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A LINK BETWEEN TWO CHARACTERISATIONS OF COMPLETE MATRIX RINGS

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We establish a link between the two characterizations, independently obtained in 1991 by Fuchs and Robson, of complete matrix rings in terms of the existence of nilpotent elements.

Every ring herein is associative with identity, and matrix ring will mean complete matrix ring of size $n \geqslant 2$.

Apart from the well known characterisations of matrix rings, Fuchs [**1**] and Robson [**3**] independently gave in 1991 new criteria for a ring to be a matrix ring. Following a characterisation of matrix rings of size 2 in [**2**, Theorem III.2], Fuchs showed in [**1**, Theorem 1] that a ring R is isomorphic to a matrix ring $\mathbb{M}_n(S)$, for some ring S, if and only if there are elements x and y in R such that

$$(1)\qquad x^{n-1} \neq 0, \quad x^n = 0 = y^2, \quad x + y \text{ is invertible} \quad \text{and} \quad \operatorname{Ann}\{x^{n-1}\} \cap Ry = \{0\},$$

where $\operatorname{Ann}\{x^{n-1}\}$ denotes the left annihilator of x^{n-1}. We first note that the condition $x^{n-1} \neq 0$ in (1) is superfluous, since if $x^{n-1} = 0$, then $y = 0$, otherwise $\operatorname{Ann}\{x^{n-1}\} \cap Ry$ contains the nonzero element y. But then x is invertible, contradicting the nilpotency of x.

Robson proved in [**3**, Theorem 2.2] that $R \cong \mathbb{M}_n(S)$, for some ring S, if and only if there are elements x and y in R such that $x^n = 0$ and

$$(2)\qquad x^{n-1}y + x^{n-2}yx + \cdots + xyx^{n-2} + yx^{n-1} = 1.$$

Robson's method of proof involved the characterisation of a matrix ring $\mathbb{M}_n(S)$ as a direct sum of n mutually isomorphic right ideals, whereas Fuchs invoked the characterisation of a matrix ring in terms of a set $\{e_{ij} : 1 \leqslant i, j \leqslant n\}$ of matrix units, that is, $\sum_{i=1}^{n} e_{ii} = 1$ and $e_{ij}e_{kl} = \delta_{jk}e_{il}$, δ_{jk} being the Kronecker delta.

This short sequel was inspired by the observation that $x + y$ is its own inverse in the proof of (i) $\Rightarrow$ (ii) in [**2**, Theorem III.2]. Consider now the following criterion,

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which is a combination of (2) and some of the conditions in (1): there are elements x and y in a ring R such that

(3)

$$x^n = 0 = y^2,\ x^{n-1}y + x^{n-2}yx + \cdots + xyx^{n-2} + yx^{n-1} = 1 \text{ and } \operatorname{Ann}\{x^{n-1}\} \cap Ry = \{0\}.$$

We first show in Theorem 1 that (3) explicitly yields, in terms of x and y, on the one hand, the inverse of $x + y$, and on the other hand, a set of matrix units. This drastically simplifies the proof by Fuchs that the mentioned set is indeed a set of matrix units. Second, we prove in Proposition 2 that the intersection condition in (3) follows from the other conditions in (3) if $n = 2$ or 3, but we show in Example 3 that this is not necessarily so if $n \geqslant 4$. Fuchs observed in [1, Corollary 4] that the intersection condition in (1) follows from the other conditions in (1) if $n = 2$. Example 3 also shows that the intersection condition in (1) does not necessarily follow from the other conditions in (1) if $n \geqslant 4$. Finally, Example 4 shows that the intersection condition in (1) does not necessarily follow from the other conditions in (1) if $n = 3$.

Before we formally state the first result, we assume that x and y are elements of a ring R such that $x^n = 0 = y^2$ and such that (2) holds, and we point out three equalities. If we premultiply (2) by y, then

$$yx^{n-1}y + yx^{n-2}yx + \cdots + yxyx^{n-2} = y, \tag{4}$$

since $y^2 = 0$. Similarly, postmultiplication of (2) by y shows that $x^{n-2}yxy + \cdots + xyx^{n-2}y + yx^{n-1}y = y$, and so by (4)

$$yx^{n-2}yx + \cdots + yxyx^{n-2} = x^{n-2}yxy + \cdots + xyx^{n-2}y. \tag{5}$$

Next, if we simultaneously premultiply (2) by yx^{n-2} and postmultiply (2) by y, then

$$yx^{n-2}(xyx^{n-2})y + yx^{n-2}(yx^{n-1})y = yx^{n-2}y, \tag{6}$$

since $x^n = 0$.

