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PRESENTATION FOR RENNER MONOIDS

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Abstract

We extend the result obtained in E. Godelle ['The braid rook monoid', *Internat. J. Algebra Comput.* **18** (2008), 779–802] to every Renner monoid: we provide a monoid presentation for Renner monoids, and we introduce a length function which extends the Coxeter length function and which behaves nicely.

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1. Introduction

The notion of a *Weyl group* is crucial in linear algebraic group theory [4]. The seminal example occurs when one considers the *algebraic group* $\mathrm{GL}_n(\mathbb{K})$. In that case, the associated Weyl group is isomorphic to the group of monomial matrices, that is, to the permutation group S_n. Weyl groups are special examples of *finite Coxeter groups*. Hence, they possess a group presentation of a particular type, and an associated length function. It turns out that this presentation and this length function are deeply related to the geometry of the associated algebraic group. Linear algebraic monoid theory, mainly developed by Putcha, Renner and Solomon, has deep connections with algebraic group theory. In particular, the *Renner monoid* [10] plays the role that the Weyl group does in linear algebraic group theory. As far as I know, in the case of Renner monoids, there is no known theory that plays the role of Coxeter group theory. Therefore it is natural to look for such a theory, and therefore to address the question of monoid presentations for Renner monoids. In [2], we considered the particular case of the *rook monoid* defined by Solomon [15]. We obtained a presentation of this monoid and introduced a length function that is nicely related to the Hecke algebra of the rook monoid. Our objective here is to consider the general case. We obtain a presentation of every Renner monoid and introduce a length function. In the case of the rook monoid, we recover the results obtained in [2]. Our length function is not the classical length function on Renner monoids [10]. We remark that the former shares with the latter several nice geometrical and combinatorial properties.

Let us postpone to the next section some definitions and notation, and state here our main results. Consider the Renner monoid $R(M)$ of a regular algebraic monoid M with a zero element. Denote by W the unit group of $R(M)$ and consider its associated Coxeter system (W, S). Denote by Λ a *cross section lattice* of the monoid $E(R(M))$ of idempotent elements of $R(M)$, and by $\Lambda_\circ$ the set of elements of Λ that are distinct from the identity. Finally, denote by λ the associated *type map* of $R(M)$; roughly speaking, this is a map that describes the action of W on $E(R(M))$.

THEOREM 1.1. *The Renner monoid $R(M)$ admits the monoid presentation whose generating set is $S \cup \Lambda_\circ$ and whose defining relations are:*

(COX1)	$s^2 = 1,$	$s \in S;$
(COX2)	$\lvert s, t\rangle^m = \lvert t, s\rangle^m,$	$(\{s, t\}, m) \in \mathcal{E}(\Gamma);$
(REN1)	$se = es,$	$e \in \Lambda_\circ,\ s \in \lambda^\star(e);$
(REN2)	$se = es = e,$	$e \in \Lambda_\circ,\ s \in \lambda_\star(e);$
(REN3)	$e\underline{w}f = e \wedge_w f,$	$e, f \in \Lambda_\circ,\ w \in \tilde{D}^\uparrow(e) \cap D^\uparrow(f).$

We define the length ℓ on $R(M)$ in the following way: if s lies in S, we set $\ell(s) = 1$; if e lies in Λ, we set $\ell(e) = 0$. Then we extend ℓ by additivity to the free monoid of words on $S \cup \Lambda_\circ$. If w lies in $R(M)$, its length $\ell(w)$ is the minimal length of its word representatives on $S \cup \Lambda_\circ$. In Section 3 we investigate the properties of this length function. In particular, we prove that it is nicely related to the classical *normal form* defined on $R(M)$, and we also prove the following proposition.

PROPOSITION 1.2. *Let T be a maximal torus of the unit group of M. Fix a Borel subgroup B that contains T. Let w lie in $R(M)$ and s lie in S. Then,*

$$BsBwB = \begin{cases} BwB & \text{if } \ell(sw) = \ell(w); \\ BswB & \text{if } \ell(sw) = \ell(w) + 1; \\ BswB \cup BwB & \text{if } \ell(sw) = \ell(w) - 1. \end{cases}$$

This article is organized as it follows. In Section 2 we first recall the background of algebraic monoid theory and of Coxeter group theory. Then we prove Theorem 1.1. In Section 3 we consider several examples of Renner monoids and deduce explicit presentations from Theorem 1.1. In Section 4 we focus on the length function and, in particular, we prove Proposition 1.2.

2. Presentation for Renner monoids

Our objective in the present section is to associate a monoid presentation to every Renner monoid. The statement of our result and its proof require some properties of algebraic monoid theory and of Coxeter group theory. In Section 2.1 we introduce Renner monoids and state the results we need about algebraic monoids. In Section 2.2 we recall the definition of Coxeter groups and some of their well-known properties. Using the two preliminary sections, we can prove Theorem 1.1 in Section 2.3. This provides a monoid presentation for every Renner monoid.

We fix an algebraically closed field $\mathbb{K}$. We denote by M_n the set of all $n \times n$ matrices over $\mathbb{K}$, and by GL_n the set of all invertible matrices in M_n. We refer to [9, 10, 14] for the general theory and proofs involving linear algebraic monoids and Renner monoids; we refer to [4] for an introduction to linear algebraic groups. If X is a subset of M_n, we denote by $\overline{X}$ its closure with respect to the Zariski topology.

2.1. Algebraic monoid theory. We introduce here the basic definitions and notation of algebraic monoid theory that we shall need later.

2.1.1. Regular monoids and reducible groups.

DEFINITION 2.1 (Algebraic monoid). An *algebraic monoid* is a submonoid of M_n, for some positive integer n, that is closed in the Zariski topology. An algebraic monoid is *irreducible* if it is irreducible as a variety.

It is very easy to construct algebraic monoids. Indeed, the Zariski closure $M = \overline{G}$ of any submonoid G of M_n is an algebraic monoid. The main example occurs when one considers for G an algebraic subgroup of GL_n. It turns out that in this case the group G is the unit group of M. Conversely, if M is an algebraic monoid, then its unit group $G(M)$ is an algebraic group. The monoid M_n is the seminal example of an algebraic monoid, and its unit group GL_n is the seminal example of an algebraic group.

One of the main differences between an algebraic group and an algebraic monoid is that the latter have idempotent elements. In the following we denote by $E(M)$ the set of idempotent elements of a monoid M. We recall that M is *regular* if $M = E(M)G(M) = G(M)E(M)$, and that M *has a zero element* if there exists an element 0 such that $0 \times m = m \times 0 = 0$ for every m in M. The next result, which is the starting point of the theory, was obtained independently by Putcha and Renner in 1982.

