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DUALITY IN NONCONVEX VECTOR MINIMUM PROBLEMS

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A nonlinear vector minimum problem is considered. Duality theorems are proved for Mond-Weir type dual and their application to a certain nonlinear fractional vector minimum problem is discussed.

1. INTRODUCTION

Consider the vector minimum problem:

$$\text{(P)} \qquad \text{minimise } f(x) \text{ subject to } g(x) \leqq 0$$

where $f\colon R^n \to R^k$ and $g\colon R^n \to R^m$ are differentiable functions. Wolfe and Mond-Weir type duality for (P) has been discussed in several papers [**1, 7, 8, 9, 10**] using different concepts of optimality, namely: weak efficient (or weak minimum), efficient (or nondominated or noninferior or Pareto optimal) and properly efficient solutions.

Weir [**8**] proved weak and strong duality theorems for Mond-Weir [**5**] type dual of (P). In the strong duality theorem he obtained an efficient solution of the dual from a properly efficient solution of the primal problem. Bector *et al* [**1**] also discussed a similar result under stronger convexity assumptions. The duality results in Singh [**7**] and Weir and Mond [**10**] are for efficient and weak efficient solutions respectively. The converse duality theorems in [**1, 7**] are proved using Kuhn-Tucker type necessary conditions of Singh [**6**] and therefore need a constraint qualification.

In the present paper we also discuss duality results for Mond-Weir type dual of (P). Our results are different than those in [**1, 7, 8, 10**]. The strong duality theorem provides a properly efficient solution of the dual while in the converse duality theorem a weak efficient solution of the dual gives a properly efficient solution of the primal problem. Moreover, the converse duality theorem is proved using Fritz John necessary conditions [**2**] which do not need a constraint qualification.

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2. Preliminaries

The following convention of vectors in R^n will be followed throughout this paper: $x \leqq y \Leftrightarrow x_i \leqq y_i$, $i = 1, 2, \ldots, n$; $x \leqslant y \Leftrightarrow x \leqq y$, $x \neq y$; $x < y \Leftrightarrow x_i < y_i$, $i = 1, 2, \ldots, n$. $\nabla g(x)$ will denote the $m \times n$ Jacobian matrix of g at x, the index set $K = \{1, 2, \ldots, k\}$ and $K_i = K - \{i\}$. For other notations and definitions we refer to Mangasarian [4].

Geoffrion [3] introduced the following scalar parametric problem:

$$(P_\lambda) \qquad \text{minimise } \lambda^T f(x) \text{ subject to } g(x) \leqq 0$$

and related its optimal solution to a properly efficient solution of (P) as follows:

Lemma 1. *Let $\lambda > 0$ be fixed. If $\bar{x}$ is an optimal solution of (P_λ), then $\bar{x}$ is a properly efficient solution of (P).*

The Comprehensive Theorem in Geoffrion [3] includes the following necessary and sufficient conditions for problem (P):

Lemma 2. (Kuhn-Tucker type necessary conditions). *Let $\bar{x}$ be a properly efficient solution of problem (P) and let g satisfy the Kuhn-Tucker constraint qualification at $\bar{x}$. Then there exist $\bar{\lambda} \in R^k$, $\bar{\mu} \in R^m$ such that*

$$\bar{\lambda}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}) = 0$$
$$\bar{\mu}^T g(\bar{x}) = 0$$
$$\bar{\lambda} > 0, \bar{\mu} \geqq 0.$$

Lemma 3. (Kuhn-Tucker type sufficient conditions). *Let f and g be convex. If there exist $\bar{\lambda} \in R^k$ and $\bar{\mu} \in R^m$ such that*

$$\bar{\lambda}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}) = 0$$
$$\bar{\mu}^T g(\bar{x}) = 0$$
$$\bar{\lambda} > 0, \bar{\mu} \geqq 0,$$

then $\bar{x}$ is a properly efficient solution of (P).

