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DERIVATIONS AND AUTOMORPHISMS OF EXTERIOR ALGEBRAS

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Dedicated to D. S. Mitrinović

0. Introduction. In this paper we study the Lie algebra $\mathscr{D}$ of derivations of the exterior algebra $\mathscr{E}$ of a vector space V over a field K of characteristic $\neq 2$, and the group A of automorphisms of $\mathscr{E}$.

Both $\mathscr{E}$ and $\mathscr{D}$ have natural $\mathbf{Z}_2$-gradings $\mathscr{D} = \mathscr{D}_0 \oplus \mathscr{D}_1$ and $\mathscr{E} = \mathscr{E}_0 \oplus \mathscr{E}_1$. Let A_0 be the subgroup of A which preserves this grading of $\mathscr{E}$. We show that $\mathscr{D}_1$ is the ideal of inner derivations of $\mathscr{E}$ except in the case when $\dim V = \aleph_0$.

For A we assume that $\dim V = n$ is finite. In the case when K is the complex field, A_0 has been determined by F. A. Berezin [**1**]. He claimed there that $A = A_0$, which is erroneous. In fact A is a semidirect product $A = N_1 \rtimes A_0$ where N_1 is the group of inner automorphisms of $\mathscr{E}$ and N_1 is abelian. All our results are established for arbitrary K of characteristic $\neq 2$.

It is important to note that $\mathscr{D}$ is the Lie algebra of *ordinary derivations* of $\mathscr{E}$. The case of graded derivations (also called *antiderivations*) is much easier and well-known. See for instance [**4**] or [**3**, p. 111–114].

1. Preliminaries. Let A be an associative algebra over a field K. With respect to the bracket operation, $[a, b] = ab - ba$, A becomes a Lie algebra over K which we will denote by A_L.

Each $a \in A$ determines a derivation D_a of A defined by $D_a(x) = [a, x] = ax - xa$ $(x \in A)$. The map $A_L \rightarrow \operatorname{Der} A$ which sends a to D_a is a homomorphism of Lie algebras. The image of this homomorphism is Inder A, the Lie algebra of inner derivations of A.

If $a \in A$ and $D \in \operatorname{Der} A$ then we have $[D, D_a] = DD_a - D_aD = D_b$ where $b = D(a)$. This shows that Inder A is an ideal of Der A.

Now let us assume that A is $\mathbf{Z}_2$-graded, i.e., $A = A_0 \oplus A_1$ is a fixed direct decomposition such that $A_iA_j \subset A_{i+j}$ $(i, j = 0, 1$; indices are added modulo $2)$. Let $\operatorname{Der}_i A$ $(i = 0, 1)$ be the subspace of Der A consisting of all derivations D such that $D(A_j) \subset A_{i+j}$ $(j = 0, 1)$.

LEMMA 1. *With the above hypotheses we have* $\operatorname{Der} A = \operatorname{Der}_0 A \oplus \operatorname{Der}_1 A$, $\operatorname{Der}_0 A$ *is a subalgebra of* Der A, *and* $[\operatorname{Der}_0 A, \operatorname{Der}_1 A] \subset \operatorname{Der}_1 A$.

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Proof. The last two assertions are obvious. It is also clear that $\mathrm{Der}_0 A \cap \mathrm{Der}_1 A = 0$ and so it remains to prove that $\mathrm{Der}\, A = \mathrm{Der}_0 A + \mathrm{Der}_1 A$.

Let $p_i: A \to A_i$ $(i = 0, 1)$ be the canonical projections. For $D \in \mathrm{Der}\, A$ we define D_i $(i = 0, 1)$ by

$$D_i|_{A_j} = p_{i+j} \circ D|_{A_j} \quad (j = 0, 1).$$

Clearly the linear transformations D_i satisfy $D_i(A_j) \subset A_{i+j}$ $(j = 0, 1)$. We claim that they are derivations of A, which will complete the proof. Thus we have to show that $D_i(xy) = (D_i x)y + x(D_i y)$ holds for $x, y \in A$. Clearly, it suffices to prove this when x and y are homogeneous, say $x = x_j \in A_j$ and $y = y_k \in A_k$ $(i, j, k = 0, 1)$. Then

$$\begin{aligned} D_i(x_j y_k) &= p_{i+j+k}(D(x_j y_k)) \\ &= p_{i+j+k}((Dx_j)y_k) + p_{i+j+k}(x_j(Dy_k)) \\ &= (p_{i+j}(Dx_j))y_k + x_j(p_{i+k}(Dy_k)) \\ &= D_i(x_j)y_k + x_j D_i(y_k). \end{aligned}$$