THEOREM 2.2. *Let M be an irreducible algebraic monoid with a zero element. Then M is regular if and only if $G(M)$ is reductive.*

The order $\leq$ on $E(M)$, defined by $e \leq f$ if $ef = fe = e$, provides a natural connection between the Borel subgroups of $G(M)$ and the idempotent elements of M.

THEOREM 2.3. *Let M be a regular irreducible algebraic monoid with a zero element. Let $\Gamma = (e_1, \ldots, e_k)$ be a maximal increasing sequence of distinct elements of $E(M)$.*

(i) *The centralizer $Z_{G(M)}(\Gamma)$ of Γ in $G(M)$ is a maximal torus of the reductive group $G(M)$.*

(ii) *Set*

$$B^+(\Gamma) = \{b \in G(M) \mid \forall e \in \Gamma,\, be = ebe\},$$
$$B^-(\Gamma) = \{b \in G(M) \mid \forall e \in \Gamma,\, eb = ebe\}.$$

Then, $B^-(\Gamma)$ and $B^+(\Gamma)$ are two opposed Borel subgroups with common torus $Z_{G(M)}(\Gamma)$.

2.1.2. Renner monoid.

DEFINITION 2.4 (Renner monoid). Let M be a regular irreducible algebraic monoid with a zero element. If T is a Borel subgroup of $G(M)$, then we denote its normalizer by $N_{G(M)}(T)$. The *Renner monoid* $R(M)$ of M is the monoid $\overline{N_{G(M)}(T)}/T$.

It is clear that $R(M)$ does not depend on the choice of the maximal torus of $G(M)$.

EXAMPLE 2.5. Consider $M = M_n(\mathbb{K})$, and choose the maximal torus $\mathbb{T}$ of diagonal matrices. The Renner monoid is isomorphic to the monoid of matrices with at most one nonzero entry, that is equal to 1, in each row and each column. This monoid is called the rook monoid R_n [16]. Its unit group is the group of monomial matrices, which is isomorphic to the symmetric group S_n.

From the definition we almost immediately have the following proposition.

PROPOSITION 2.6. *Let M be a regular irreducible algebraic monoid with a zero element, and fix a maximal torus T of $G(M)$. The Renner monoid $R(M)$ is a finite factorizable inverse monoid. In particular, the set $E(R(M))$ is a commutative monoid and a lattice for the partial order $\leq$ defined by $e \leq f$ when $ef = e$. Furthermore, there is a canonical order-preserving isomorphism of monoids between $E(R(M))$ and $E(\overline{T})$.*

2.2. Coxeter group theory. Here we recall some well-known facts about Coxeter groups. We refer to [1] for general theory and proofs.

DEFINITION 2.7 (Coxeter system). Let Γ be a finite simple labelled graph whose labels are positive integers greater than or equal to 3. We denote by S the vertex set of Γ. We denote by $\mathcal{E}(\Gamma)$ the set of pairs $(\{s, t\}, m)$ such that either $\{s, t\}$ is an edge of Γ labelled by m or $\{s, t\}$ is not an edge of Γ and $m = 2$. When $(\{s, t\}, m)$ belongs to $\mathcal{E}(\Gamma)$, we denote by $|s, t\rangle^m$ the word $sts\cdots$ of length m. The *Coxeter group* $W(\Gamma)$ associated with Γ is defined by the group presentation

$$\left\langle S \;\middle|\; \begin{array}{ll} s^2 = 1 & s \in S \\ |s, t\rangle^m = |t, s\rangle^m & (\{s, t\}, m) \in \mathcal{E}(\Gamma) \end{array} \right\rangle.$$

We say that $(W(\Gamma), S)$ is a *Coxeter system*.

PROPOSITION 2.8. *Let M be a regular irreducible algebraic monoid with a zero element, and denote by G its unit group. Fix a maximal torus T and a Borel subgroup B that contains T. Then:*

(i) *the* Weyl group *$W = N_G(T)/T$ of G is a finite Coxeter group;*
(ii) *the unit group of $R(M)$ is the Weyl group W.*

REMARK 2.9. Combining the results of Propositions 2.6 and 2.8, we get

$$R(M) = E(\overline{T}) \cdot W = W \cdot E(\overline{T}).$$

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DEFINITION 2.10. Let (W, S) be a *Coxeter system*. Let w belong to W. The *length* $\ell(w)$ of w is the minimal integer k such that w has a word representative of length k on the alphabet S. Such a word is called a *reduced word representative* of w.

In the following, we use the following classical result [1].

PROPOSITION 2.11. *Let (W, S) be a Coxeter system and I, J be subsets of S. Let W_I and W_J be the subgroups of W generated by I and J, respectively.*

(i) *The pairs (W_I, I) and (W_J, J) are Coxeter systems.*
(ii) *For every element w which belongs to W there exists a unique element $\hat{w}$ of minimal length in the double class $W_J w W_I$. Furthermore, there exist w_1 in W_I and w_2 in W_J such that $w = w_2 \hat{w} w_1$ with $\ell(w) = \ell(w_1) + \ell(\hat{w}) + \ell(w_2)$.*

Note that (ii) holds when I or J is empty.

2.3. Cross section. Our objective here is to prove Theorem 1.1. We first need to make precise the notation used in this theorem. Throughout this section, we assume that M is a regular irreducible algebraic monoid with a zero element. We denote by G the unit group of M. We fix a maximal torus T of G and a Borel subgroup B that contains T. We denote by W the Weyl group $N_G(T)/T$ of G. We denote by S the standard generating set associated with the canonical Coxeter structure of the Weyl group W.

2.3.1. The cross section lattice. To describe the generating set of our presentation, we need to introduce the *cross section lattice*, which is related to Green's relations. The latter are classical tools in semigroup theory. Let us recall the definition of relation $\mathcal{J}$. The $\mathcal{J}$-class of an element a in M is the double coset MaM. The set $\mathcal{U}(M)$ of $\mathcal{J}$-classes carries a natural partial order $\leq$ defined by $MaM \leq MbM$ if $MaM \subseteq MbM$. It turns out that the map $e \mapsto MeM$ from $E(M)$ to $\mathcal{U}(M)$ induces a one-to-one correspondence between the set of W-orbits on $E(\overline{T})$ and the set $\mathcal{U}(M)$. The existence of this one-to-one correspondence leads to the following definition.

DEFINITION 2.12 (Cross section lattice). A subset Λ of $E(\overline{T})$ is a *cross section lattice* if the map $\Lambda \to \mathcal{U}(M)$, $e \mapsto MeM$ is an order-preserving bijection.