3. Duality

In relation to (P) we consider the following Mond-Weir type dual:

(D) maximise $f(y)$

(1) subject to $\lambda^T \nabla f(y) + \mu^T \nabla g(y) = 0$

(2) $\mu^T g(y) \geqq 0$

(3) $\lambda > 0, \quad \mu \geqq 0.$

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Theorem 3.1. (Weak Duality). *Let x be feasible for (P) and (y, λ, μ) be feasible for (D). If $\lambda^T f$ is pseudoconvex and $\mu^T g$ is quasiconvex at y, then*

$$\lambda^T f(x) \geqq \lambda^T f(y). \tag{4}$$

Proof: Since $g(x) \leqq 0$ and $\mu \geqq 0$,

$$\mu^T g(x) \leqq 0 \leqq \mu^T g(y).$$

Using quasiconvexity of $\mu^T g$ at y, we get

$$\mu^T \nabla g(y)(x - y) \leqq 0.$$

Therefore from equation (1)

$$\lambda^T \nabla f(y)(x - y) \geqq 0.$$

But $\lambda^T f$ is pseudoconvex at y. Hence

$$\lambda^T f(x) \geqq \lambda^T f(y).$$

□

Remark 1. It may be noted that the inequality (4) implies $f(x) \nleq f(y)$.

Theorem 3.2. *Let $\bar{x}$ be feasible for (P) and $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be feasible for (D) such that*

$$\bar{\lambda}^T f(\bar{x}) = \bar{\lambda}^T f(\bar{y}). \tag{5}$$

If $\bar{\lambda}^T f$ is pseudoconvex and $\bar{\mu}^T g$ is quasiconvex at $\bar{y}$, then $\bar{x}$ is properly efficient for (P).

Proof: Let x be any feasible solution for (P). From the weak duality theorem,

$$\bar{\lambda}^T f(x) \geqq \bar{\lambda}^T f(\bar{y}).$$

Using (5), we get

$$\bar{\lambda}^T f(x) \geqq \bar{\lambda}^T f(\bar{x}).$$

Thus $\bar{x}$ is optimal for $(P_{\bar{\lambda}})$. Hence by Lemma 1, $\bar{x}$ is properly efficient for (P). □

Theorem 3.3. *Let $\bar{x}$ be feasible for (P) and $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be feasible for (D) such that*

$$f(\bar{x}) = f(\bar{y}). \tag{6}$$

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If $\lambda^T f$ is pseudoconvex and $\mu^T g$ is quasiconvex at y for each dual feasible (y, λ, μ), then $(\overline{y}, \overline{\lambda}, \overline{\mu})$ is properly efficient for the dual problem (D).

PROOF: First we show that $(\overline{y}, \overline{\lambda}, \overline{\mu})$ is efficient for (D). Assume that it is not efficient, then there exists (y^*, λ^*, μ^*) feasible for (D) such that

$$f_r(y^*) > f_r(\overline{y}) \text{ for some } r \in K$$
$$f_i(y^*) \geqq f_i(\overline{y}) \text{ for all } i \in K_r.$$

Therefore

$$\lambda^{*T} f(y^*) > \lambda^{*T} f(\overline{y})$$

or using (6),

$$\lambda^{*T} f(y^*) > \lambda^{*T} f(\overline{x}),$$

a contradiction to the weak duality theorem. Hence $(\overline{y}, \overline{\lambda}, \overline{\mu})$ is efficient for (D). Assume now that it is not properly efficient. Then there exists a dual feasible solution (y^*, λ^*, μ^*) and an $i \in K$ such that $f_i(y^*) > f_i(\overline{y})$ and

$$f_i(y^*) - f_i(\overline{y}) > M(f_j(\overline{y}) - f_j(y^*))$$

for all $M > 0$ and all $j \in K_i$ satisfying $f_j(\overline{y}) > f_j(y^*)$. This means that $f_i(y^*) - f_i(\overline{y})$ can be made arbitrary large whereas $f_j(\overline{y}) - f_j(y^*)$ is finite for all $j \in K_i$. Therefore