LEMMA 2. *Assume moreover that A is anticommutative, i.e.,*

$$yx = (-1)^{ij}xy \text{ for } x \in A_i,\ y \in A_j\ (i, j = 0, 1).$$

Then Inder A *is an abelian ideal of* Der A *and* Inder $A \subset \mathrm{Der}_1 A$.

Proof. We have mentioned before that Inder A is an ideal of Der A. Recall that we have a surjective Lie algebra homomorphism $A_L \to$ Inder A whose kernel is clearly the center Z_A of A. Thus Inder $A \cong A_L/Z_A$ as Lie algebras. Since $A_0 \subset Z_A$ (by anticommutativity) and $[A, A] \subset A_0$ we see that A_L/A_0 is abelian and also A_L/Z_A is abelian. Thus Inder A is an abelian ideal of Der A.

If $a \in A_0$ then $D_a = 0$ and if $a \in A_1$ then $D_a \in \mathrm{Der}_1$A. This proves that Inder $A \subset \mathrm{Der}_1 A$.

2. Derivations of exterior algebras. Let V be a vector space over a field K and let $\mathscr{E}$ be the exterior algebra of V. $\mathscr{E}$ has a **Z**-grading

$$\mathscr{E} = \oplus_{i \geqq 0} \mathscr{E}^i$$

where $\mathscr{E}^i$ is the *i-th* exterior power of V. In particular, $\mathscr{E}^0 = K$ and $\mathscr{E}^1 = V$.

We shall be more interested in the induced $\mathbf{Z}_2$-grading

$$\mathscr{E} = \mathscr{E}_0 \oplus \mathscr{E}_1$$

where

$$\mathscr{E}_0 = \sum_{i \geqq 0} \mathscr{E}^{2i}, \quad \mathscr{E}_1 = \sum_{i \geqq 0} \mathscr{E}^{2i+1}.$$

If char $K = 2$ then $\mathscr{E}$ is commutative and it follows from [**2**, Chapter III, §10, Prop. 14] that every linear map $V \to \mathscr{E}$ extends uniquely to a derivation of $\mathscr{E}$.

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Therefore *we shall assume from now on that* char $K \neq 2$.

Since $\mathscr{E}$ is anticommutative, we have $\mathscr{E}_0 \subset \mathscr{Z}$ where $\mathscr{Z}$ is the center of $\mathscr{E}$. In fact it is known that we have equality $\mathscr{E}_0 = \mathscr{Z}$ except in the case when $\dim V = n$ is finite and odd. In the exceptional case we have $\mathscr{Z} = \mathscr{E}_0 \oplus \mathscr{E}^n$.

We shall write $\mathscr{D} = \operatorname{Der} \mathscr{E}$, $\mathscr{I} = \operatorname{Inder} \mathscr{E}$, and $\mathscr{D}_i = \operatorname{Der}_i \mathscr{E}$ $(i = 0, 1)$. We know from Lemmas 1 and 2 that $\mathscr{D} = \mathscr{D}_0 \oplus \mathscr{D}_1$, $\mathscr{I} \subset \mathscr{D}_1$ and that $\mathscr{I}$ is an abelian ideal of $\mathscr{D}$.