Note that such a cross section lattice is a transversal of $E(\overline{T})$ for the action of W. It is not immediately clear that such a cross section lattice exists. Indeed it does, and the following theorem holds.

THEOREM 2.13 [9, Theorem 9.10]. *For every Borel subgroup $\mathbb{B}$ of G that contains T, we set*

$$\Lambda(\mathbb{B}) = \{e \in E(\overline{T}) \mid \forall b \in \mathbb{B},\ be = ebe\}.$$

The map $B \mapsto \Lambda(\mathbb{B})$ is a bijection between the set of Borel subgroups of G that contain T and the set of cross section lattices of $E(\overline{T})$.

EXAMPLE 2.14. Consider $M = M_n$. Consider the Borel subgroup $\mathbb{B}$ of invertible upper triangular matrices and $\mathbb{T}$ the maximal torus of invertible diagonal matrices.

Denote by e_i the diagonal matrix $\begin{pmatrix} \mathrm{Id}_i & 0 \\ 0 & 0 \end{pmatrix}$ of rank i. Then, the set $\Lambda(\mathbb{B})$ is $\{e_0, \ldots, e_n\}$. For every index i, we have $e_i \leq e_{i+1}$.

REMARK 2.15.

(i) Let Γ be a maximal chain of idempotent elements of $\overline{T}$ and consider the Borel subgroup $B^+(\Gamma)$ defined in Theorem 2.3. It follows from the definitions that $\Gamma \subseteq \Lambda(B^+(\Gamma))$.

(ii) [9, Definition 9.1] A cross section lattice is a sublattice of $E(\overline{T})$.

2.3.2. Type map. In order to state the defining relations of our presentation, we now turn to the notion of a type map. Recall that we have fixed a Borel subgroup B of G that contains T. We write Λ for $\Lambda(B)$. We consider Λ as a sublattice of $E(R(M))$ (see Proposition 2.6).

NOTATION 2.16 [10]. Let e belong to Λ.

(i) The *type map* $\lambda : e \mapsto \lambda(e)$ of the regular monoid M is defined by

$$\lambda(e) = \{s \in S \mid se = es\}.$$

(ii) We set

$$\lambda_\star(e) = \bigcap_{f \leq e} \lambda(f) \quad \text{and} \quad \lambda^\star(e) = \bigcap_{f \geq e} \lambda(f).$$

(iii) We set

$$W(e) = \{w \in W \mid we = ew\}, \qquad W_\star(e) = \{w \in W(e) \mid we = e\}.$$

We denote by $W^\star(e)$ the subgroup of W generated by $\lambda^\star(e)$.

PROPOSITION 2.17 [10, Lemma 7.15]. *With the above notation:*

(i) $\lambda_\star(e) = \{s \in S \mid se = es = e\}$ *and* $\lambda^\star(e) = \{s \in S \mid se = es \neq e\}$*;*

(ii) *the sets* $W(e)$*,* $W_\star(e)$ *and* $W^\star(e)$ *are the* standard parabolic subgroups *of* W *generated by the sets* $\lambda(e)$*,* $\lambda_\star(e)$ *and* $\lambda^\star(e)$*, respectively. Furthermore,* $W(e)$ *is the direct product of* $W_\star(e)$ *and* $W^\star(e)$*.*

NOTATION 2.18 [10]. By Propositions 2.11 and 2.17, for every w in W and every e, f in Λ, each of the sets $wW(e)$, $W(e)w$, $wW_\star(e)$, $W_\star(e)w$ and $W(e)wW(f)$ has a unique element of minimal length. We denote by $D(e)$, $\tilde{D}(e)$, $D_\star(e)$ and $\tilde{D}_\star(e)$ the set of elements w of W that are of minimal length in their classes $wW(e)$, $W(e)w$, $wW_\star(e)$ and $W_\star(e)w$, respectively. Note that the set of elements w of W that are of minimal length in their double class $W(e)wW(f)$ is $\tilde{D}(e) \cap D(f)$.

2.3.3. Properties of the cross section lattice. As in previous sections, we fix a Borel subgroup B of G that contains T, and denote by Λ the associated cross section lattice contained in $E(R(M))$. We use the notation $\mathcal{E}(\Gamma)$ of Section 2.2. We set $\Lambda_\circ = \Lambda - \{1\}$. To make the statement of Proposition 2.24 clear we need a preliminary result.

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LEMMA 2.19. *Let e_1, e_2 lie in $E(\overline{T})$ such that $e_1 \leq e_2$. There exist f_1, f_2 in Λ with $f_1 \leq f_2$ and w in W such that $wf_1w^{-1} = e_1$ and $wf_2w^{-1} = e_2$.*

PROOF. Let Γ be a maximal chain of $E(\overline{T})$ that contains e_1 and e_2. The Borel subgroup $B^+(\Gamma)$ contains the maximal torus T. Therefore, there exists w in W such that $w^{-1}B^+(\Gamma)w = B$. This implies that $w^{-1}\Lambda(B^+(\Gamma))w = \Lambda$. We conclude using Remark 2.15(i). □

LEMMA 2.20. *Let h, e belong to Λ such that $h \leq e$. Then, $W(h) \cap \tilde{D}(e) \subseteq W_\star(h)$ and $W(h) \cap D(e) \subseteq W_\star(h)$.*

PROOF. Let w lie in $W(h) \cap \tilde{D}(e)$. We can write $w = w_1w_2 = w_2w_1$ where w_1 lies in $W_\star(h)$ and w_2 lies in $W^\star(h)$. Since $h \leq e$, we have $\lambda^\star(h) \subseteq \lambda^\star(e)$ and $W^\star(h) \subseteq W^\star(e)$. Since w belongs to $\tilde{D}(e)$, this implies that $w_2 = 1$. The proof of the second inclusion is similar. □

PROPOSITION 2.21. *Let e, f lie in $\Lambda_\circ$ and w lie in $\tilde{D}(e) \cap D(f)$. There exists h in $\Lambda_\circ$ with $h \leq e \wedge f$ such that w belongs to $W_\star(h)$ and $ewf = hw = h$.*

To prove the above proposition, we use the existence of a normal decomposition in $R(M)$.

PROPOSITION 2.22 [10, Section 8.6]. *For every w in $R(M)$ there exists a unique triple (w_1, e, w_2) with $e \in \Lambda$, $w_1 \in D_\star(e)$ and $w_2 \in \tilde{D}(e)$ such that $w = w_1ew_2$.*

Following [10], we call the triple (w_1, e, w_2) the *normal decomposition* of w.