$$\lambda_i^*(f_i(y^*) - f_i(\overline{y})) > \sum_{j \in K_i} \lambda_j^*(f_j(\overline{y}) - f_j(y^*))$$

or

$$\lambda^{*T} f(y^*) > \lambda^{*T} f(\overline{y}).$$

Using (6), we get

$$\lambda^{*T} f(y^*) > \lambda^{*T} f(\overline{x}).$$

This again contradicts the weak duality theorem. Hence $(\overline{y}, \overline{\lambda}, \overline{\mu})$ is a properly efficient solution for (D). ∎

THEOREM 3.4. (Strong Duality). *Let $\overline{x}$ be a properly efficient solution of Problem (P) and let g satisfy the Kuhn-Tucker constraint qualification at $\overline{x}$. Then there exists $(\overline{\lambda}, \overline{\mu})$, such that $(\overline{y} = \overline{x}, \overline{\lambda}, \overline{\mu})$ is a feasible solution for (D) and the objective values of (P) and (D) are equal. Also, if $\lambda^T f$ is pseudoconvex and $\mu^T g$ is quasiconvex at y for every dual feasible solution (y, λ, μ), then $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is a properly efficient solution for (D).*

PROOF: Since $\overline{x}$ is a properly efficient solution for (P) at which the Kuhn-Tucker constraint qualification is satisfied, by Lemma 2 there exist $\overline{\lambda} \in R^k$ and $\overline{\mu} \in R^m$ such that

$$\overline{\lambda}^T \nabla f(\overline{x}) + \overline{\mu}^T \nabla g(\overline{x}) = 0$$
$$\overline{\mu}^T g(\overline{x}) = 0$$
$$\overline{\lambda} > 0, \quad \overline{\mu} \geqq 0.$$

Therefore $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (D). Also, the two vector objectives are equal. Hence, by Theorem 3.3, $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a properly efficient solution for (D). □

THEOREM 3.5. (Converse Duality). *Let $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be a weak efficient solution for (D), the $n\times n$ Hessian matrix $\nabla^2\left(\bar{\lambda}^T f(\bar{y}) + \bar{\mu}^T g(\bar{y})\right)$ be positive or negative definite and $\nabla f_i(\bar{y})$, $i = 1, 2, \ldots, k$ be linearly independent. If $\lambda^T f$ is pseudoconvex and $\mu^T g$ is quasiconvex at $\bar{y}$, then $\bar{y}$ is a properly efficient solution for (P).*

PROOF: Since $(\bar{y}, \bar{\lambda}, \bar{\mu})$ is a weak efficient solution for (D), by Theorem 1 [2] there exist $\bar{u} \in R^k$, $\bar{v} \in R^n$, $\bar{w} \in R$, $\bar{\eta} \in R^k$ and $\bar{\nu} \in R^m$ such that

$$\bar{u}^T\nabla f(\bar{y}) + \bar{v}^T\nabla^2\left(\bar{\lambda}^T f(\bar{y}) + \bar{\mu}^T g(\bar{y})\right) + \bar{w}\,\bar{\mu}^T\nabla g(\bar{y}) = 0 \tag{7}$$

$$[\nabla f(\bar{y})]\bar{v} + \bar{\eta} = 0 \tag{8}$$

$$[\nabla g(\bar{y})]\bar{v} + \bar{w}g(\bar{y}) + \bar{\nu} = 0 \tag{9}$$

$$\bar{w}[\bar{\mu}^T g(\bar{y})] = 0 \tag{10}$$

$$\bar{\eta}^T\bar{\lambda} = 0 \tag{11}$$

$$\bar{\nu}^T\bar{\mu} = 0 \tag{12}$$

$$(\bar{u}, \bar{w}, \bar{\eta}, \bar{\nu}) \geqq 0, \qquad (\bar{u}, \bar{v}, \bar{w}, \bar{\eta}, \bar{\nu}) \neq 0. \tag{13}$$