Let $\mathscr{M}$ be the maximal ideal of $\mathscr{E}$, i.e.,

$$\mathscr{M} = \sum_{i \geqq 1} \mathscr{E}^i.$$

We shall denote by $\hat{\mathscr{E}}$ the $\mathscr{M}$-completion of $\mathscr{E}$. Clearly, $\hat{\mathscr{E}}$ inherits a $\mathbf{Z}_2$-grading from $\mathscr{E}$; $\hat{\mathscr{E}} = \hat{\mathscr{E}}_0 \oplus \hat{\mathscr{E}}_1$. In fact $\mathscr{E}$ is a subalgebra of $\hat{\mathscr{E}}$ and is dense in $\hat{\mathscr{E}}$ for the $\mathscr{M}$-topology. $\hat{\mathscr{E}}_i$ is the closure of $\mathscr{E}_i$ $(i = 0, 1)$ in $\hat{\mathscr{E}}$ for the same topology. The elements of $\hat{\mathscr{E}}$ can be identified with the formal infinite series

$$(1) \quad x = \sum_{i \geqq 0} x_i, \quad x_i \in \mathscr{E}^i.$$

If $x_i = 0$ for all i except finitely many of them, then $x \in \mathscr{E}$.

Let $\tilde{\mathscr{E}}$ be the *idealizer* of $\mathscr{M}$ in $\hat{\mathscr{E}}$, i.e., $\tilde{\mathscr{E}}$ consists of all $x \in \hat{\mathscr{E}}$ such that $x\mathscr{M} \subset \mathscr{M}$ and $\mathscr{M}x \subset \mathscr{M}$. Clearly $\tilde{\mathscr{E}}$ is a subalgebra of $\hat{\mathscr{E}}$ containing $\mathscr{E}$. It is easy to see that if $x \in \hat{\mathscr{E}}$ satisfies $xV \subset \mathscr{M}$ and $Vx \subset \mathscr{M}$ then in fact $x \in \tilde{\mathscr{E}}$.

Thus if x is given by (1) then $x \in \tilde{\mathscr{E}}$ if and only if for every $y \in V$ $x_i y = 0$ for all but finitely many $i \geqq 0$.

For $a \in \tilde{\mathscr{E}}$ let D_a be the corresponding inner derivation of $\tilde{\mathscr{E}}$. From the definition of $\tilde{\mathscr{E}}$ it follows that $D_a(\mathscr{E}) \subset \mathscr{E}$. Hence we have a restriction homomorphism $\operatorname{Inder} \tilde{\mathscr{E}} \to \mathscr{D}$. This is clearly injective and we denote by $\tilde{\mathscr{I}}$ the image in $\mathscr{D}$ of this homomorphism. It is clear that $\tilde{\mathscr{I}} \subset \mathscr{D}_1$.

THEOREM 3. *We have* $\tilde{\mathscr{E}} = \mathscr{E}$ *except when a basis of* V *has cardinality* $\aleph_0$.

THEOREM 4. W*e have* $\tilde{\mathscr{I}} = \mathscr{D}_1$. *Moreover, if the cardinality of a basis of* V *is not* $\aleph_0$ *then* $\mathscr{I} = \mathscr{D}_1$.

Let us first introduce some notation. Let a_i $(i \in I)$ be a basis of V and assume that the index set I is totally ordered. By $\mathscr{F}$ we shall denote the set of all finite subsets of I. We have a partition of $\mathscr{F}$ into $\mathscr{F}_0$ and $\mathscr{F}_1$ where $\mathscr{F}_0$ (resp. $\mathscr{F}_1$) consists of those $S \in \mathscr{F}$ whose cardinality is an even (resp. odd) integer.

The algebra $\mathscr{E}$ has a basis $\{a_S \mid S \in \mathscr{F}\}$ where if $S = \{i_1, i_2, \ldots, i_k\}$ with $i_1 < i_2 \ldots < i_k$ then

$$a_S = a_{i_1} a_{i_2} \ldots a_{i_k}.$$

In particular $a_\phi = 1$ is the identity element of $\mathscr{E}$ and $a_{\{i\}} = a_i$ for $i \in I$.

Proof of Theorem 3. If I is a finite set then $\hat{\mathscr{E}} = \mathscr{E}$ and consequently $\tilde{\mathscr{E}} = \mathscr{E}$.