PROOF OF PROPOSITION 2.21. Consider the normal decomposition (w_1, h, w_2) of ewf. Then w_1 belongs to $D_\star(h)$ and w_2 belongs to $\tilde{D}(h)$. The element $w^{-1}ewf$ is equal to $w^{-1}w_1hw_2$ and belongs to $E(R(M))$. Since w_2 lies in $\tilde{D}(h)$, this implies that $w_3 = w_2w^{-1}w_1$ lies in $W_\star(h)$, and that $f \geq w_2^{-1}hw_2$. By Lemma 2.19, there exists w_4 in W and f_1, h_1 in $\Lambda_\circ$, with $f_1 \geq h_1$, such that $w_4^{-1}f_1w_4 = f$ and $w_4^{-1}h_1w_4 = w_2^{-1}hw_2$. Since Λ is a cross section for the action of W, we have $f_1 = f$ and $h_1 = h$. In particular, w_4 belongs to $W(f)$. Since w_2 belongs to $\tilde{D}(h)$, we deduce that there exists r in $W(h)$ such that $w_4 = rw_2$ with $\ell(w_4) = \ell(w_2) + \ell(r)$. Then w_2 lies in $W(f)$. Now write $w_1 = w'_1w''_1$ where w''_1 lies in $W^\star(h)$ and w'_1 belongs to $D_\star(h)$. Then $ewf = w'_1hw''_1w_2$, and w''_1w_2 lies in $D(h)$. By symmetry, we get that w'_1 belongs to $W(e)$. Hence, ${w'}_1^{-1}ww_2^{-1}$ is equal to $w''_1w_3^{-1}$ and belongs to $W(h)$. By hypothesis w lies in $\tilde{D}(e) \cap D(f)$. Then we must have

$$\ell(w''_1w_3^{-1}) = \ell(w_2^{-1}) + \ell({w'}_1^{-1}) + \ell(w).$$

Since $w''_1w_3^{-1}$ belongs to $W(h)$, it follows that w_2 and w'_1 belong to $W(h)$ too. This implies that $w_2 = w'_1 = 1$ and $w = w''_1w_3^{-1}$. Therefore, $ewf = hw''_1 = hw = wh$. Finally, w belongs to $W_\star(h)$ by Lemma 2.20. □

2.3.4. A presentation for $R(M)$.

NOTATION 2.23.

(i) For each w in W, we fix a reduced word representative $\underline{w}$.

(ii) We denote by $e \wedge_w f$ the unique element in $\Lambda_\circ$ that represents the element h in Proposition 2.21.

Note that for s in S, we have $\underline{s} = s$. We recall that Λ is a sublattice of $E(\overline{T})$ for the order $\leq$ defined by $e \leq f$ if $ef = fe = e$. We are now ready to state a monoid presentation for $R(M)$.

PROPOSITION 2.24. *The Renner monoid has the following monoid presentation whose generating set is $S \cup \Lambda_\circ$ and whose defining relations are:*

(COX1)	$s^2 = 1,$	$s \in S;$
(COX2)	$\lvert s, t\rangle^m = \lvert t, s\rangle^m,$	$(\{s, t\}, m) \in \mathcal{E}(\Gamma);$
(REN)	$se = es,$	$e \in \Lambda_\circ,\ s \in \lambda^\star(e);$
(REN2)	$se = es = e,$	$e \in \Lambda_\circ,\ s \in \lambda_\star(e);$
(REN3′)	$e\underline{w}f = e \wedge_w f,$	$e,\ f \in \Lambda_\circ,\ w \in \tilde{D}(e) \cap D(f).$

The reader should note that the relations of type (REN3) in Theorem 1.1 are a special case of the relations of type (REN3′). Note that when $e \leq f$ and $w = 1$, then relation (REN3′) becomes $ef = fe = e$. More generally, $e \wedge_1 f = e \wedge f$.

PROOF. By Remark 2.9 the submonoids $E(R(M))$ and W generate the monoid $R(M)$. As S is a generating set for W, it follows from the definition of Λ that the set $S \cup \Lambda_\circ$ generates $R(M)$ as a monoid. Clearly, relations (COX1) and (COX2) hold in W, and relations (REN1) and (REN2) hold in $R(M)$. Relations (REN3′) hold in $R(M)$ by Proposition 2.21. It remains to prove that we obtain a presentation of the monoid $R(M)$. Let w belong to $R(M)$ with (w_1, e, w_2) as normal form. Consider any word ω on the alphabet $S \cup \Lambda_\circ$ that represents w. We claim that, starting from ω, one can obtain the word $\underline{w_1}e\underline{w_2}$ using the relations of the above presentation only. This is almost obvious by induction on the number j of letters of the word ω that belong to $\Lambda_\circ$. The property holds for $j = 0$ (in this case $w = w_1$ and $e = w_2 = 1$) because (COX1) and (COX2) are the defining relations of the presentation of W. The case $j = 1$ is also clear, applying relations (COX1), (COX2), (REN1) and (REN2). Now, for $j \geq 2$, the case j can be reduced to the case $j - 1$ using relations (REN3′) (and the other relations). □

The presentation in Proposition 2.24 is not minimal; some relations can be removed in order to obtain the presentation stated in Theorem 1.1. Let us introduce some notation used in this theorem.

NOTATION 2.25. If e lies in Λ, we denote by $\tilde{D}^\uparrow(e)$ the set

$$\tilde{D}(e) \cap \left(\bigcap_{f > e} W(f) \right).$$

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Similarly, we denote by $D^{\uparrow}(e)$ the set

$$D(e) \cap \Big(\bigcap_{f>e} W(f)\Big).$$

REMARK 2.26.

(i)
$$\Big(\bigcap_{f>e} \lambda(f)\Big) \cap \lambda_{\star}(e) = \emptyset$$

by Proposition 2.17.

(ii)
$$\tilde{D}^{\uparrow}(e) = \tilde{D}(e) \cap W_{\cap_{f>e}\lambda(f)} \quad \text{and} \quad D^{\uparrow}(e) = D(e) \cap W_{\cap_{f>e}\lambda(f)}.$$

The reader may note that, for $e \leq f$, $\tilde{D}^{\uparrow}(e) \cap D^{\uparrow}(f) = \{1\}$.