Theorem 1. *If a ring R contains elements x and y such that $x^n = 0 = y^2$, $x^{n-1}y + x^{n-2}yx + \cdots + xyx^{n-2} + yx^{n-1} = 1$ and $\operatorname{Ann}\{x^{n-1}\} \cap Ry = \{0\}$ for some $n \geqslant 2$, then $(x+y)^{-1} = x^{n-1} + x^{n-2}y + x^{n-3}yx + \cdots + xyx^{n-3} + yx^{n-2}$ and $\{x^{i-1}yx^{n-j} : 1 \leqslant i,j \leqslant n\}$ is a set of matrix units in R (which implies that R is a matrix ring).*

Proof: We first show that

$$yx^{n-1}y = y, \tag{7}$$

and, if $n \geqslant 3$, that

$$yx^jy = 0, \quad j = 1, \dots, n-2. \tag{8}$$

It follows directly from (2) (or (4)) that $yxy = y$ in case $n = 2$. If $n \geqslant 3$, then by (5) we have $(yx^{n-2}y + \cdots + yxyx^{n-3})x = (x^{n-2}yx + \cdots + xyx^{n-2})y \in \mathrm{Ann}\{x^{n-1}\} \cap Ry = \{0\}$. Therefore (4) implies (7). We conclude from (6) and (7) that $yx^{n-2}y + yx^{n-2}y = yx^{n-2}y$, that is, $yx^{n-2}y = 0$, and so (8) holds in case $n = 3$. If $n \geqslant 4$, then let $2 < k \leqslant n-1$, and suppose now that $yx^{n-i}y = 0$ for $i = 2, \dots, k-1$. Then, again since $x^n = 0$, simultaneous premultiplication of (2) by yx^{n-k} and postmultiplication of (2) by y shows that $yx^{n-k}(x^{k-1}yx^{n-k})y + yx^{n-k}(x^{k-2}yx^{n-k+1})y + \cdots + yx^{n-k}(xyx^{n-2})y + yx^{n-k}(yx^{n-1})y = (yx^{n-1}y)x^{n-k}y + (yx^{n-2}y)x^{n-(k-1)}y + \cdots + (yx^{n-(k-1)}y)x^{n-2}y + yx^{n-k}(yx^{n-1})y = yx^{n-k}y$. Hence, by (7) and the induction hypothesis we have $yx^{n-k}y + yx^{n-k}y = yx^{n-k}y$, and so $yx^{n-k}y = 0$. Therefore induction establishes (8).

For $1 \leqslant i,j \leqslant n$, we set $e_{ij} := x^{i-1}yx^{n-j}$. Let $1 \leqslant k,l \leqslant n$. We conclude from (7) that $e_{ij}e_{jl} = x^{i-1}yx^{n-1}yx^{n-l} = e_{il}$. If $j \neq k$, then (8) and the fact that $x^n = 0 = y^2$ imply that $e_{ij}e_{kl} = x^{i-1}yx^{n-j+k-1}yx^{n-l} = 0$. By (2) we have $\sum_{i=1}^{n} e_{ii} = 1$, and so $\{e_{ij} : 1 \leqslant i,j \leqslant n\}$ is a set of matrix units. Finally, if one invokes (2), (8) and the nilpotency of x and y, then direct verification shows that $(x+y)(x^{n-1} + x^{n-2}y + x^{n-3}yx + \cdots + xyx^{n-3} + yx^{n-2}) = 1 = (x^{n-1} + x^{n-2}y + x^{n-3}yx + \cdots + xyx^{n-3} + yx^{n-2})(x+y)$. $\square$

PROPOSITION 2. *Let $n = 2$ or 3. If R is a ring containing elements x and y such that $x^n = 0 = y^2$ and $x^{n-1}y + x^{n-2}yx + \cdots + xyx^{n-2} + yx^{n-1} = 1$, then $\mathrm{Ann}\{x^{n-1}\} \cap Ry = \{0\}$.*