Now let us assume that I is not countable. Let $x \in \tilde{\mathscr{E}}$ be arbitrary and write $x = x_0 + x_1 + x_2 + \ldots$ with $x_r \in \mathscr{E}^r$. Let $P = \{r \mid x_r \neq 0\}$. For $r \in P$ let $I_r = \{i \in I \mid a_i x_r = 0\}$. Then each I_r is a finite set and consequently their

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union is countable. Therefore there exists an $i \in I$ such that $i \notin I_r$ for all $r \in P$. Then $a_i x_r \neq 0$ for all $r \in P$. Since

$$a_i x = a_i x_0 + a_i x_1 + a_i x_2 + \ldots \in \mathscr{E}$$

and $a_i x_r \in \tilde{\mathscr{E}}^{r+i}$ $(r \geqq 0)$, this implies that P is a finite set. Consequently $x \in \mathscr{E}$ and so $\tilde{\mathscr{E}} = \mathscr{E}$.

Finally, let us assume that I has cardinality $\aleph_0$. Then we may assume that I is the set of positive integers. It is easy to see that the element

$$x = \sum_{i \geqq 0} a_1 a_2 \ldots a_i$$

is in $\tilde{\mathscr{E}}$ but is not in $\mathscr{E}$.

Proof of Theorem 4. Let $D \in \mathscr{D}_1$ and write $Da_i = b_i$. Since $a_i \in V = \mathscr{E}^1 \subset \mathscr{E}_1$ we have $b_i \in \mathscr{E}_0$. From $a_i^2 = 0$ we obtain $(Da_i)a_i + a_i(Da_i) = 0$, i.e., $b_1 a_1 + a_1 b_1 = 0$. Since $b_i \in \mathscr{E}_0 \subset \mathscr{Z}$ this gives $2a_i b_i = 0$ and since char $K \neq 2$ we have $a_i b_i = 0$.

For $i, j \in I$ let $I^{(i)} = I \backslash \{i\}$ and $I^{(i,j)} = I \backslash \{i, j\}$. We denote $V^{(i)}$ (resp. $V^{(i,j)}$) the subspace of V spanned by a_k for $k \in I^{(i)}$ (resp. $k \in I^{(i,j)}$). Further, $\mathscr{E}^{(i)}$ (resp. $\mathscr{E}^{(i,j)}$) will be the exterior algebra of $V^{(i)}$ (resp. $V^{(i,j)}$). We also put $\mathscr{F}^{(i)} = \{S \in \mathscr{F} \mid i \notin S\}$, $\mathscr{F}^{(i,j)} = \mathscr{F}^{(i)} \cap \mathscr{F}^{(j)}$. Finally, we define $\mathscr{F}_0^{(i)} = \mathscr{F}^{(i)} \cap \mathscr{F}_0$, $\mathscr{F}_1^{(i)} = \mathscr{F}^{(i)} \cap \mathscr{F}_1$ and similarly $\mathscr{F}_0^{(i,j)}$ and $\mathscr{F}_1^{(i,j)}$.

It follows from $a_i b_i = 0$ and $b_i \in \mathscr{E}_0$ that $b_i = a_i c_i$ where $c_i \in \mathscr{E}_1^{(i)}$.

From $a_i a_j + a_j a_i = 0$ we obtain

$$(Da_i)a_j + a_i(Da_j) + (Da_j)a_i + a_j(Da_i) = 0, \quad \text{or}$$

$$b_i a_j + a_i b_j + b_j a_i + a_j b_i = 0.$$

Since $b_i \in \mathscr{E}_0 \subset \mathscr{Z}$ and char $K \neq 2$ this gives $a_i b_j + a_j b_i = 0$. Using $b_i = a_i c_i$ and $b_j = a_j c_j$ we obtain

(2) $\quad a_i a_j (c_j - c_i) = 0.$

Using the basis $\{a_S \mid S \in \mathscr{F}_1^{(i)}\}$ of $\mathscr{E}_1^{(i)}$, we can write

(3) $\quad c_i = \sum \alpha_S{}^i a_S, \ (S \in \mathscr{F}_1^{(i)}).$

The coefficients $\alpha_S{}^i \in K$ are defined for $S \in \mathscr{F}_1$ and $i \in I \backslash S$. It follows from (2) and (3) that $\alpha_S{}^i = \alpha_S{}^j$ whenever $S \in \mathscr{F}_1$ and $i, j \in I \backslash S$. Therefore for each $S \in \mathscr{F}_1$ there is a scalar $\alpha_S \in K$ such that $\alpha_S{}^i = \alpha_S$ for all $i \in I \backslash S$.

