A Property of the Harmonic Triangle of the Complete Quadrangle formed by the four points of contact of the four tangents to the Cubic Curve from a point on it.

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§ 1. The following two propositions are necessary for what is to follow, and are very easy to prove.

I. *If A, B, C be three points such that B and C are conjugate with respect to the Polar Conic of A, then are C and A conjugate points with respect to the Polar Conic of B, and similarly with regard to the third vertex.*

A triangle possessing the property defined above, viz. that each pair of vertices are conjugate points with respect to the Polar Conic of the third vertex, is called an *Apolar Triangle.*

II. *If the cubic curve be referred to an Apolar Triangle as the triangle of reference, the term in xyz is absent from its equation.*

§ 2. Let O be a point on the cubic curve, and let the points of contact of the four tangents from O to the curve be P, Q, R, S.

Let the Harmonic Triangle of PQRS be ABC. Then it is a known property of the cubic curve that A, B, C lie on the curve; and if we refer the curve to ABC as triangles of reference, its equation will be of the form

$$px^2(y-z)+qy^2(z-x)+rz^2(x-y)=0 \qquad (1)$$

Hence triangles of the species defined in this article are Apolar Triangles.

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§ 3. Since it is a well-known proposition that if $O \equiv u$, where u is the Elliptic Parameter of O in the representation

$$x = \wp(u), \quad y = \wp'(u),$$

then

$$A \equiv u + \pi_1, \quad B \equiv u + \pi_2, \quad C \equiv u + \pi_3,$$

where π_1, π_2, π_3 are the ordinary semi-elliptic periods, we shall for convenience call the triangle ABC, defined in § 2, "the Elliptic Triangle corresponding to the point O."

§ 4. The proposition of Article 2 is really a particular case of the following more general theorem :

If P, Q, R, S be four points on a Cubic Curve, the Harmonic Triangle, ABC, is an Apolar Triangle.

Take the equation to the cubic curve in the form

$$\begin{aligned} &ax^3 + by^3 + cz^3 + 3a_2x^2y + 3a_3x^2z \\ &\quad + 3b_1y^2x + 3b_3y^2z + 3c_1z^2x + 3c_2z^2y + 6kxyz = 0, \qquad (2), \end{aligned}$$

and refer it to the triangle ABC, where

$$P \equiv (1, 1, 1); \; Q \equiv (-1, 1, 1); \; R \equiv (1, -1, 1); \; S \equiv (1, 1, -1).$$

Substituting in (2) we get

$$a + b + c + 3a_2 + 3a_3 + 3b_1 + 3b_3 + 3c_1 + 3c_2 + 6k = 0 \qquad (3)$$

$$-a + b + c + 3a_2 + 3a_3 - 3b_1 + 3b_3 - 3c_1 + 3c_2 - 6k = 0 \qquad (4)$$

$$a - b + c - 3a_2 + 3a_3 + 3b_1 + 3b_3 + 3c_1 - 3c_2 - 6k = 0 \qquad (5)$$

$$a + b - c + 3a_2 - 3a_3 + 3b_1 - 3b_3 + 3c_1 + 3c_2 - 6k = 0 \qquad (6)$$

Changing the sign of (3) and adding the four above equations we get $k = 0$, whence the result stated follows by II., and the application to Elliptic Triangles is evident in virtue of their definition in Article 3.

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§ 5. *Elliptic Triangles are Apolar Triangles with respect to all Cubic Curves having the same inflexions as the given Cubic, and to all Class-Cubics having the same Cuspidal Tangents as the Cayleyan of the given curve.*

Since all Cubics having the same Inflexional Points as a given Cubic pass through the intersections of a Cubic and its Hessian, we have merely to show that the Elliptic Triangles are Apolar Triangles with respect to both the Cubic and its Hessian.

Take the equation to the Cubic in the form (1).

Then the co-efficient of the term xyz in the equation to the Hessian is given in Salmon's *Higher Plane Curves* to be

$$abc-(ab_3c_2+bc_1a_3+ca_2b_1)+2k^3-2k(b_1c_1+c_2a_2+a_3b_3)$$
$$+3(a_2b_3c_1+a_3b_1c_2),$$

which evidently vanishes in the case of (1).

Hence the first portion of our proposition is established in virtue of II.

§ 6. Let us next prove that they are Apolar Triangles with respect to the Cayleyan.

Again quoting from Salmon, the co-efficient of lmn in the Cayleyan is

$$abc-(ab_3c_2+bc_1a_3+ca_2b_1)-4k^3+4k(b_1c_1+c_2a_2+a_3b_3)$$
$$-3(a_2b_3c_1+a_3b_1c_2),$$

which again evidently vanishes in the case of (1).

[*Corollary.*

We may here call attention to a corollary which has no immediate bearing on our subject, but which follows at once from the formulae quoted in Articles 5 and 6.

If an Apolar Triangle inscribed in a cubic be Apolar also to the Hessian, it is Apolar to the Cayleyan.]

§ 7. Lastly, we wish to prove that Elliptic Triangles are Apolar Triangles with respect to the Hessian of the Cayleyan.

The equation to the Cayleyan of (1) is

$$\begin{aligned}&qr(q-r)l^3+rp(r-p)m^3+pq(p-q)n^3\\&+r(qr+pq-2rp)l^2m-q(qr+rp-2pq)l^2n\\&+p(qr+rp-2pq)m^2n-r(rp+pq-2qr)m^2l\\&+q(rp+pq-2qr)n^2l-p(qr+pq-2rp)n^2m=0.\qquad(7)\end{aligned}$$

Now if we use the same notation with regard to the letters A, B, C, K, A_2, etc., in connexion with (7) as adopted in the point equation of Article 4, we see, on reference to Salmon, that the co-efficient of lmn in the Hessian of the Cayleyan (7) will vanish if (see Art. 6, and note that in (7) $K=0$)

$$A_2B_3C_1+A_3B_1C_2=0,$$

which is evident from (7).

Hence *Elliptic Triangles are Apolar Triangles with respect to the Cubic itself, its Hessian, its Cayleyan, and the Hessian of the Cayleyan;* whence the result enunciated in Article 5 is established.