PROOF OF THEOREM 1.1. We need to prove that every relation $e\underline{w}f = e \wedge_w f$ of type (REN3′) in Proposition 2.24 can be deduced from relations (REN3) of Theorem 1.1, using the other common defining relations of type (COX1), (COX2), (REN1) and (REN2). We prove this by induction on the length of w. If $\ell(w) = 0$ then w is equal to 1 and therefore belongs to $\tilde{D}^{\uparrow}(e) \cap D^{\uparrow}(f)$. Assume that $\ell(w) \geq 1$ and that w does not belong to $\tilde{D}^{\uparrow}(e) \cap D^{\uparrow}(f)$. Assume, furthermore, that w does not lie in $\tilde{D}^{\uparrow}(e)$ (the other case is similar). Choose e_1 in $\Lambda_{\circ}$ such that $e_1 > e$ and w does not lie in $W(e_1)$. Then, applying relations (REN3), we can transform the word $e\underline{w}f$ into the word $ee_1\underline{w}f$. Using relations (COX2), we can transform the word $\underline{w}$ into a word $\underline{w_1}\,\underline{w_2}$ where w_1 belongs to $W(e_1)$ and w_2 belongs to $\tilde{D}^{\uparrow}(e_1)$. Then, applying relations (COX2) and (REN1), we can transform the word $ee_1\underline{w}f$ into the word $e\underline{w_1}e_1\underline{w_2}f$. By hypothesis on w, we have $w_2 \neq 1$ and, therefore, $\ell(w_1) < \ell(w)$. Assume that w_2 belongs to $D^{\uparrow}(f)$. We can apply relation (REN3) to transform $e\underline{w_1}e_1\underline{w_2}f$ into $e\underline{w_1}(e_1 \wedge_{w_2} f)$. Using relations (COX2), we can transform $\underline{w_1}$ into a word $\underline{w_1'}\,\underline{w_1''}\,\underline{w_1'''}$ with w_1''' in $W_{\star}(e_1 \wedge_{w_2} f)$, w_2'' in $W^{\star}(e_1 \wedge_{w_2} f)$ and w_1' in $D(e_1 \wedge_{w_2} f)$. Then $e\underline{w_1}(e_1 \wedge_{w_2} f)$ can be transformed into $e\underline{w_1'}(e_1 \wedge_{w_2} f)\underline{w_1''}$. Since $\ell(w_1') \leq \ell(w_1) < \ell(w)$, we can apply an induction argument to transform the word $e\underline{w_1}(e_1 \wedge_{w_2} f)$ into the word $e \wedge_{w_1} (e_1 \wedge_{w_2} f)\underline{w_1''}$. Now, by the uniqueness of the normal decomposition, w_1'' has to belong to $W_{\star}(\overline{e \wedge_{w_1} (e_1 \wedge_{w_2} f)})$. Therefore we can transform $e \wedge_{w_1} (e_1 \wedge_{w_2} f)\underline{w_1''}$ into $e \wedge_{w_1} (e_1 \wedge_{w_2} f)$ using relations (REN2). Note that the letters $e \wedge_{w_1} (e_1 \wedge_{w_2} \overline{f})$ and $e \wedge_w f$ have to be equal as they represent the same element in Λ. Assume, finally, that w_2 does not belong to $D^{\uparrow}(f)$. By similar arguments we can, applying relations (COX2) and (REN1), transform the word $e\underline{w_1}e_1\underline{w_2}f$ into a word $e\underline{w_1}e_1\underline{w_3}f_1\underline{w_4}$ where $f_1 > f$ in $\Lambda_{\circ}$ and $w_2 = w_3w_4$ with w_3 in $D^{\uparrow}(f_1)$ and w_4 in $W(f_1)$. At this stage we are in position to apply relation (REN3). Thus, we can transform the word $e\underline{w_1}e_1\underline{w_2}f$ into the word $e\underline{w_1}(e_1 \wedge_{w_3} f_1)\underline{w_4}f$. Since we have $\ell(w_1) + \ell(w_4) < \ell(w)$ we can apply an induction argument to conclude as in the first case. □

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3. Some particular Renner monoids

Here we focus on some special Renner monoids considered in [5–7]. In each case, we provide an explicit monoid presentation using the general presentation obtained in Section 1.

3.1. The rook monoid. Consider $M = M_n$ and choose $\mathbb{B}$ for the Borel subgroup (see Example 2.14 and Figure 1). In this case, the Weyl group is the symmetric group S_n. Its generating set S is $\{s_1, \ldots, s_{n-1}\}$ where s_i is the transposition matrix corresponding to $(i, i+1)$. The cross section lattice $\Lambda = \{e_0, \ldots, e_{n-1}, e_n\}$ is linear ($e_j \leq e_{j+1}$ for every j). For every j,

$$\lambda_\star(e_j) = \{s_i \mid j+1 \leq i\} \quad \text{and} \quad \lambda^\star(e_j) = \{s_i \mid i \leq j-1\}.$$

In particular, $\tilde{D}^\uparrow(e_i) \cap D^\uparrow(e_i) = \{1, s_i\}$ and, for $i \neq j$, $\tilde{D}^\uparrow(e_i) \cap \tilde{D}^\uparrow(e_j) = \{1\}$.

Therefore, we recover the monoid presentation of the rook monoid $R(M)$ stated in [2]: the generating set is $\{s_1, \ldots, s_{n-1}, e_0, \ldots, e_{n-1}\}$ and the defining relations are

$$\begin{array}{rl}
s_i^2 = 1, & 1 \leq i \leq n-1;\\
s_i s_j = s_j s_i, & 1 \leq i, j \leq n-1 \text{ and } |i-j| \geq 2;\\
s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, & 1 \leq i \leq n-1;\\
e_j s_i = s_i e_j & 1 \leq i < j \leq n-1;\\
e_j s_i = s_i e_j = e_j & 0 \leq j < i \leq n-1;\\
e_i e_j = e_j e_i = e_{\min(i,j)} & 0 \leq i, j \leq n-1;\\
e_i s_i e_i = e_{i-1} & 1 \leq i \leq n-1.
\end{array}$$