Let m be an odd positive integer and let $\mathscr{F}^m$ be the set of all $S \in \mathscr{F}$ of cardinality m. We claim that $\alpha_S \neq 0$ for only finitely many $S \in \mathscr{F}^m$. Indeed, let $i_1, i_2, \ldots, i_m$ be distinct elements of I. Since $c_{i_1} \in \mathscr{E}$ there are only finitely many $S \in \mathscr{F}^m$ such that $\alpha_S \neq 0$ and $i_1 \notin S$. Similar statements are valid for indices $i_2, \ldots, i_m$. Hence there are only finitely many $S \in \mathscr{F}^m$ such that $\alpha_S \neq 0$ and $\{i_1, i_2, \ldots, i_m\} \not\subset S$. This proves our claim. Thus each sum

$$\sum \alpha_S a_S (S \in \mathscr{F}^m, m \text{ odd})$$

is in fact finite and so

$$c = \sum \alpha_S a_S (S \in \mathscr{F}_1)$$

is an element of $\hat{\mathscr{E}}$.

We have

$$-ca_i = a_i c = \sum_{S \in \mathscr{F}_1} \alpha_S a_i a_S = \sum_{S \in \mathscr{F}_1(i)} \alpha_S{}^i a_i a_S = a_i c_i = b_i \in \mathscr{E} \quad (i \in I),$$

which proves that $c \in \tilde{\mathscr{E}}$. The same computation gives $D_c(a_i) = -2b_i = -2D(a_i)$, and so $D \in \tilde{\mathscr{I}}$.

We have proved that $\mathscr{D}_1 \subset \tilde{\mathscr{I}}$ and since we remarked before that $\tilde{\mathscr{I}} \subset \mathscr{D}_1$, we have $\tilde{\mathscr{I}} = \mathscr{D}_1$. The second assertion now follows from Theorem 3.

3. Automorphisms of exterior algebras. In this section we assume that $\dim V = n$ is finite and char $K \neq 2$. As before, $\mathscr{E}$ is the exterior algebra of V.

Let A be the group of automorphisms of $\mathscr{E}$ (considered just as a K-algebra) and let A_0 be the subgroup of A consisting of those automorphisms σ which preserve the $\mathbf{Z}_2$-grading of $\mathscr{E}$, i.e., such that $\sigma(\mathscr{E}_i) = \mathscr{E}_i$ $(i = 0, 1)$.

Recall that $\mathscr{E}$ is a local algebra with the maximal ideal $\mathscr{M} = \sum_{i \geqq 1} \mathscr{E}_i$ and that

$$\mathscr{M}^k = \sum_{i \geqq k} \mathscr{E}^i (k \geqq 0).$$

Therefore, every $\sigma \in A$ stabilizes the chain

$$\mathscr{E} = \mathscr{M}^0 \supset \mathscr{M}^1 \supset \mathscr{M}^2 \supset \ldots \supset \mathscr{M}^n \supset 0.$$

Hence every $\sigma \in A$ induces an automorphism σ_i of the vector space $\mathscr{M}^i/\mathscr{M}^{i+1}$. Since the canonical map $\mathscr{E}^i \rightarrow \mathscr{M}^i/\mathscr{M}^{i+1}$ is an isomorphism we can consider σ_i as operating in $\mathscr{E}^i$.

The map f_i: $A \rightarrow \mathrm{GL}(\mathscr{E}^i)$ defined by $f_i(\sigma) = \sigma_i$ is clearly a homomorphism. In particular, σ_0 is the identity for every $\sigma \in A$; i.e., f_0 is the trivial homomorphism and it is well-known that f_1 is surjective. In fact every automorphism τ of V extends uniquely to an automorphism $g(\tau)$ of $\mathscr{E}$. Thus if $N = \ker(f_1)$ than we have a short exact sequence

$$1 \rightarrow N \rightarrow A \underset{g}{\overset{f_1}{\rightleftarrows}} GL(V) \rightarrow 1$$

with g a section, i.e., $f_1 \circ g$ = identity.

Let G be the image of g in A. Then A is a semidirect product $A = N \rtimes G$.