PROOF: If $n = 2$ and $ry \in \mathrm{Ann}\{x\} \cap Ry$, $r \in R$, then $ry = ry(xy + yx) = 0$. Next, let $n = 3$. Then (5) states that

$$yxyx = xyxy. \tag{9}$$

Also, if we premultiply (2) by yxy, then $yxyx^2y + yxyxyx = yxy$, and so by (9)

$$yxyx^2y = yxy. \tag{1}$$

Similarly, postmultiplication of (2) by yxy shows that

$$yx^2yxy = yxy. \tag{11}$$

But (6) states that $yx^2yxy + yxyx^2y = yxy$, and so by (10) and (11) we have

$$yxy = 0. \tag{12}$$

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Consequently, if $ry \in \operatorname{Ann}\{x^2\} \cap Ry$, $r \in R$, then $ry = ry(x^2y + xyx + yx^2) = (ryx^2)y + r(yxy)x + ry^2x^2 = 0$. ∎

We now exhibit, for every $n \geqslant 4$, a ring $R(n)$ containing elements x and y such that $x^{n-1} \neq 0$, $x^n = 0 = y^2$, $x + y$ is invertible and $x^{n-1}y + x^{n-2}yx + \cdots + xyx^{n-2} + yx^{n-1} = 1$, but $\operatorname{Ann}\{x^{n-1}\} \cap R(n)y \neq \{0\}$, which shows, firstly, that the intersection condition does not necessarily follow from the other conditions in (3) if $n \geqslant 4$, and, secondly, that the intersection condition neither necessarily follows from the other conditions in (1) if $n \geqslant 4$.

EXAMPLE 3. Let S be any ring and set $R(n) := \mathbb{M}_n(S)$, $n \geqslant 4$. We use the standard notation $E_{i,j}$ for the matrix in $\mathbb{M}_n(S)$ with 1_S (the identity of S) in position (i,j) and zeros elsewhere. Set $x := E_{2,1} + E_{3,2} + \cdots + E_{n,n-1} - E_{4,1}$ and $y := E_{1,n} + E_{2,n-1}$. Then

$$x^k = \begin{cases} E_{k+1,1} + E_{k+2,2} + \cdots + E_{n,n-k} - E_{k+3,1}, & \text{if } 1 \leqslant k \leqslant n-3; \\ E_{n-1,1} + E_{n,2}, & \text{if } k = n-2; \\ E_{n,1}, & \text{if } k = n-1; \\ 0, & \text{if } k = n. \end{cases} \tag{13}$$

Hence

$$x^{n-1}y = E_{n,n}, \quad yx^{n-1} = E_{1,1}, \quad x(yx^{n-2}) = x(E_{1,2} + E_{2,1}) = E_{2,2} + E_{3,1} - E_{4,2}, \tag{14}$$

and

$$(x^{n-2}y)x = (E_{n-1,n} + E_{n,n-1})x = \begin{cases} E_{3,3} + E_{4,2} - E_{3,1}, & \text{if } n = 4; \\ E_{4,4} + E_{5,3} - E_{5,1}, & \text{if } n = 5; \\ E_{n-1,n-1} + E_{n,n-2}, & \text{if } n \geqslant 6. \end{cases} \tag{15}$$

If $n \geqslant 5$ and $2 \leqslant k \leqslant n-3$, then $2 \leqslant n-1-k \leqslant n-3$, and so by (13)

$$\begin{aligned} &(x^k y)x^{n-1-k} \\ &= (E_{k+1,n} + E_{k+2,n-1} - E_{k+3,n})(E_{n-k,1} + E_{n-k+1,2} + \cdots + E_{n,k+1} - E_{n-k+2,1}) \\ &= E_{k+1,k+1} - E_{k+1,n}E_{n-k+2,1} + E_{k+2,k} - E_{k+2,n-1}E_{n-k+2,1} \\ &\qquad\qquad - E_{k+3,k+1} + E_{k+3,n}E_{n-k+2,1} \\ &= \begin{cases} E_{3,3} - E_{3,1} + E_{4,2} - E_{5,3} + E_{5,1}, & \text{if } k = 2; \\ E_{4,4} + E_{5,3} - E_{5,1} - E_{6,4}, & \text{if } n \geqslant 6 \text{ and } k = 3; \\ E_{k+1,k+1} + E_{k+2,k} - E_{k+3,k+1}, & \text{if } n \geqslant 7 \text{ and } 4 \leqslant k \leqslant n-3. \end{cases} \end{aligned} \tag{16}$$