Now $\bar{\lambda} > 0$, $\bar{\eta}^T\bar{\lambda} = 0 \Rightarrow \bar{\eta} = 0$. Therefore from (8)

$$[\nabla f(\bar{y})]\bar{v} = 0 \tag{14}$$

which with equation (1) gives

$$\bar{\mu}^T\nabla g(\bar{y})\bar{v} = 0. \tag{15}$$

On multipling (7) by $\bar{v}$ from the right and using equations (14) and (15), we get

$$\bar{v}^T[\nabla^2\left(\bar{\lambda}^T f(\bar{y}) + \bar{\mu}^T g(\bar{y})\right)]\bar{v} = 0. \tag{16}$$

Since $\nabla^2\left(\bar{\lambda}^T f(\bar{y}) + \bar{\mu}^T g(\bar{y})\right)$ is assumed to be positive or negative definite, equation (16) gives $\bar{v} = 0$.

Now suppose $\bar{w} = 0$. Therefore from equation (9), $\bar{\nu} = 0$. Also, equation (7) gives

$$\bar{u}^T\nabla f(\bar{y}) = 0. \tag{17}$$

Since $\nabla f_i(\bar{y})$ are assumed to be linearly independent, (17) implies $\bar{u} = 0$. Thus we get

$$(\bar{u}, \bar{v}, \bar{w}, \bar{\eta}, \bar{\nu}) = 0,$$

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a contradiction to (13). Hence $\overline{w} > 0$.

Now from equation (9),

$$g(\overline{y}) = \frac{-\overline{\nu}}{\overline{w}} \leqq 0.$$

Therefore $\overline{y}$ is a feasible solution for (P), and Theorem 3.2 implies that $\overline{y}$ is a properly efficient solution for (P).

The following result gives sufficient conditions for a weak efficient solution of the dual problem (D) to be properly efficient. It follows immediately from Theorems 3.3 and 3.5. ∎

COROLLARY. *Let $(\overline{y}, \overline{\lambda}, \overline{\mu})$ be a weak efficient solution for (D) and let the hypotheses of Theorem 3.5 be satisfied. If $\lambda^T f$ is pseudoconvex and $\mu^T g$ is quasiconvex at y for each dual feasible (y, λ, μ), then $(\overline{y}, \overline{\lambda}, \overline{\mu})$ is a properly efficient solution of the dual problem (D).*

REMARK 2. It may be noted that in Weir and Mond [10] the dual variable $\lambda \geq 0$. Therefore, their implication $w^T\lambda_0 = 0 \Rightarrow w = 0$ in the converse duality theorem is erroneous.

THEOREM 3.6. (Strict Converse Duality Theorem). *Let $\overline{x}$ and $(\overline{y}, \overline{\lambda}, \overline{\mu})$ be feasible solutions for (P) and (D) respectively such that*

$$\overline{\lambda}^T f(\overline{x}) = \overline{\lambda}^T f(\overline{y}). \tag{18}$$

If $\overline{\lambda}^T f$ is strictly pseudoconvex and $\overline{\mu}^T g$ is quasiconvex at $\overline{y}$, then $\overline{x} = \overline{y}$ and $\overline{y}$ is properly efficient for (P).

PROOF: We will assume $\overline{x} \neq \overline{y}$ and exhibit a contradiction.

