SOME SERIES AND RECURRENCE RELATIONS FOR MACROBERT'S *E*-FUNCTION

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1. Introductory. Since [3]

$$\Gamma(\alpha)\,\Gamma(\beta)W_{k,\,m}(z) = z^k e^{-\frac{1}{2}z}E(\alpha,\beta::z), \tag{1.1}$$

where $\alpha = \frac{1}{2}-k+m$, $\beta = \frac{1}{2}-k-m$, a result involving $W_{k,\,m}(z)$ can be transformed into a result involving MacRobert's E-function. Further this result can be generalised with the help of the known integrals for E-functions.

The object of this paper is to use this method to obtain some recurrence relations and series for MacRobert's E-functions.

2. Formulae required in the proof. We have [2]

$$W_{k+n,\,m}(z) = (-1)^n\,\Gamma(m+k+n+\tfrac{1}{2})n!\sum_{r=0}^{n}\frac{(-1)^r z^{r/2}\,W_{k+r/2,\,m+r/2}(z)}{(n-r)!\,r!\,\Gamma(m+k+r+\frac{1}{2})}, \tag{2.1}$$

where Re $(\frac{1}{2}-k+m) > 0$, and

$$W_{k-n,\,m}(z) = \frac{(-1)^n\,\Gamma(m+k-n+\frac{1}{2})n!}{\Gamma(m+k+\frac{1}{2})}\sum_{r=0}^{n}\frac{(-1)^r\,z^{r/2}\,W_{k-r/2,\,m+r/2}(z)}{(n-r)!\,r!}, \tag{2.2}$$

where Re $(\frac{1}{2}-k+m) > 0$.

The author [1] has obtained the following recurrence relations for the Whittaker confluent hypergeometric function

$$W_{k,\,m-1}(z)+(\tfrac{1}{2}-k+m)W_{k-1,\,m}(z) = (\tfrac{3}{2}-k-m)W_{k-1,\,m-1}(z)+W_{k,\,m}(z). \tag{2.3}$$

and

$$(m+k-z-\tfrac{1}{2})W_{k,\,m}(z) = \{m^2-(k-\tfrac{1}{2})^2\}W_{k-1,\,m}(z)-z^{\frac{1}{2}}W_{k+\frac{1}{2},\,m-\frac{1}{2}}(z). \tag{2.4}$$

There is a misprint in [1]; in (2.8) read $\Gamma(m+k-n+\frac{1}{2})$ for $\Gamma(m+k+n+\frac{1}{2})$.

3. Series for MacRobert's *E*-function. On using (1.1), (2.1) becomes

$$\frac{z^n}{\Gamma(\alpha-n)}E(\alpha-n,\beta-n::z) = \frac{1}{\Gamma(\alpha)}\sum_{t=0}^{n}\binom{n}{t}z^t E(\alpha,\beta-t::z). \tag{3.1}$$

Now in (3.1) replace α and β by α_1 and α_2, generalise and so obtain

$$\frac{z^n}{\Gamma(\alpha_1-n)}E(p\,;\alpha_r-n:q\,;\rho_s-n:z) = \frac{1}{\Gamma(\alpha_1)}\sum_{t=0}^{n}\binom{n}{t}z^t\,E\left(\begin{matrix}\alpha_1,\alpha_2-t,\,\dots,\alpha_p-t \\ q\,;\rho_s-t\end{matrix}:z\right). \tag{3.2}$$

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Similarly, from (2.2) it can be deduced that

$$[\Gamma(\alpha_1+n)]^{-1} z^{-n} E(p\,;\alpha_r+n : q\,;\rho_s+n : z)$$
$$= \sum_{t=0}^{n} (-1)^t [\Gamma(\alpha_1+t)]^{-1} \binom{n}{t} E\left(\begin{matrix}\alpha_1+t, \alpha_2, \ldots, \alpha_p \\ q\,;\rho_s\end{matrix} : z\right). \quad (3.3)$$

4. Recurrence formulae for MacRobert's *E*-function. On using (1.1), (2.3) becomes

$$(\alpha_1-1)\, E(\alpha_1-1, \alpha_2+1 :: z)+z^{-1}\, E(\alpha_1+1, \alpha_2+1 :: z)$$
$$= z^{-1}\, E(\alpha_1, \alpha_2+2 :: z)+\alpha_2\, E(\alpha_1, \alpha_2 :: z). \quad (4.1)$$

On generalising, this becomes

$$(\alpha_1-1)\, E\left(\begin{matrix}\alpha_1-1, \alpha_2+1, \alpha_3, \ldots, \alpha_p \\ q\,;\rho_s\end{matrix} : z\right)+z^{-1}\, E\left(\begin{matrix}p\,;\alpha_r+1 \\ q\,;\rho_s+1\end{matrix} : z\right)$$
$$= z^{-1}\, E\left(\begin{matrix}\alpha_1, \alpha_2+2, \alpha_3+1, \alpha_4+1, \ldots, \alpha_p+1 \\ q\,;\rho_s+1\end{matrix} : z\right)+\alpha_2\, E(p\,;\alpha_r : q; \rho_s : z). \quad (4.2)$$

Again using (1.1), we get from (2.4)

$$\beta E(\alpha, \beta :: z)+zE(\alpha, \beta :: z)= z^{-1}E(\alpha+1, \beta+1 :: z)+(\alpha-1)zE(\alpha-1, \beta :: z). \quad (4.3)$$

On generalising we obtain

$$\alpha_2\, E(p\,;\alpha_r : q\,;\rho_s : z)+zE\left(\begin{matrix}\alpha_1, \alpha_2, \alpha_3-1, \alpha_4-1, \ldots, \alpha_p-1 \\ q\,;\rho_s-1\end{matrix} : z\right)$$
$$= z^{-1}\, E(p\,;\alpha_r+1 : q\,;\rho_s+1 : z)+(\alpha_1-1)zE\left(\begin{matrix}\alpha_1-1, \alpha_2, \alpha_3-1, \alpha_4-1, \ldots, \alpha_p-1 \\ q\,;\rho_s-1\end{matrix} : z\right). \quad (4.4)$$

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