LEMMA 5. *If $\sigma \in N$ then σ_i is the identity for all i.*

Proof. We always have σ_0 = identity and by hypothesis we also have σ_1 = identity. Now let $2 \leqq k \leqq n$ and let $x_1, \ldots, x_k \in V$. Then $\sigma(x_i) =$

$x_i + y_i$ where $y_i \in \mathscr{M}^2$ and consequently

$$\sigma(x_1x_2 \dots x_k) - x_1x_2 \dots x_k = (x_1 + y_1)(x_2 + y_2) \dots (x_k + y_k) - x_1x_2 \dots x_k \in \mathscr{M}^{k+1}.$$

This proves that σ_k = identity.

LEMMA 6. *N is a unipotent group.*

Proof. Let 1 be the identity map of $\mathscr{E}$. It follows from Lemma 5 that for $\sigma \in N$ we have $(\sigma - 1)(\mathscr{M}^i) \subset \mathscr{M}^{i+1}$ $(i \geqq 0)$ and so $\sigma - 1$ is nilpotent, i.e., σ is unipotent.

Let $N_0 = N \cap A_0$. Since $A = N \rtimes G$ and $G \subset A_0$ it follows that $A_0 = N_0 \times G$.

Recall that every inner derivation of $\mathscr{E}$ is of the form D_a $(a \in \mathscr{E}_1)$.

LEMMA 7. *If $a, b \in \mathscr{E}_1$ then $D_aD_b = 0$.*

Proof. It suffices to check that $D_aD_b(x) = 0$ for $x \in \mathscr{E}_i$ $(i = 0, 1)$. Indeed

$$\begin{aligned} D_aD_b(x) &= a(bx - xb) - (bx - xb)a \\ &= (abx + xba) - (axb + bxa). \end{aligned}$$

This is zero because $xba = bax = -abx$ and $bxa = (-1)^{i+1}xab = -axb$.

In particular, it follows from this lemma that $D_a^2 = 0$ for $a \in \mathscr{E}_1$ and so $\exp(D_a) = 1 + D_a \in A$. Since for $a \in \mathscr{E}_1$

$$(1 + D_a)(x) = x + ax - xa \quad (x \in \mathscr{E})$$

it is clear that $1 + D_a \in N$.

If $a, b \in \mathscr{E}_1$ then by Lemma 7

$$(1 + D_a)(1 + D_b) = 1 + D_{a+b} = (1 + D_b)(1 + D_a).$$

Hence, the automorphisms $1 + D_a$ $(a \in \mathscr{E}_1)$ form an abelian subgroup of N which we will denote by N_1. The map $\mathscr{E}_1 \to N_1$ sending a to $1 + D_a$ is a homomorphism of the additive group of $\mathscr{E}_1$ onto N_1 with kernel $\mathscr{E}_1 \cap \mathscr{Z}$.

For $k \geqq 1$ let M_k be the subgroup of N_1 consisting of all automorphisms $1 + D_a$ with $a \in \mathscr{E}^{2k-1}$.

THEOREM 8. *For each $k \geqq 1$ the product $M^{(k)} = M_k M_{k+1} \dots$ is a normal subgroup of A. In particular, $N_1 \lhd A$.*

Proof. For $a \in \mathscr{E}$ and $\sigma \in A$ we have $\sigma D_a \sigma^{-1} = D_{\sigma(a)}$. It remains to notice that $M^{(k)}$ consists of all $1 + D_a$ with $a \in \mathscr{M}^{2k-1}$, and that $\mathscr{M}^{2k-1}$ is σ-stable.

Now let us define for $k \geqq 1$ the subgroup $N^{(k)}$ of N. It consists of all $\sigma \in N$ such that

$$\sigma(x) \in \mathscr{P}_k = \mathscr{E}_1 + \mathscr{M}^{2k} \quad \text{for } x \in V.$$

It is clear that

$$N = N^{(1)} \supset N^{(2)} \supset \dots \supset N^{(m)} \supset N^{(m+1)} = N_0$$

where $m = \left[\frac{n}{2}\right]$, and that

$$N^{(k)} \cap N_1 = M^{(k)} \quad (k \geqq 1).$$

THEOREM 9. *We have*
(i) $N^{(k)} = N^{(k+1)} \times M_k \quad (k \geqq 1)$,
(ii) $N = N_1 \rtimes N_0$,
(iii) $A = N_1 \rtimes A_0$.

