THE NORMAL VIBRATIONS OF A RIGID SPHERICAL PUNCH ON THE SURFACE OF AN ELASTIC HALF-SPACE

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Summary. A rigid spherical punch vibrates normally on the surface of a semi-infinite isotropic elastic half-space. The essential novelty of this problem, which is treated within the context of classical elasticity, is that of a changing boundary; the radius of the circle of contact on the free surface varies with time. The geometrical co-ordinates are modified to yield a boundary value problem with fixed boundaries. However the governing differential equations become more complicated. These equations are solved by a perturbation procedure for the case where the contact radius $a(t)$ is of the form

$$a(t) = a_0(1+\eta(t))$$

where a_0 is constant and $|\eta(t)| \ll 1$. Finally the normal stress and the total load under the punch are evaluated in the form of series which are valid for sufficiently slowly varying $\eta(t)$.

1. Introduction

There seem to be few examples in the literature of elastodynamic problems involving mixed boundary conditions where the boundaries themselves are time dependent. The present paper is concerned with one of the simplest problems of this type; the problem is that of a rigid spherical punch oscillating normally on the surface of an isotropic elastic half-space. Here the contact radius $a(t)$ divides the surface $Z = 0$ into two regions in one of which the normal displacement is specified, while for the second region the normal component of stress vanishes.

For problems of this type the boundary conditions specify a partial stress history and a partial displacement history for some regions of the bounding surface. In the circumstances it is not possible to construct integral transforms of the boundary conditions so that direct application of the principal tool of elastodynamics is no longer possible. Similar difficulties occur in quasistatic viscoelastic problems (e.g., see (**1**), (**2**)) and to date there is no systematic method of attacking such problems.

The method of solution adopted in the present paper entails changing the independent space variables from the axisymmetric coordinates, R and Z, to co-ordinates

$$r = R/a(t), \quad z = Z/a(t) \tag{1.1}$$

so that boundary conditions imposed on $Z = 0$ (i.e., $z = 0$) now relate to fixed regions $0 \leqq r \leqq 1$ and $r > 1$. Thus the boundary conditions are simplified at the expense of complicating the governing differential equations. In the present paper, the resulting differential equations are solved by a perturbation procedure for the problem where $a(t)$ is of the form

$$a(t) = a_0(1+\eta(t)) \tag{1.2}$$

where $|\eta| \ll 1$ and where a_0 is a constant.

The derivation of the basic equations and perturbation procedure follow immediately in § 2; the solution of the first order perturbation scheme is given in § 3.

2. Basic equations

Defining $u(R, Z, t)$ and $w(R, Z, t)$ to be respectively the radial and axial components of displacement, the axisymmetric equations of elastodynamics are

$$\frac{\partial^2 u}{\partial R^2} + \frac{1}{R}\frac{\partial u}{\partial R} - \frac{u}{R^2} + \frac{(1-2\nu)}{2(1-\nu)}\frac{\partial^2 u}{\partial Z^2} + \frac{1}{2(1-\nu)}\frac{\partial^2 w}{\partial R \partial Z} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{2.1}$$

$$\frac{1-2\nu}{2(1-\nu)}\left(\frac{\partial^2 w}{\partial R^2} + \frac{1}{R}\frac{\partial w}{\partial R}\right) + \frac{\partial^2 w}{\partial Z^2} + \frac{1}{2(1-\nu)}\left(\frac{\partial^2 u}{\partial R \partial Z} + \frac{1}{R}\frac{\partial u}{\partial Z}\right) = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} \tag{2.2}$$

where ν denotes Poisson's ratio and where c is the dilatational wave velocity

$$c = \{(\lambda+2\mu)/\rho\}^{\frac{1}{2}}$$

in which expression λ and μ are the Lamé constants and ρ is the density.

For the problem of an axisymmetric punch of profile

$$Z = -f(R) \tag{2.3}$$

where f is a specified function, we require the solution of equations (2.1), (2.2) subject to the conditions

$$Z = 0. \quad \tau_{rz} = \mu\left(\frac{\partial u}{\partial Z} + \frac{\partial w}{\partial R}\right) = 0. \tag{2.4}$$

$$Z = 0,\ R < a(t). \quad w = Q(t) - f(R) \tag{2.5}$$

$$Z = 0,\ R > a(t). \quad \sigma_{zz} = \frac{2\mu}{(1-2\nu)}\left[\nu\left(\frac{\partial u}{\partial R} + \frac{u}{R}\right) + (1-\nu)\frac{\partial w}{\partial Z}\right] = 0. \tag{2.6}$$

In writing down (2.4) we assume a smooth punch so that τ_{rz} vanishes for all R on $Z = 0$. In (2.5) we assume without loss of generality, that $f(0) = 0$ so that $Q(t)$ measures directly the penetration of the tip of the punch into the half space.