3.2. The sympletic algebraic monoid. Let n be a positive even integer and Sp_n be the *symplectic algebraic group* [4, p. 52]: write $n = 2\ell$ where ℓ is a positive integer, and consider the matrix $J_\ell = \left(\begin{smallmatrix} & 1\\ 1 & \end{smallmatrix}\right)$ in M_ℓ. Let $J = \left(\begin{smallmatrix} 0 & J_\ell\\ -J_\ell & 0\end{smallmatrix}\right)$ in M_n. Then Sp_n is equal to $\{A \in M_n \mid A^t J A = J\}$, where A^t is the transpose matrix of A. We set $M = \overline{\mathbb{K}^\times \mathrm{Sp}_n}$. This monoid is a regular monoid with 0 whose associated reductive algebraic unit group is $\mathbb{K}^\times\ \mathrm{Sp}_n$. It is called the *symplectic algebraic monoid* [7]. Let $\mathbb{B}$ be the Borel subgroup of GL_n as defined in Example 2.14, and set $B = \mathbb{K}^\times(\mathbb{B} \cap \mathrm{Sp}_n)$. This is a Borel subgroup of the unit group of M. It is shown in [7] that the cross section lattice Λ of M is $\{e_0, e_1, \ldots, e_\ell, e_n\}$ where the elements e_i correspond to the matrices of M_n defined in Example 2.14 (see Figure 2). In particular, the cross section lattice Λ is linear. In this case, the Weyl group is a Coxeter group of type B_ℓ. In other words, the group W is isomorphic to the subgroup of S_n generated by the permutation matrices $s_1, \ldots, s_\ell$ corresponding to $(1, 2)(n-1, n)$, $(2, 3)(n-2, n-1), \ldots, (\ell-1, \ell)(\ell+1, \ell+2)$, and $(\ell, \ell+1)$, respectively. We have $\lambda_\star(e_i) = \{s_{i+1}, \ldots, s_\ell\}$ and $\lambda^\star(e_i) = \{s_1, \ldots, s_{i-1}\}$. Therefore, $\tilde{D}^\uparrow(e_\ell) \cap \tilde{D}^\uparrow(e_\ell) = \{1, s_\ell, s_\ell s_{\ell-1} s_\ell\}$ and, for i in $\{1, \ldots, \ell-1\}$, $\tilde{D}^\uparrow(e_i) \cap D^\uparrow(e_i) = \{1, s_i\}$. A direct calculation proves that $e_i s_i e_i = s_i e_{i-1}$ for every i, and $e_\ell s_\ell s_{\ell-1} s_\ell e_\ell = e_{\ell-2}$. Hence, a monoid presentation of $R(M)$ is given by the generating set $\{s_1, \ldots, s_\ell, e_0, \ldots, e_\ell\}$

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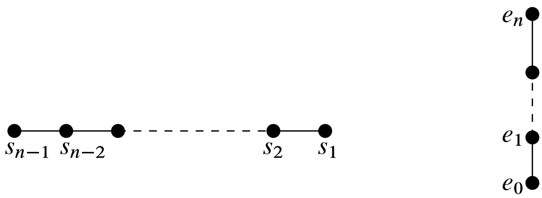

FIGURE 1. Coxeter graph and Hasse diagram for M_n.

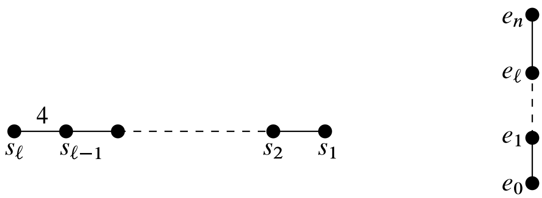

FIGURE 2. Coxeter graph and Hasse diagram for Sp_n.

and the defining relations

$$\begin{array}{rll}
s_i^2 &= 1, & 1 \le i \le \ell; \\
s_i s_j &= s_j s_i, & 1 \le i,\ j \le \ell \text{ and } |i-j| \ge 2; \\
s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, & 1 \le i \le \ell - 2; \\
s_\ell s_{\ell-1} s_\ell s_{\ell-1} &= s_{\ell-1} s_\ell s_{\ell-1} s_\ell; & \\
e_j s_i &= s_i e_j & 1 \le i < j \le \ell; \\
e_j s_i &= s_i e_j = e_j, & 0 \le j < i \le \ell; \\
e_i e_j &= e_j e_i = e_{\min(i,j)}, & 0 \le i,\ j \le \ell; \\
e_i s_i e_i &= e_{i-1}, & 1 \le i \le \ell; \\
e_\ell s_\ell s_{\ell-1} s_\ell e_\ell &= e_{\ell-2}. &
\end{array}$$

3.3. The special orthogonal algebraic monoid. Let n be a positive integer and J_n be defined as in Section 3.2. The *special orthogonal group* SO_n is defined as $\mathrm{SO}_n = \{A \in \mathrm{SL}_n \mid g^T J_n g = J_n\}$. The group $\mathbb{K}^\times\,\mathrm{SO}_n$ is a connected reductive group. Following [5, 6], we define the *special orthogonal algebraic monoid* to be the Zariski closure $M = \overline{\mathbb{K}^\times \mathrm{SO}_n}$ of $\mathbb{K}^\times\,\mathrm{SO}_n$. This is an algebraic monoid [5, 6], and $B = \mathbb{B} \cap M$ is a Borel subgroup of its unit group. In this case, the cross section lattice depends on the parity of n. Furthermore, the Weyl group is a Coxeter group whose type depends on the parity of n too.

Assume that $n = 2\ell$ is even. In this case, W is a Coxeter group of type D_ℓ. The standard generating set of W is $\{s_1, \ldots, s_\ell\}$ where, for $1 \le i \le \ell - 1$, the element s_i is the permutation matrix associated with $(i, i+1)(n-i, n-i+1)$, and s_ℓ is the permutation matrix associated with $(\ell-1, \ell+1)(\ell, \ell+2)$. It is shown in [6] that the cross section Λ is equal to $\{e_0, e_1, \ldots, e_\ell, f_\ell, e_n\}$. The elements e_i correspond to the matrices of M_n defined in Example 2.14; the element f_ℓ is the diagonal matrix $e_{\ell+1} + e_{\ell-1} - e_\ell$. The Hasse diagram of Λ is as represented in Figure 3. For j

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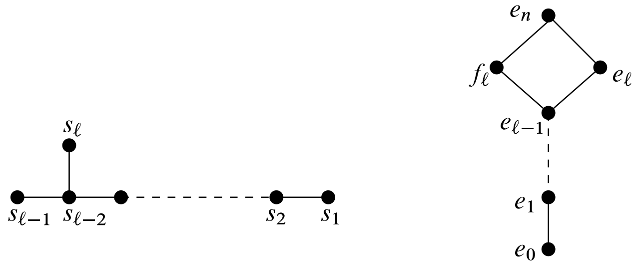

FIGURE 3. Coxeter graph and Hasse diagram for $\mathrm{SO}_{2\ell}$.

