Notes

108.19 Countable lists of rational numbers by removing digits

It is well-known that the rational numbers are countable, with proofs shown in [1], [2] and others. One way to list all the rationals between 0 and 1 is shown on the website "An easy proof that rational numbers are countable" [3]. The demonstration shows rationals in a table where the fractions' numerators and denominators correspond to row and column numbers. These rational numbers are then ordered by a zig-zag diagonal pattern that proceeds through the table while eliminating both duplicates and numbers greater than 1. The resulting countable list, List A includes all unique rational numbers in the half-open interval (0, 1]. The first ten rationals in List A are shown in the table below, which gives both fractional and decimal forms. When a rational is in decimal form, the digits to the right of the decimal point are referred to as the mantissa.

List A:

Position in List	1	2	3	4	5	6	7	8	9	10
Rational Number (as a fraction)	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{3}{4}$	$\frac{2}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
Rational Number (as a decimal)	$0.\bar{9}$	0.5	$0.\bar{3}$	$0.\bar{6}$	0.25	0.2	0.75	0.4	$0.1\bar{6}$	$0.\overline{142857}$

The complete list is countable since there is a one-to-one correspondence between the natural numbers and the rationals. It is easy to create a new countable list of these unique rationals by simply interchanging numbers in the list. But is there another way to generate a new countable list of all rationals in (0, 1] using List A?

Yes. As we show, we can generate another countable list of all the rational numbers in (0, 1] by removing any fixed number of digits from the mantissa of every rational number of List A.

This can be established by a proof by contradiction. List B can be generated by removing a positive integer i number of digits from the mantissa of every rational number in List A. Suppose, for the sake of later contradiction, that List A has some rational number r_1 which is not in List B. This rational number can be expressed in decimal form as $r_1 = 0.d_1d_2d_3\ldots$, where each d_k is a digit from 0 to 9 for positive integer k. Note that List A does not include 0, so not all of the digits of the mantissa are 0. List A includes every rational in (0, 1], so it also includes the rational number

$$r_2 = 0.000\ldots d_1d_2d_3d_4$$

which has 0 repeated for i digits after the decimal point, followed by the mantissa from r_1. Since r_2 is in List A, then when the first i digits of its mantissa are removed (all 0s), then it would become r_1. This is a contradiction showing that r_1 must also be in List B. Accordingly, List B has all the rational numbers that List A has.

As an example, we take the following simple case where 1 digit is removed from the mantissa of List A to generate List B. The following rational is in List A:

$$\tfrac{1}{7} = 0.\overline{142857}.$$

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Suppose, for the sake of later contradiction, that this rational was not in List B. The following rational is also in List A:

$$\tfrac{1}{70} = 0.0\overline{142857}.$$

Eliminating the tenths place of the latter number shows that List B must contain the former number, which contradicts the assumption, and shows that the rational number $\frac{1}{7}$ must be in List B. This example extends to any rational number in List A, showing that List B contains every rational number that List A has for the interval (0, 1].

Although List B has all the rationals from List A, it also contains an infinite number of duplicate numbers. For example, List A must have the following rational numbers $0.d_1 20813$, where d_1 can be any digit from 0 to 9. List B, which has digit d_1 removed, has rational number 0.20813 listed 10 times. In fact, every rational number in List A is listed 10 times in List B. The infinite number of duplicates in List B can be removed by a process of elimination starting with the first number in the list (as was done in [3]).

Of course, the process that was used to remove digits from List A can also be used in List B by removing any part of its mantissa to create yet another countable list of all rationals for (0, 1]. However, removing decimal places from the mantissa and adding decimal places to the mantissa have very different results. Notice that it is *not* possible to create a countable list of rationals by inserting a decimal place in the mantissa of the unique rationals of List A. In the following example, part of List A is on the top and part of List C, with a new tenths place inserted, is on the bottom:

List A	0.5	0.333333…	0.666666…	0.25	0.2
List C	$0.a5$	$0.b333333…$	$0.c666666…$	$0.d25$	$0.e2$

In List C, any one digit can be chosen for a, b, c, d and e. Say that you pick $b = 1$ for 0.333333 (i.e. 0.1333333). List C cannot possibly have the number 0.2333333 because the list cannot possibly have another number with all 3's starting in the second decimal place. In fact, for each number in List C, there are 9 rationals not in the list. So, all rationals in the interval (0,1] cannot be obtained by adding digits to the mantissa. This shows that removing decimal places from the mantissa and adding decimal places produces very different results.

To summarise: Starting with the countable list of all unique rationals in (0, 1], another countable infinite list of all the rationals can be achieved by removing the same number of decimal places from the mantissa of each original number. However, adding digits to the mantissa produces a very different result; the resulting list no longer has all rationals from the original interval.

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108.20 Euler's totient theorem and Fermat's little theorem are generalisations of one another!

Let us consider a non-familiar converse for the obvious fact that if $a \equiv 1 \pmod{n}$, then n divides $a^n - 1$, which is also related to Fermat's little theorem (briefly, Fermat's theorem). For example if $n = p$ is prime, then by Fermat's theorem, p divides $a^p - 1$ if, and only if, $a \equiv 1 \pmod{p}$. In fact, $a \equiv 1 \pmod{p}$ if, and only if, $a^p \equiv 1 \pmod{p^2}$. Indeed, for any natural numbers a, n, if $a \equiv 1 \pmod{n}$ then $a^n \equiv 1 \pmod{n^2}$ and by applying an induction on a natural number m we have $a^{n^m} \equiv 1 \left(\bmod n^{m+1}\right)$. In the last step of this induction, one may write

$$a^{n^{m+1}} - 1 = \left(a^{n^m} - 1\right)\left(a^{n^m(n-1)} + a^{n^m(n-2)} + \dots + 1\right),$$

assuming $a^{n^m} \equiv 1 \left(\bmod n^{m+1}\right)$, by the induction hypothesis, and noticing that the sum in the previous parenthesis is divisible by n [note, still $a \equiv 1 \left(\bmod n\right)$], we then immediately infer that $a^{n^{m+1}} \equiv 1 \left(\bmod n^{m+2}\right)$. In this Note we like to formulate a few results related to the above non-familiar converse and obtain some useful consequences including the unusual fact in the title. Indeed, this fact is a rare occurrence between any two theorems in mathematics, even between the equivalent ones (see my concluding comments, briefly). Using the above simple facts, and invoking Fermat’s theorem, one may observe that if we replace n by a prime number p in the above congruences, then $a^{p^m} \equiv 1 \left(\bmod p^{m+1}\right)$ if, and only if, $a \equiv 1 \left(\bmod p\right)$. In particular, if $p = 2$, then $a^{2^m} \equiv 1 \left(\bmod 2^{m+2}\right)$ if, and only if, a is odd, where $m \geqslant 1$. We show that the latter two cases can be unified and obtained as consequences of either Corollary 1 or Corollary 2, below. Before presenting the results, let us recall that if p is the least prime divisor of a natural number n, then $\left(n, p - 1\right) = 1$. Motivated by this we define a *quasi-prime* number to be a natural number n such that $\left(n, p - 1\right) = 1$, where p is any prime divisor of n. It is evident that n is *quasi-prime* if, and