From physical considerations the two functions $a(t)$ and $Q(t)$ are related, as in the corresponding static problem, by the condition that the normal stress distribution under the punch involves no physically unacceptable singularities.

Because equations (2.5) and (2.6) refer to a moving boundary $R = a(t)$, it proves convenient to utilise a new dimensionless radial co-ordinate

$$r = R/a(t). \tag{2.7}$$

To preserve homogeneity in the differential equations we make a similar change of variable for the axial co-ordinate

$$z = Z/a(t) \tag{2.8}$$

and at the same time replace t by a dimensionless time

$$\tau = ct/a_0. \tag{2.9}$$

Here a_0 is a quantity of dimension length, conveniently chosen to be the constant length parameter a_0 appearing in (1.2).

In terms of the modified independent variables, equations (2.1) and (2.2) become

$$N(u, w) = (a/a_0)^2\ddot{u} - 2(a\dot{a}/a_0^2)L\dot{u} + (\dot{a}/a_0)^2L(L+1)u - (a\ddot{a}/a_0^2)Lu \tag{2.10}$$

$$M(u, w) = (a/a_0)^2\ddot{w} - 2(a\dot{a}/a_0^2)L\dot{w} + (\dot{a}/a_0)^2L(L+1)w - (a\ddot{a}/a_0^2)Lw \tag{2.11}$$

where N and M are operators associated with the elastostatic problem

$$N(u, w) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{(1-2\nu)}{2(1-\nu)}\frac{\partial^2 u}{\partial z^2} + \frac{1}{2(1-\nu)}\frac{\partial^2 w}{\partial r \partial z} \tag{2.12}$$

$$M(u, w) = \frac{1-2\nu}{2(1-\nu)}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2} + \frac{1}{2(1-\nu)}\left(\frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r}\frac{\partial u}{\partial z}\right) \tag{2.13}$$

and where the fluxion dots denote differentiation with respect to τ. Finally L is the operator.

$$L = r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z}. \tag{2.14}$$

The corresponding forms of the boundary conditions (2.4), (2.5) and (2.6) are respectively

$$z = 0. \quad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = 0 \tag{2.15}$$

$$z = 0,\ r < 1. \quad w = q(\tau) - f(ra(\tau)) \tag{2.16}$$

$$z = 0,\ r > 1. \quad \nu\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + (1-\nu)\frac{\partial w}{\partial z} = 0 \tag{2.17}$$

where

$$q(\tau) \equiv Q(t).$$

So far, in the context of the theory of linear elasticity, no approximations have been introduced. However, the basic equations (2.10) and (2.11) are too complicated to solve in the absence of simplifying approximations. Various approximations are suggested by the forms of (2.10) and (2.11). For example, for sufficiently slow and slowly accelerated motion we could assume equations (2.10) and (2.11) to be dominated by the left sides, treating the right sides in a perturbation approximation. The problem considered here is slightly different and is defined by

$$a(\tau) = a_0(1+\eta(\tau)) \tag{2.18}$$

where $|\eta| \ll 1$ and a_0 is a fixed constant; thus we employ an amplitude perturbation scheme rather than one based on a low frequency approximation.

To derive the perturbation scheme we write

$$a = a_0[1+\varepsilon\eta(\tau)] \tag{2.19}$$

$$q(\tau) = q_0[1+\varepsilon\theta_1(\tau)+\varepsilon^2\theta_2(\tau)+\dots] \tag{2.20}$$

$$u = u_0+\varepsilon u_1+\varepsilon^2 u_2+\dots \tag{2.21}$$

$$w = w_0+\varepsilon w_1+\varepsilon^2 w_2+\dots. \tag{2.22}$$

In these expansions ε is an ordering parameter, introduced to identify terms of the same order, and subsequently equated to unity. The leading terms u_0, w_0 in (2.21) and (2.22) are taken to be the (known) elastostatic solutions for contact radius a_0 and penetration depth q_0; in this static solution a_0 and q_0 are related in a known fashion

$$q_0 = \gamma(a_0). \tag{2.23}$$

For example, for the case of a spherical (or more precisely "parabolic") punch of radius B for which

$$f(R) = -R^2/2B \tag{2.24}$$

we have [e.g. see (3) or (4)]

$$\gamma(a_0) = a_0^2/B. \tag{2.25}$$

Substituting from (2.19), (2.21) and (2.22) into (2.10) and (2.11) and picking out the coefficients of ε^0, ε^1, ε^2 etc., leads to the perturbation scheme