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We conclude from (13)-(16) that

$$
\begin{aligned}
&x^{n-1}y + x^{n-2}yx + \cdots + xyx^{n-2} + yx^{n-1} \\
&= \begin{cases}
E_{4,4} + (E_{3,3} + E_{4,2} - E_{3,1}) + (E_{2,2} + E_{3,1} - E_{4,2}) + E_{1,1}, & \text{if } n = 4; \\
E_{5,5} + (E_{4,4} + E_{5,3} - E_{5,1}) + (E_{3,3} - E_{3,1} + E_{4,2} & \\
\qquad -E_{5,3} + E_{5,1}) + (E_{2,2} + E_{3,1} - E_{4,2}) + E_{1,1}, & \text{if } n = 5; \\
E_{6,6} + (E_{5,5} + E_{6,4}) + (E_{4,4} + E_{5,3} - E_{5,1} - E_{6,4}) + (E_{3,3} & \\
\qquad -E_{3,1} + E_{4,2} - E_{5,3} + E_{5,1}) + (E_{2,2} + E_{3,1} - E_{4,2}) + E_{1,1}, & \text{if } n = 6; \\
E_{n,n} + (E_{n-1,n-1} + E_{n,n-2}) + \sum_{k=4}^{n-3} (E_{k+1,k+1} + E_{k+2,k} - E_{k+3,k+1}) & \\
\qquad +(E_{4,4} + E_{5,3} - E_{5,1} - E_{6,4}) + (E_{3,3} - E_{3,1} + E_{4,2} - E_{5,3} + E_{5,1}) & \\
\qquad +(E_{2,2} + E_{3,1} - E_{4,2}) + E_{1,1}, & \text{if } n \geqslant 7;
\end{cases} \\
&= E_{1,1} + E_{2,2} + \cdots + E_{n,n} = 1_{R(n)} \text{ for every } \ n \geqslant 4.
\end{aligned}
$$

However, $\mathrm{Ann}\{x^{n-1}\} \cap R(n)y = SE_{1,n-1} + SE_{2,n-1} + \cdots + SE_{n,n-1} \neq \{0\}$. Also note that

$$
\begin{aligned}
&(x+y)^{-1} \\
&= \begin{cases}
E_{1,2} - E_{1,n} + E_{2,3} + E_{3,2} + E_{3,4} - E_{3,n} & \\
\qquad\qquad +E_{4,5} + \cdots + E_{n-1,n} + E_{n,1}, & \text{if } n \geqslant 5; \\
\frac{1}{2}(E_{1,2} - E_{1,4}) + E_{2,3} + \frac{1}{2}(E_{3,2} + E_{3,4}) + E_{4,1}, & \text{if 2 is invertible in } S \text{ and } n = 4.
\end{cases}
\end{aligned}
$$

Finally we construct a ring $R(3)$ containing elements x and y such that $x^2 \neq 0$, $x^3 = 0 = y^2$ and $x + y$ is invertible, but $\mathrm{Ann}\{x^2\} \cap R(3)y \neq \{0\}$.

EXAMPLE 4. Here $\mathbb{Z}$ denotes the ring of integers and $\mathbb{Z}_4$ denotes the ring $\mathbb{Z}/4\mathbb{Z}$ with elements 0,1,2,3. Set $R(3) := \mathbb{M}_3(\mathbb{Z}_4)$ and

$$
x := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad y := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

Then $x^2 \neq 0$, $x^3 = 0 = y^2$,

$$
(x+y)^{-1} = \begin{pmatrix} 0 & 3 & 2 \\ 0 & 1 & 3 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathrm{Ann}\{x^2\} \cap R(3)y = \begin{pmatrix} 0 & 2\mathbb{Z}_4 & 0 \\ 0 & 2\mathbb{Z}_4 & 0 \\ 0 & 2\mathbb{Z}_4 & 0 \end{pmatrix}.
$$

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