2.7) we have

$$\left|x - \overline{H}_{mn}(g,x)\right| = \left|\sum_{k=0}^{n} \ell_k(x)^m B_{1k}(x)(x - x_k)\right|.$$

Now, from (2.12), $B_{1k}(x) > 0$ for all x, and so

$$\operatorname{sgn}\ [\ell_k(x)^m B_{1k}(x)(x - x_k)] = \begin{cases} +1 & (j \geqslant k), \\ -1 & (j < k). \end{cases}$$

Therefore, on putting $n' = [n/2]$, we obtain

$$\ell_{n'}(x)^m B_{1n'}(x)(x_{n'} - x) - \left(\sum_{k=0}^{j} \ell_k(x)^m B_{1k}(x)(x - x_k)\right) \leqslant \left|x - \overline{H}_{mn}(g,x)\right| \leqslant \sum_{k=0}^{n} \ell_k(x)^m B_{1k}(x)\,|x - x_k|\,. \tag{3.1}$$

We work firstly with the right hand side of (3.1). Since $|x - x_k| \leqslant 2$, and $k!(n-k)! \geqslant (n')!(n-n')!$, we have from (2.16) and (2.18),

$$\left|x - \overline{H}_{mn}(g,x)\right| \leqslant \frac{2d_m(n\log n)^{m-2}}{((n')!(n-n')!)^m} \sum_{k=0}^{n} \left(\frac{(\theta)_{j+1}(1-\theta)_{n-j}}{j+\theta-k}\right)^m .$$

Now $(\theta(1-\theta))/(|j+\theta-k|) \leqslant 1$ for all k, and so

$$\left|x - \overline{H}_{mn}(g,x)\right| \leqslant 2d_m(n+1)(n\log n)^{m-2} \left(\frac{(1+\theta)_j(2-\theta)_{n-j-1}}{(n')!(n-n')!}\right)^m .$$

Thus, with $q(x)$ given by (2.19), we have

$$\frac{1}{n}\log\left|x - \overline{H}_{mn}(g,x)\right| \leqslant \frac{m}{n}\log\left(\frac{q(x)}{(n')!(n-n')!}\right) + O\left(\frac{\log n}{n}\right). \tag{3.2}$$

From (2.22) it follows that

$$\frac{1}{n}\log\left((n')!(n-n')!\right) = \log n - \log 2 - 1 + O\left(\frac{\log n}{n}\right). \tag{3.3}$$

Hence, by (2.20), we can conclude from (3.2) that

$$\limsup_{n\to\infty} \frac{1}{n}\log\left|x - \overline{H}_{mn}(g,x)\right| \leqslant \frac{m}{2}\left[(1+x)\log(1+x) + (1-x)\log(1-x)\right]. \tag{3.4}$$

Next consider the summation term on the left hand side of (3.1). Since $\lim_{n\to\infty} j/n = (1+x)/2 < 1/2$, there exists a number $\alpha < 1/2$ so that $j < \alpha n$ for all n large enough. Then, because $k!(n-k)! = \Gamma(k+1)\,\Gamma(n-k+1) > \Gamma(\alpha n+1)\,\Gamma((1-\alpha)n+1)$ for $0 \leqslant$

$k \leqslant j$, we have (as with the derivation of (3.2)),

$$\frac{1}{n}\log\left|\sum_{k=0}^{j}\ell_k(x)^m B_{1k}(x)(x-x_k)\right| \leqslant \frac{m}{n}\log\left(\frac{q(x)}{\Gamma(\alpha n+1)\,\Gamma((1-\alpha)n+1)}\right) + O\left(\frac{\log n}{n}\right).$$

By (2.22) we can write

$$\frac{1}{n}\log\left(\Gamma(\alpha n+1)\,\Gamma((1-\alpha)n+1)\right) = \log n + \log\left(\alpha^\alpha(1-\alpha)^{1-\alpha}\right) - 1 + O\left(\frac{\log n}{n}\right),$$

and so by (2.20) we have

(3.5)

$$\limsup_{n\to\infty}\frac{1}{n}\log\left(\sum_{k=0}^{j}\ell_k(x)^m B_{1k}(x)(x-x_k)\right) \leqslant \frac{m}{2}\left[(1+x)\log(1+x)+(1-x)\log(1-x)\right]+km,$$

where $k = -\log 2 - \log\left(\alpha^\alpha(1-\alpha)^{1-\alpha}\right) < 0$.

It remains to consider the expression $\ell_{n'}(x)^m B_{1n'}(x)(x_{n'}-x)$ on the left hand side of (3.1). Because $|x_{n'} - x_{n'\pm 1}| = 2/n$, (2.12) gives

$$B_{1n'}(x) \geqslant c\left(\frac{n}{2}\right)^{m-2}(x_{n'}-x)^{m-2} = c(n'-j-\theta)^{m-2},$$

where c depends only on m. Thus

$$\ell_{n'}(x)^m B_{1n'}(x)(x_{n'}-x) \geqslant \frac{2c}{n(n'-j-\theta)}\left(\frac{q(x)}{(n')!(n-n')!}\right)^m(\theta(1-\theta))^m. \tag{3.6}$$

By Berman [1, Lemma 1] for each x there exists an increasing sequence $\{k_n\}_{n=1}^\infty$ of positive integers, and a number $a(x)$ with $0 < a(x) < 1/2$, such that if we write

$$x = -1 + \frac{2}{k_n}(j+\theta),$$

where $0 \leqslant \theta < 1$, then $a(x) \leqslant \theta \leqslant 1 - a(x)$ for all n. Hence we can assume $\theta(1-\theta)$ has a positive lower bound, and then, on using $n' - j - \theta \leqslant n$, (3.6) can be written in the form

$$\ell_{n'}(x)^m B_{1n'}(x)(x_{n'}-x) \geqslant \frac{c'}{n^2}\left(\frac{q(x)}{(n')!(n-n')!}\right)^m,$$

where c' depends only on m. By (2.20) and (3.3) we can conclude that

(3.7)

$$\limsup_{n\to\infty}\frac{1}{n}\log\left(\ell_{n'}(x)^m B_{1n'}(x)(x_{n'}-x)\right) \geqslant \frac{m}{2}\left[(1+x)\log(1+x)+(1-x)\log(1-x)\right].$$

To complete the proof of Theorem 1, we observe that the estimates (3.5) and (3.7) for the terms on the left hand side of (3.1) yield

$$(3.8) \qquad \limsup_{n\to\infty} \frac{1}{n} \log \left| x - \overline{H}_{mn}(g,x) \right| \geqslant \frac{m}{2} \left[(1+x)\log(1+x) + (1-x)\log(1-x) \right].$$

The required statement (1.3) then follows from (3.4) and (3.8).

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