Proof. By Theorem 8, $N_1 \lhd A$. If $a \in \mathscr{E}_1$ and $1 + D_a \in A_0$ then for $x \in V$ we must have $D_a(x) = 0$. Thus $D_a = 0$ and so $N_1 \cap A_0 = 1$. Hence in order to prove (ii) and (iii) it suffices to show that $N = N_1N_0$ and $A = N_1A_0$. Since $A = NG$ and $G \subset A_0$ it suffices to prove only that $N = N_1N_0$. This last equality clearly follows from (i), which we now proceed to prove.

If $a \in \mathscr{E}^{2k-1}$ then for $x \in V$ we have $(1 + D_a)x - x = D_ax \in \mathscr{E}^{2k}$. Thus if $1 + D_a \in N^{(k+1)}$ then $D_ax = 0$ for all $x \in V$, i.e., $D_a = 0$. Therefore $N^{(k+1)} \cap M_k = 1$.

We claim that M_k normalizes $N^{(k+1)}$. For this purpose let $\sigma \in N^{(k+1)}$, $a \in \mathscr{E}^{2k-1}$, $x \in V$. Then we have to show that

$$(1 - D_a)\sigma(1 + D_a)x \in \mathscr{P}_{k+1}.$$

We have

$$(1 - D_a)\sigma(1 + D_a)x = x + (\sigma x - x) - D_a\sigma D_ax + (\sigma D_a - D_a\sigma)x.$$

Since $x \in \mathscr{P}_{k+1}$, $\sigma x - x \in \mathscr{P}_{k+1}$ and $D_a\sigma D_ax \in \mathscr{M}^{2k+1} \subset \mathscr{P}_{k+1}$ we need only show that $(\sigma D_a - D_a\sigma)x \in \mathscr{P}_{k+1}$. This is so because $D_a(\sigma x - x) \in D_a(\mathscr{M}^2) \subset \mathscr{M}^{2k+1} \subset \mathscr{P}_{k+1}$ and $\sigma D_ax - D_ax \in \mathscr{P}_{k+1}$. This last relation holds because $D_ax \in \mathscr{M}^{2k}$ and $\sigma_{2k} =$ identity.

It remains to show that $N^{(k)} = M_kN^{(k+1)}$. Let $\sigma \in N^{(k)}$. For $x \in V$ we can write uniquely

$$\sigma(x) = x + \tau(x) + z$$

where $\tau(x) \in \mathscr{E}^{2k}$ and $z \in \mathscr{M}^2 \cap \mathscr{P}_{k+1}$.

Since $x^2 = 0$ we have

$$0 = (\sigma x)^2 = (x + \tau(x) + z)^2 = 2x\tau(x) + u$$

where $u \in \mathscr{M}^{2k+2}$. Thus $x\tau(x) = 0$ for all $x \in V$. By [**2**, Chapter III, §10, Prop. 14] τ extends to a unique derivation D of $\mathscr{E}$. Clearly $D \in D_1$ and since $\mathscr{D}_1 = \mathscr{I}$ by Theorem 4, there exists an $a \in \mathscr{E}_1$ such that $D_a = D$. Since $\tau(x) = Dx = D_ax = ax - xa \in \mathscr{E}^{2k}$ for all $x \in V$, we may assume that $a \in \mathscr{E}^{2k-1}$.

We finish the proof by showing that $(1 - D_a)\sigma \in N^{(k+1)}$. This is equivalent to

$$(1 - D_a)\sigma x \in \mathscr{P}_{k+1} \quad \text{for } x \in V.$$

We have

$$\begin{aligned}(1 - D_a)\sigma x &= x + \tau(x) + z - D_a(x) - D_a\tau(x) - D_a z\\ &= x + z - D_a z\end{aligned}$$

because $D_a(x) = \tau(x)$ and $D_a\tau(x) = D_a{}^2(x) = 0$ by Lemma 7. Since $x \in \mathscr{E}_1$, $z \in \mathscr{P}_{k+1}$ and $D_a z \in D_a(\mathscr{M}^2) \subset \mathscr{M}^{2k+1} \subset \mathscr{P}_{k+1}$, the proof is complete.