in $\{0, \ldots, \ell-2\}$ one has $\lambda_\star(e_j) = \{s_i \mid j+1 \le i\}$ and $\lambda^\star(e_j) = \{s_i \mid i \le j-1\}$. Furthermore, one can verify that

$$\begin{aligned}
\lambda_\star(e_{\ell-1}) = \lambda_\star(f_\ell) = \lambda_\star(e_\ell) &= \emptyset, \\
\lambda^\star(e_{\ell-1}) = \lambda^\star(f_\ell) &= \{s_i \mid i \le \ell-2\}, \\
\lambda^\star(e_\ell) &= \{s_i \mid i \le \ell-1\}, \\
\lambda^\star(f_\ell) &= \{s_i \mid i \ne \ell-1\}.
\end{aligned}$$

Therefore, for i in $\{1, \ldots, \ell-2\}$, we have $\tilde{D}^\uparrow(e_i) \cap D^\uparrow(e_i) = \{1, s_i\}$. Furthermore,

$$\begin{gathered}
\tilde{D}^\uparrow(e_{\ell-1}) \cap D^\uparrow(e_{\ell-1}) = \{1\} \quad \text{and} \\
\tilde{D}^\uparrow(f_\ell) \cap D^\uparrow(f_\ell) = \{1, s_{\ell-1}\}; \qquad \tilde{D}^\uparrow(e_\ell) \cap \tilde{D}^\uparrow(e_\ell) = \{1, s_\ell\}; \\
\tilde{D}^\uparrow(e_\ell) \cap D^\uparrow(f_\ell) = \{1, s_\ell s_{\ell-2} s_{\ell-1}\}; \qquad \tilde{D}^\uparrow(f_\ell) \cap \tilde{D}^\uparrow(e_\ell) = \{1, s_{\ell-1} s_{\ell-2} s_\ell\}.
\end{gathered}$$

The monoid $R(M)$ has a presentation with generating set $\{s_1, \ldots, s_\ell, e_0, \ldots, e_\ell, f_\ell\}$ and defining relations

$$\begin{aligned}
s_i^2 &= 1, && 1 \le i \le \ell; \\
s_i s_j &= s_j s_i, && 1 \le i, j \le \ell \text{ and } |i-j| \ge 2; \\
s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, && 1 \le i \le \ell-2; \\
s_\ell s_{\ell-2} s_\ell &= s_{\ell-2} s_\ell s_{\ell-2}; && \\
e_j s_i &= s_i e_j, && 1 \le i < j \le \ell; \\
e_j s_i &= s_i e_j = e_j, && 0 \le j < i \le \ell; \\
e_i e_j &= e_j e_i = e_{\min(i,j)}, && 0 \le i, j \le \ell; \\
f_\ell e_\ell &= e_\ell f_\ell = e_{\ell-1}; && \\
e_i s_i e_i &= e_{i-1}, && 1 \le i \le \ell-1; \\
e_\ell s_\ell e_\ell &= f_\ell s_{\ell-1} f_\ell = e_{\ell-2}; && \\
e_\ell s_\ell s_{\ell-2} s_{\ell-1} f_\ell &= f_\ell s_{\ell-1} s_{\ell-2} s_\ell e_\ell = e_{\ell-3}. &&
\end{aligned}$$

Assume that $n = 2\ell+1$ is odd. In that case, W is a Coxeter group of type B_ℓ. It is shown in [5] that the cross section lattice is linear as in the case of the symplectic algebraic monoid. It turns out that the Renner monoid of $\mathrm{SO}_{2\ell+1}$ is isomorphic to the Renner monoid of symplectic algebraic monoid $\overline{\mathbb{K}^\times \mathrm{Sp}_{2\ell}}$, and that we obtain the same presentation as in the latter case.

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3.4. More examples: adjoint representations. Let G be a simple algebraic group, and denote by $\mathfrak{g}$ its Lie algebra. Let M be the algebraic monoid $\overline{\mathbb{K}^{\times}\mathrm{Ad}(G)}$ in $\mathrm{End}(\mathfrak{g})$. The cross section lattice of M and the type map of M have been calculated for each Dynkin diagram (see [10, Section 7.4]). Therefore one can deduce a monoid presentation for each of the associated Renner monoid.

3.5. More examples: $\mathcal{J}$-irreducible algebraic monoids. In [11], Renner and Putcha consider among regular irreducible algebraic monoids those that are $\mathcal{J}$-*irreducible*, that is, those whose cross section lattices have a unique minimal nonzero element. It is easy to see that the $\mathcal{J}$-irreducibility property is related to the existence of irreducible rational representations [11, Proposition 4.2]. Renner and Putcha determined the cross section lattice of those $\mathcal{J}$-irreducible that arise from special kinds of dominant weights [11, Figures 2, 3]. Using [11, Theorem 4.13], one can deduce the associated type maps and therefore a monoid presentation of each corresponding Renner monoids.

4. A length function on $R(M)$

In this section we extend the length function defined in [2] to any Renner monoid. Throughout this section, we assume that M is a regular irreducible algebraic monoid with a zero element. We denote by G the unit group of M. We fix a maximal torus T of G and a Borel subgroup B that contains T. We denote by W the Weyl group $N_G(T)/T$ of G. We denote by S the standard generating set associated with the canonical Coxeter structure of the Weyl group W. We denote by Λ the associated cross section lattice contained in $R(M)$. As before, we set $\Lambda_\circ = \Lambda - \{1\}$.

4.1. Minimal word representatives. The definition of the length function on W and of a reduced word is given in Section 2.2.

DEFINITION 4.1.

(i) We set $\ell(s) = 1$ for s in S and $\ell(e) = 0$ for e in Λ. Let $x_1, \ldots, x_k$ be in $S \cup \Lambda_\circ$ and consider the word $\omega = x_1 \cdots x_k$. Then, the *length* of the word ω is the integer $\ell(\omega)$ defined by $\ell(\omega) = \sum_{i=1}^{k} \ell(x_i)$.

(ii) The *length* of an element w which belongs to $R(M)$ is the integer $\ell(w)$ defined by

$$\ell(w) = \min\{\ell(\omega) \mid \omega \text{ is a word representative of } w \text{ over } S \cup \Lambda_\circ\}.$$

The following properties are direct consequences of the definition.

PROPOSITION 4.2. *Let w belong to $R(M)$.*

(i) *The length function ℓ on $R(M)$ extends the length function ℓ defined on W.*

(ii) $\ell(w) = 0$ *if and only if w lies in Λ.*

(iii) *If s lies in S then* $|\ell(sw) - \ell(w)| \leq 1$.

(iv) *If w' belongs to $R(M)$, then* $\ell(ww') \leq \ell(w) + \ell(w')$.

PROOF. (i) and (ii) are clear: the letters of every representative word of an element which lies in W are in S. If w lies in $R(M)$ and s lies to S, then $\ell(sw) \leq \ell(w) + 1$.

