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A BANACH SPACE IN WHICH A BALL IS CONTAINED IN THE RANGE OF SOME COUNTABLY ADDITIVE MEASURE IS SUPERREFLEXIVE

BY

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ABSTRACT. A nonstandard proof of the fact that a Banach space in which a ball is contained in the range of a countably additive measure is superreflexive is given. The proof is an application of a general method in which we first transfer certain standard objects to the nonstandard hull of a Banach space and then, using the quite well developed theory of nonstandard hulls, derive results about the objects in the original Banach space. It also provides us with an example of the applications of the theory of nonstandard hull valued measures.

1. **Introduction.** The range of a vector measure has displayed many intriguing connections with the geometry of subsets of Banach spaces. In [7] Kaczmarz and Steinhaus presented Banach's elegant proof of the fact that the unit ball of l^2 is the range of a countably additive vector measure. More generally Bregtagnolle, Dacunha-Castelle and Krivine (1966) and Rosenthal (1973) showed that the unit ball of $L^p[0,1]$ and l^p for $2 \leqq p < \infty$ is the range of a countably additive vector measure. But if $1 < p < 2$ the unit ball of $L^p[0,1]$ and of l^p is not the range of a countably additive vector measure (see [2]). Based on these facts, it is natural to ask what conclusions we can draw about X if the unit ball of it is the range of a countably additive vector measure. Let Ω be a nonempty set, Σ a σ-algebra of subsets of Ω, X a Banach space and ν a countably additive X-valued measure on (Ω, Σ). Bartle, Dunford and Schwartz showed that $\nu(\Sigma)$ is relatively weakly compact (see [3]). Thus if $\nu(\Sigma)$ is the unit ball of X or more generally contains a ball in X, then X is reflexive. In[2] $\nu(\Sigma)$ is proved to have the Banach–Saks property, i.e. every sequence in $\nu(\Sigma)$ has a subsequence whose arithmetic means converge in norm. Hence if a ball is contained in $\nu(\Sigma)$, then X has the Banach–Saks property. In this note a stronger result is proved along that line: if $(\nu(\Sigma))^0 \neq \emptyset$, then the Banach space X is superreflexive. Note that if X is superreflexive then X has the Banach–Saks property and therefore is reflexive (see [15]). Nonstandard analysis is applied to the proof. We will see that the nonstandard proof is quite elementary and natural and it also shows how the nonstandard hull valued measures are used to study standard Banach space valued measures. Basic

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definitions can be found in [3] and [5].

2. **The Proof.**

DEFINITION 1. *Let X and Y be Banach spaces. We say that Y is finitely representable in X, if for every $\epsilon > 0$ and for every finite-dimensional subspace $F \subset Y$ there exists a linear transformation T from F into X satisfying*

$$(1-\epsilon)\|y\| \leqq \|Ty\| \leqq (1+\epsilon)\|y\| \text{ for all } y \in F.$$

DEFINITION 2. *A Banach space X is said to be superreflexive if Y finitely representable in X implies that Y is reflexive.*

It is a well known fact that a Banach space is reflexive if and only if every separable subspace of it is reflexive (see [6]). Thus X is superreflexive if and only if every separable Banach space finitely representable in X is reflexive. The following lemma, which can be found in ([3], P. 28), is central to the proof.

LEMMA 3. *Let $\mathcal{A}$ be an algebra of subsets of Ω. Any one of the following three statements about a bounded vector measure F from $(\Omega, \mathcal{A})$ to a Banach space X implies the others.*

(1) *There exists à finitely additive nonnegative real-valued measure μ on $\mathcal{A}$ such that F is μ-continuous (μ is called a control measure of F).*

(2) *F is strongly additive.*

(3) *$F(\mathcal{A})$ is a relatively weakly compact subset of X.*

Now we adopt the usual framework of nonstandard analysis from [5]. The nonstandard model we consider will be $\aleph_1$-saturated. Let ν be the X-valued measure as in the introduction and assume $(\nu(\Sigma))^0 \neq \emptyset$. Let μ be a control measure of ν on (Ω, Σ). We transfer everything to the nonstandard universe; $^*\nu$ and $^*\mu$ are a finitely additive internal $^*X$-valued measure and positive measure on $(^*\Omega, ^*\Sigma)$ respectively. Let $\hat{X}$ be the nonstandardhull of X. The following lemma is a simplified version of a result of Henson and Moore (see section 3 of [4]).

LEMMA 4. *If $\hat{X}$ is reflexive, then X is superreflexive.*

PROOF.Let Y be any separable Banach space finitely representable in X. There is an increasing family $\{Y_n : n = 1, 2, \ldots\}$ of finite dimensional subspaces of Y such that the dimension of Y_n is n and $\bigcup_{n\in\mathbb{N}} Y_n$ is dense in Y. For each n there exists a linear transformation T_n from Y_n into X such that

$$(1-1/n)\|y\| \leqq \|T_n y\| \leqq (1+1/n)\|y\| \text{ for all } y \in Y_n.$$

Consider the star transformations $\{^*T_n\}_{n\in^*\mathbb{N}}$ and $\{^*Y_n\}_{n\in^*\mathbb{N}}$ of $\{T_n\}$ and $\{Y_n\}$. Pick $\omega \in {}^*\mathbb{N}_\infty$. Let

$$(^*Y_\omega)\hat{} = \{[y] \in \hat{Y} : y \in {}^*Y_\omega\}.$$

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Define $\hat{T}_\omega : ({}^*Y_\omega)\hat{} \longrightarrow \hat{X}$ by $\hat{T}_\omega([y]) = ({}^*T_\omega y)\hat{}$. It is clear that $\hat{T}_\omega$ is an isometry of $({}^*Y_\omega)\hat{}$ into $\hat{X}$. Since $\bigcup_{n\in\mathbb{N}} Y_n$ is naturally contained in $({}^*Y_\omega)\hat{}$. The extension to Y of the restriction of $\hat{T}_\omega$ to $\bigcup_{n\in\mathbb{N}} Y_n$ gives the isometry of Y into $\hat{X}$. So Y is reflexive. Therefore X is superreflexive. □