Since $\overline{x}$ and $(\overline{y}, \overline{\lambda}, \overline{\mu})$ are feasible for (P) and (D) respectively, we have

$$\overline{\mu}^T g(\overline{x}) \leqq 0 \leqq \overline{\mu}^T g(\overline{y}).$$

The quasiconvexity of $\overline{\mu}^T g$ at $\overline{y}$ implies

$$\overline{\mu}^T \nabla g(\overline{y})(\overline{x} - \overline{y}) \leqq 0. \tag{19}$$

Equations (1) and (19) imply

$$\overline{\lambda}^T \nabla f(\overline{y})(\overline{x} - \overline{y}) \geqq 0.$$

But $\overline{\lambda}^T f$ is strictly pseudoconvex at $\overline{y}$. Hence

$$\overline{\lambda}^T f(\overline{x}) > \overline{\lambda}^T f(\overline{y}),$$

a contradiction to (18). Hence $\overline{x} = \overline{y}$. Proper efficiency of $\overline{y}$ for (P) now follows from Theorem 3.2. ∎

4. Application

We now apply our results to obtain a dual for the following nonlinear fractional vector minimum problem:

$$\text{(FP)} \qquad \begin{aligned} &\text{minimise } f(x) = \left[\frac{\phi_1(x)}{\psi(x)}, \frac{\phi_2(x)}{\psi(x)}, \dots, \frac{\phi_k(x)}{\psi(x)}\right] \\ &\text{subject to } g(x) \leqq 0, \quad x \in S \end{aligned}$$

where

(i) $S \subseteq R^n$ is an open convex set,
(ii) $\phi\colon S \to R^k$, $g\colon S \to R^m$ are differentiable convex functions on S and $\psi\colon S \to R$ is a differentiable concave function on S, and
(iii) $\phi(x) \geqq 0$ and $\psi(x) > 0$ on S.

Therefore for each $\lambda > 0$, $\lambda^T f$ is pseudoconvex and since g is convex, convexity hypotheses in this paper are satisfied. Hence the dual problem (D) becomes

$$\begin{aligned} &\text{maximise } f(y) = \left[\frac{\phi_1(y)}{\psi(y)}, \frac{\phi_2(y)}{\psi(y)}, \dots, \frac{\phi_k(y)}{\psi(y)}\right] \\ &\text{subject to } \sum_{i=1}^{k} \lambda_i \left[\frac{\nabla\phi_i(y)}{\psi(y)} - \frac{\phi_i(y)}{\psi^2(y)}\nabla\psi(y)\right] + \mu^T \nabla g(y) = 0 \\ &\qquad \mu^T g(y) \geqq 0, \quad \lambda > 0,\ \mu \geqq 0,\ y \in S. \end{aligned}$$

On simplification, we get the following dual for (FP):

$$\text{(FD)} \qquad \begin{aligned} &\text{maximise} \quad f(y) \\ &\text{subject to} \\ &\qquad \lambda^T \nabla\phi(y) - \left(\sum_{i=1}^{k} v_i\right) \nabla\psi(y) + w^T \nabla g(y) = 0 \\ &\qquad w^T g(y) \geqq 0 \\ &\qquad \lambda_i \phi_i(y) - v_i \psi(y) = 0, \quad i = 1, 2, \dots, k \\ &\qquad \lambda > 0,\ v \geqq 0,\ w \geqq 0,\ y \in S \end{aligned}$$

where λ, $v \in R^k$ and $w \in R^m$.

In particular when $k = 1$, we get the following pair of scalar nonlinear program and its dual

Primal Problem:

$$\text{minimise } \frac{\phi(x)}{\psi(x)}$$
$$\text{subject to } g(x) \geqq 0, \qquad x \in S$$

Dual Problem:

$$\text{maximise} \quad f(y) = \frac{\phi(y)}{\psi(y)}$$

subject to

$$\nabla\phi(y) - v\,\nabla\psi(y) + w^T\nabla g(y) = 0$$
$$w^T g(y) \geqq 0$$
$$\phi(y) - v\psi(y) = 0$$
$$v,\, w \geqq 0,\, y \in S$$

where $\phi\colon S \to R$ and $v \in R$. Other notations are as in (FD).

REMARK 3. If ψ is linear, then $\phi(x) \geqq 0$ is not required and the dual variable v in the above dual problems is unrestricted in sign.

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