Since $w = s^2 w = s(sw)$, the inequality $\ell(w) \leq \ell(sw) + 1$ holds too. Point (iii) follows, and (iv) is a direct consequence of (iii). □

PROPOSITION 4.3. *Let w belong to $R(M)$.*

(i) *If (w_1, e, w_2) is the normal decomposition of w, then $\ell(w) = \ell(w_1) + \ell(w_2)$.*
(ii) *If ω_1, ω_2 are two representative words of w on $S \cup \Lambda_\circ$ such that the equalities $\ell(w) = \ell(\omega_1) = \ell(\omega_2)$ hold, then using the defining relations of the presentation of $R(M)$ in Proposition 2.24, we can transform ω_1 into ω_2 without increasing the length.*

PROOF. (i) Let ω be a representative word w on $S \cup \Lambda_\circ$ such that $\ell(w) = \ell(\omega)$. It is clear that we can repeat the argument of the proof of Proposition 2.24 without using (COX1). Therefore

$$\ell(\omega) \geq \ell(w_1 e w_2) = \ell(w_1) + \ell(w_2) \geq \ell(w).$$

(ii) This is a direct consequence of the proof of (i). □

COROLLARY 4.4. *Let w lie in $R(M)$ and e belong to $\Lambda_\circ$. Denote by (w_1, f, w_2) the normal decomposition of w.*

(i) *$\ell(we) \leq \ell(w)$ and $\ell(ew) \leq \ell(w)$.*
(ii) *$\ell(we) = \ell(w)$ if and only if the normal decomposition of we is $(w_1, e \wedge f, w_2)$. Furthermore, in this case, w_2 lies in $W^\star(e)$.*

PROOF. (i) This is a direct consequence of the definition of the length and of Proposition 4.3(i):

$$\ell(we) = \ell(w_1 f w_2 e) \leq \ell(w_1) + 0 + \ell(w_2) + 0 = \ell(w).$$

The same arguments prove that $\ell(ew) \leq \ell(w)$.

(ii) Decompose w_2 as a product $w_2' w_2'' w_2'''$ where w_2''' lies in $W_\star(f)$, w_2'' lies in $W^\star(f)$, w_2' lies in $D(f)$ and $\ell(w_2) = \ell(w_2') + \ell(w_2'') + \ell(w_2''')$. Then

$$we = w_1 f w_2 e = w_1 f w_2' e w_2'' = w_1 (f \wedge_{w_2'} e) w_2''.$$

In particular, $\ell(we) \leq \ell(w_1) + \ell(w_2'')$. Assume that $\ell(we) = \ell(w)$. We must have $w_2' = w_2''' = 1$. The element w_2'' (that is, w_2) must belong to $D^\star(f \wedge_1 e) = D^\star(f \wedge e)$, and the element w_1 must belong to $\tilde{D}^\star(f \wedge e)$. In particular, w_2 lies in $W^\star(e)$. Furthermore, w_2 lies in $D(f \wedge e)$ since $\lambda^\star(f \wedge e) \subseteq \lambda(f)$ by Proposition 2.17(i). Conversely, if the the normal decomposition of we is $(w_1, e \wedge f, w_2)$, then $\ell(we) = \ell(w_1) + \ell(w_2) = \ell(w)$. □

4.2. Geometrical formula. In Proposition 4.6 below we provide a geometrical formula for the length function ℓ defined in the previous section. This formula extends naturally the geometrical definition of the length function on a Coxeter group. Another length function on Renner monoids has already been defined and investigated [8, 10, 13]. This length function has nice properties, which are similar to those in Propositions 4.2, 4.3(i) and 4.6. This alternative length function was first

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introduced by Solomon [13] in the special case of rook monoids in order to verify a combinatorial formula that generalizes Rodrigues' formula [12]. That is why we call this length function the *Solomon length function* in the following. We proved in [2] that our length function for the rook monoid satisfies the same combinatorial formula. We also proved in [2] that in the case of the rook monoid, our presentation of $R(M)$ and our length function are related to the Hecke algebra. In a forthcoming paper, we will prove that this remains true in the general case.

LEMMA 4.5. *Let w belong to $R(M)$ and denote by (w_1, e, w_2) its normal decomposition. Let s be in S.*

(i) *We have one of the two following cases:*

 (a) *there exists t in $\lambda_\star(e)$ such that $sw_1 = w_1t$. In this case, $sw = w$;*

 (b) *the element sw_1 lies in $D_\star(e)$ and (sw_1, e, w_2) is the normal decomposition of sw.*

(ii) *Denote by $\tilde{l}$ the Solomon length function on $R(M)$. Then*

$$\ell(sw) - \ell(w) = \tilde{l}(sw) - \tilde{l}(w).$$

PROOF. (i) If sw_1 lies in $D_\star(e)$, then by Proposition 2.22, the triple (sw_1, e, w_2) is the normal decomposition of sw. Assume now that sw_1 does not belong to $D_\star(e)$. In that case, e cannot be equal to 1. Since w_1 belongs to $D_\star(e)$, by the exchange lemma, there exists t in $\lambda(e)$ such that $sw_1 = w_1t$. Therefore,

$$sw = sw_1ew_2 = w_1tew_2 = w_1ew_2 = w.$$

(ii) The Solomon length $\tilde{l}(w)$ of an element w in $R(M)$ can be defined by the formula $\tilde{l}(w) = \ell(w_1) - \ell(w_2) + \tilde{\ell}_e$ where (w_1, e, w_2) is the normal decomposition of w and $\tilde{\ell}_e$ is a constant that depends on e only [8, Definition 4.1]. Therefore the result is a direct consequence of (i). □

As a direct consequence of Lemma 4.5(ii) and [10, Theorem 8.18] we get Proposition 1.2.

PROPOSITION 4.6. *Let w belong to $R(M)$, and (w_1, e, w_2) be its normal decomposition. Then*

$$\ell(w) = \dim(Bw_1eB) - \dim(Bew_2B).$$

When w lies in S_n, then $e = w_2 = 1$, and we recover the well-known formula

$$\ell(w) = \dim(BwB) - \dim(B).$$

PROOF. By [8, Section 4], for every normal decomposition (v_1, f, v_2) we have the equality

$$\dim(Bv_1fv_2B) = \ell(v_1) - \ell(v_2) + k_f,$$

where k_f is a constant that depends on f only. Therefore,

$$\dim(Bw_1eB) - \dim(Bew_2B) = \ell(w_1) + k_e - (-\ell(w_2) + k_e) = \ell(w).$$

This concludes the proof. □

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