PROOF OF THE MAIN RESULT. For each $A \in {}^*\Sigma$, define

$$\hat{\nu}(A) = ({}^*\nu(A))\hat{}\text{ and } {}^0\mu(A) = ({}^*\mu(A))^0.$$

Then $\hat{\nu}$ is a finitely additive $\hat{X}$-valued measure and ${}^0\mu$ a finitely additive nonnegative real-valued measure respectively. For any given ϵ, there exists δ such that

$$(\forall B \in \Sigma)[\mu(B) < \delta \longrightarrow \|\nu(B)\| < \epsilon/2].$$

Hence $(\forall B \in {}^*\Sigma)[{}^*\mu(B) < \delta \longrightarrow \|{}^*\nu(B)\| < \epsilon/2]$ is true in the nonstandard model by the Transfer Principle. Thus for any $A \in {}^*\Sigma, {}^0\mu(A) < \delta \longrightarrow {}^*\mu(A) < \delta \longrightarrow \|{}^*\nu(A)\| < \epsilon/2 \longrightarrow \|\hat{\nu}(A)\| = (\|{}^*\nu(A)\|)^0 < \epsilon$. So $\hat{\nu}$ is ${}^0\mu$-continuous. By Lemma 3 $\hat{\nu}({}^*\Sigma) = ({}^*(\nu(\Sigma)))\hat{}$ is relatively weakly compact. But $\nu(\Sigma)$ contains a ball in $\hat{X}$. Therefore $\hat{X}$ is reflexive. By Lemma 4 X is superreflexive. □

3. **Remarks.**

1. According to a correspondence with Professor Joseph Diestel, the main result of this note can be proved in the standard framework of functional analysis. He presented the author a proof which is based on the following deep result of Rosenthal (see [12]). If $1 \leqq p < 2$, then every subspaces of L^p either contains a complemented isomorph of l^p, or imbeds in (is linearly homeomorphic to a subspace of) $L^{p'}$ for some $p < p' < \infty$. Thus every reflexive subspace of L^1 imbeds in L^p for some $1 < p < \infty$ since Pelczynski has shown that infinite dimensional complemented subspaces of l^1 are isomorphic to l^1 (see [3], p. 114). Here we give the key points of his proof. Let μ and ν be the measures as in Section 2. Define operator T_ν from $L^\infty(\mu)$ to X by $T_\nu g = \int_\Omega g\, d\nu$ for all $g \in L^\infty(\mu)$. Then T_ν is surjective. Consider the dual operator T_ν^* from X^* (dual space of X) to $L^\infty(\mu)^*$. It is clear that T_ν^* is injective. Since X^* is reflexive, T_ν^* is an isomorphism of X^* onto a reflexive subspace of $L^\infty(\mu)^*$. We can prove that $T_\nu^*(X^*)$ is a subset of $L^1(\mu)$. Then by Rosenthal's result we can get the superreflexivity of X^*. Then X is superreflexive.

2. By comparing the two proofs, we see that the nonstandard proof is elementary. On the contrary, the other proof needs the main structural result of the paper [12] which is 30 pages long.

3. If we assume that ν is strongly additive and the closed convex hull of $\nu(\Sigma)$ contains a ball in X, X is still superreflexive.

4. In [8], [11], [13] and [14], nonstandard vector measure theory is studied; and some applications are given. The techniques used there may have some further applications to a study of vector measures.

5. The author is indebted to the anonymous referee for his suggestions. He pointed out that the ball of a Banach space is contained in the range of some vector measure if and only if the Banach space is a quotient of L^p for some $(2 \leqq p \leqq \infty)$. Here is the proof.

We know that the unit ball of any $L^p(2 \leqq p \leqq \infty)$ is the range of a vector measure. It follows easily that

(1) The unit ball of any quotient of $L^p(2 \leqq p \leqq \infty)$ is contained in the range of a vector measure.

(2) Any Banach space whose ball is contained in the range of a vector measure is actually a quotient of L^p for some $(2 \leqq p \leqq \infty)$.

STEP 1. As Diestel remarked the hypothesis implies that X is the quotient of $L^\infty(\mu)$ for some μ. Moreover this quotient $T_\nu : L^\infty(\mu) \longrightarrow X$ is weak* to weak continuous. Hence X is reflexive and $T_\nu^*(X^*)$ is a reflexive closed subspace of $(L^\infty)^*$, hence superreflexive by the mentioned result of Rosenthal and $T_\nu^*(X^*)$ embeds in $L^{q'}$ for some $1 < q' < \infty$. It follows that X has a nontrivial cotype $p' \geqq 2$.

STEP 2. (i) If $p' = 2$ then T_ν is a 2-summing operator in view of a well known extension of a theorem of Grothendiek. See [Pisier, Theorem 4.1]. (ii) If $p' > 2$ then, T_ν is $p' + \epsilon$-summing (for any $\epsilon > 0$) in view of a result of Maurey [9].

STEP 3. In both cases, one can conclude by a classical result of Pietsch stating that the p-summing $(2 \leqq p < \infty)$ operator $T_\nu : L^\infty \longrightarrow X$ factors through L^p [Pisier, Corollary 1.5]. Hence X is a quotient of L^p.

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