4. Inner automorphisms of $\mathscr{E}$. Our hypotheses about K, V, $\mathscr{E}$ will be the same as in the preceding section.

Since $\mathscr{E}$ is a local algebra, an element $x \in \mathscr{E}$ is invertible if and only if $x \notin \mathscr{M}$. We shall denote by U the group of units of $\mathscr{E}$, i.e., $U = \mathscr{E}\backslash\mathscr{M}$. Clearly $U_0 = U \cap \mathscr{E}_0$ is a subgroup of U. We put

$$U_1 = U \cap (1 + \mathscr{E}_1) = 1 + \mathscr{E}_1.$$

Of course, U_1 is *not a subgroup* (in general) but we have

$$aU_1a^{-1} = U_1 \quad \text{for } a \in U_0.$$

The center Z of U is contained in the center $\mathscr{Z}$ of $\mathscr{E}$ and so we have

$$Z = U \cap \mathscr{Z}.$$

Since $\mathscr{E}_0 \subset \mathscr{Z}$ we have $U_0 \subset Z$. In fact $U_0 = Z$ except when $\dim V = n$ is odd. In the exceptional case we have

$$Z = U_0 \cdot (1 + \mathscr{E}^n).$$

THEOREM 10. *U_1 is a system of coset representatives of U_0 in U.*

Proof. Let $x, y \in \mathscr{E}_1$. Then $1 + x$, $1 + y$ are in U_1 and

$$(1 + x)^{-1}(1 + y) = (1 - x)(1 + y) = 1 - x + y - xy.$$

If this product belongs to U_0 then since $1 - xy \in \mathscr{E}_0$ and $y - x \in \mathscr{E}_1$ we must have $y - x = 0$, i.e., $y = x$. This shows that if $x \neq y$ then $(1 + x)U_0 \neq (1 + y)U_0$.

It remains to show that $U = U_0U_1$. Letting $a \in U$ we have to show that $a \in U_0U_1$. Clearly we may assume that $a = 1 + b$ with $b \in \mathscr{M}$. If $b \in \mathscr{E}_1$ then $a \in U_1$ and there is nothing to prove. So let $b = b_0 + b_1$ with $b_i \in \mathscr{E}_i \cap \mathscr{M}$, and $b_0 \neq 0$. We can write

$$b_0 = c_{2k} + c_{2k+2} + \ldots$$

where $c_{2i} \in \mathscr{E}^{2i}$ and $c_{2k} \neq 0$ $(k \geqq 1)$. We shall say that $2k$ is the *order* of the element a. Now it is clear that $(1 - c_{2k})a$ has order $> 2k$ and our claim follows by induction.

The automorphisms of $\mathscr{E}$ of the form $x \to axa^{-1}$ $(a \in U, x \in \mathscr{E})$ are called *inner*. The inner automorphisms of $\mathscr{E}$ form a group Inaut $\mathscr{E}$ and we have a short exact sequence

$$1 \to Z \to U \to \text{Inaut } \mathscr{E} \to 1.$$

THEOREM 11. *Let N_1 be the group defined in the previous section. We have*

$$N_1 = \text{Inaut } \mathscr{E}.$$

Proof. Let $a \in U$. By Theorem 10 we can write $a = (1 + b)c$ with $b \in \mathscr{E}_1$ and $c \in U_0$. Since $U_0 \subset Z \subset \mathscr{Z}$ we have, for $x \in \mathscr{E}$,

$$\begin{aligned} axa^{-1} &= (1 + b)cxc^{-1}(1 - b) = (1 + b)x(1 - b) \\ &= x + bx - xb = (1 + D_b)(x). \end{aligned}$$

Note that $bxb = 0$ for all $x \in \mathscr{E}$ because $b \in \mathscr{E}_1$.

This proves that Inaut $\mathscr{E} \subset N_1$.

Conversely, if $a \in \mathscr{E}_1$ then $1 + D_a$ is simply conjugation by $1 + a \in U$.

This Theorem gives an alternative proof of the assertion $N_1 \triangleleft A$.

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