A MINIMAL CUBIC GRAPH OF GIRTH SEVEN

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A "cubic" graph is one with three edges incident on each vertex. Let v and e be the number of vertices and edges, respectively. Then $3v = 2e$ for a cubic graph. The girth of a graph is the smallest number of edges in any non-trivial polygon. A minimal graph is one with the smallest number of edges with its particular properties. The minimal cubic graphs of girths one to eight, excluding seven, are discussed in Tutte's paper [1]. A minimal cubic graph of girth seven is given here.

A basic feature of cubic graphs of odd girth is the "necessary subgraph" shown in figure 1 for girths three, five and seven. If we assume that the required graph has only the vertices of the "necessary" subgraph, we may construct the graph itself by starting from one exterior vertex and drawing in the remaining lines to "fill up" each vertex in turn, remembering not to form any polygons with too few edges. This works for girths three and five. However, for girth seven, we can draw in only six such lines, and utilizing the symmetry of the graph, we can show that it is not possible to complete the graph in this way. Now, the necessary subgraph for girth seven has twenty-two vertices, so we need more. Every cubic graph has an even number of vertices ($3v = 2e$) and so we need at least twenty-four vertices. Coxeter's graph [2], with twenty-eight vertices, had the record. However, by applying the criterion for the necessary subgraph at successive inner vertices, we can obtain a graph with twenty-four vertices (shown in figure 2) and so it is minimal.

This graph has a Hamiltonian circuit, and is redrawn in figure 3 to exhibit this. From figure 3, we see 45° rotational symmetry about the centre, and reflection symmetry about the dark lines. This shows the equivalence under symmetry operations of edges marked with the same letter. Further, on

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redrawing, using the Hamiltonian circuit through the edges cb instead of ca as the outside ring, we see the equivalence of edges marked b and a.

The proof of the inadequacy of a graph with twenty-two vertices, and the suggestion for the search for this graph, were given by Dr. W.T. Tutte during his lectures on linear graphs.

REFERENCES

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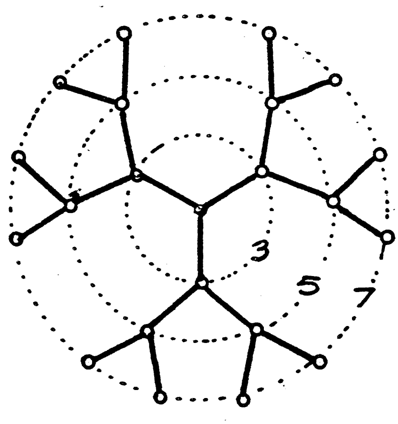

FIGURE 1

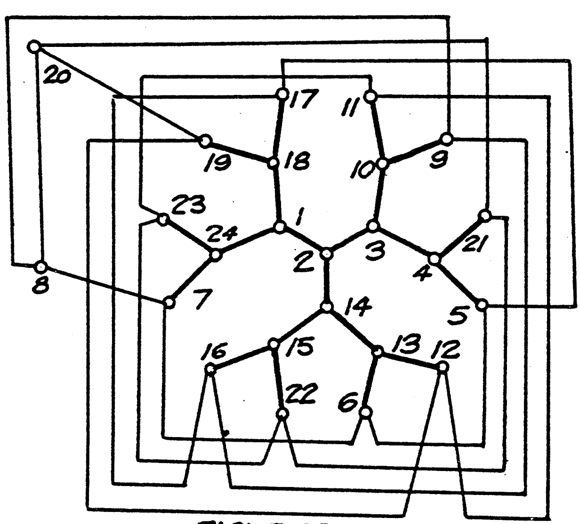

FIGURE 2

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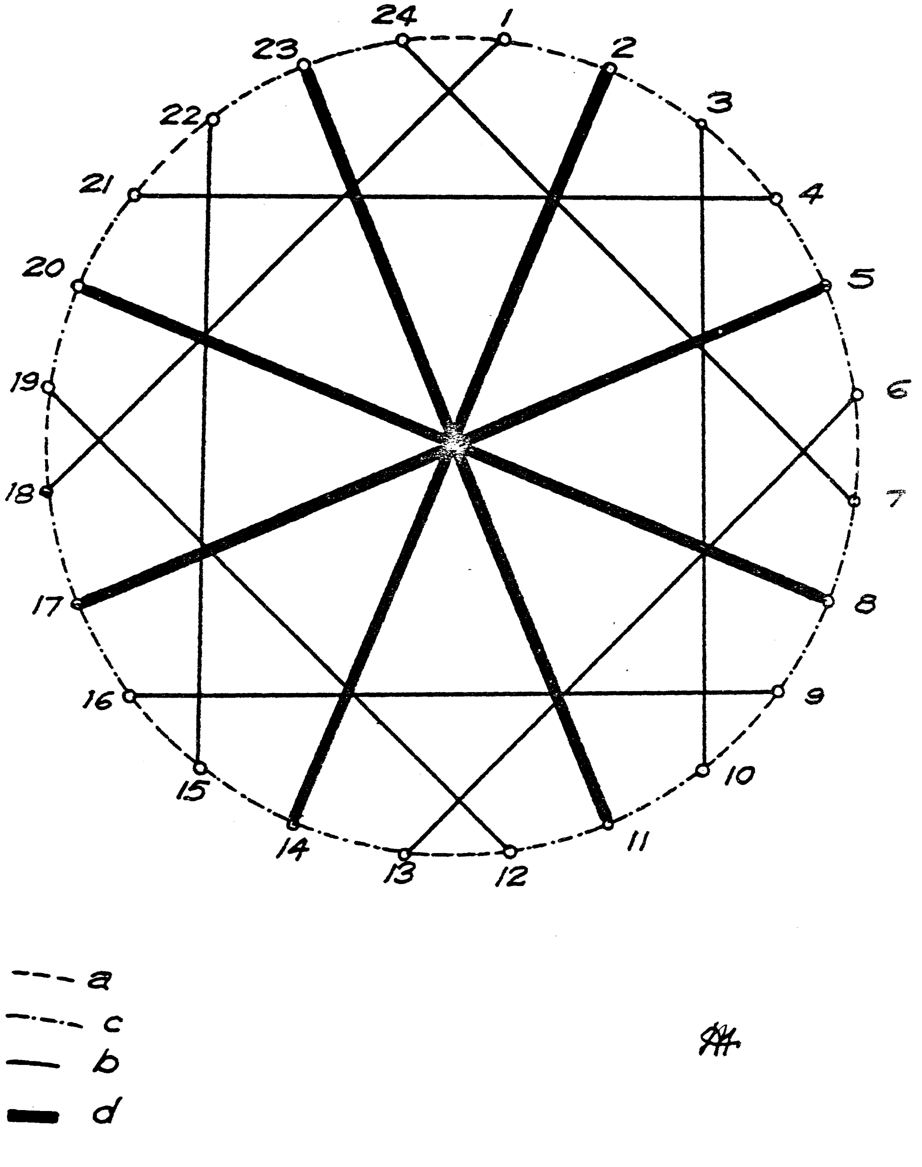
FIGURE 3



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