$$N(u_0, w_0) = 0, \qquad M(u_0, w_0) = 0 \tag{2.26}$$

$$N(u_1, w_1)-\ddot{u}_1 = -\ddot{\eta}Lu_0, \quad M(u_1, w_1)-\ddot{w}_1 = -\ddot{\eta}Lw_0 \tag{2.27}$$

$$N(u_2, w_2)-\ddot{u}_2 = 2\eta\ddot{u}_1-2\dot{\eta}L\dot{u}_1-\ddot{\eta}Lu_1+\dot{\eta}^2L(L+1)u_0-\eta\ddot{\eta}Lu_0 \tag{2.28a}$$

$$M(u_2, w_2)-\ddot{w}_2 = 2\eta\ddot{w}_2-2\dot{\eta}L\dot{w}_1-\ddot{\eta}Lw_1+\dot{\eta}^2L(L+1)w_0-\eta\ddot{\eta}Lw_0 \tag{2.28b}$$

etc.

Thus, as hypothesized, u_0, w_0 are solutions of the elastostatic equations and, according to the first terms in (2.19) and (2.20), solve the static problem for contact radius a_0 and penetration depth q_0. The first and second perturbation equations are respectively (2.27) and (2.28). Clearly these equations, and those of higher order, are inhomogeneous wave equations in which the inhomogeneous terms on the right-hand sides are of lower order than the terms

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on the left-hand sides. Thus in principle the perturbation equations may be solved successively. In practise only the first order equations (2.27) are readily solvable and in the present paper we confine discussion to these equations.

In any event, it is doubtful whether we would be justified in considering equations (2.28) as valid second order approximations, since, in deriving these equations, we have already neglected terms comparable with u_2, w_2 in imposing boundary conditions on the unperturbed surface $z = 0$; it is possible that terms of comparable magnitude would also arise from a finite elastic calculation.

Equations (2.27) are solved in § 3 below. We note here properties of the static solution u_0, w_0 for the case of the spherical punch defined by (2.24). The properties in question, of importance subsequently, are the values of $\sigma_{zz}^{(0)}$, $\tau_{rz}^{(0)}$ and w_0 on the surface $z = 0$

$$\sigma_{zz}^{(0)} = \begin{cases} -\dfrac{4\mu a_0^2}{\pi(1-\nu)Ba}(1-r^2)^{\frac{1}{2}} & (r<1) \\ 0 & (r>1) \end{cases} \tag{2.29}$$

$$\tau_{rz}^{(0)} = 0 \qquad (r>1) \tag{2.30}$$

$$w_0 = \frac{a_0^2}{B}(1-\tfrac{1}{2}r^2) \qquad (0<r \leqq 1) \tag{2.31}$$

Of these results (2.30) and (2.31) follow directly from the imposed boundary conditions together with equation (2.25). The remaining result (2.29) is well known in the Hertz contact theory (e.g. see (**4**)), although the appearance of both a and a_0 may seem puzzling. Essentially the appearance of both a and a_0 in (2.29) is because of the assumption that (u_0, w_0) provide an appropriate zeroth order approximation to the displacement field for all values of a. Formally the zeroth order stress field is linear in a^{-1} (as indicated by (2.29)) because in computing strains from u_0, w_0 we make use of equations (1.1); for example

$$\varepsilon_{zz}^{(0)} = \frac{\partial w_0}{\partial Z} = \frac{1}{a}\frac{\partial w_0}{\partial z}$$

and similarly for the other strain components.

The choice (u_0, w_0) for the zeroth order displacement field is not unique. Alternative possibilities [e.g. $(a/a_0)^2(u_0, w_0)$] differ from (u_0, w_0) by terms of order $\eta(u_0, w_0)$; of course a different choice for the zeroth order approximations involves modified perturbation equations. Ultimately the sum of the zeroth and first order terms of the solution is necessarily independent [to order $\eta(u_0, w_0)$] of the choice of zeroth order approximation. The present choice for the zeroth order approximation involves a time independent displacement field and this leads to the simplest first order perturbation scheme, i.e. equations (2.27).

3. Solution of the inhomogeneous wave equations (2.27)

Particular integrals of equations (2.27) are very easily found by virtue of certain properties possessed by the operator L. The properties are that if u, w are solutions of an elastostatic half-space problem for which τ_{rz} vanishes on $z = 0$ then the displacement components

$$\hat{u} = Lu, \quad \hat{w} = Lw$$

are also solutions of a half-space problem for which τ_{rz} vanishes on $z = 0$. These theorems are proved in the Appendix.

In particular

$$\hat{u} = Lu_0, \quad \hat{w} = Lw_0$$

satisfy

$$N(\hat{u}, \hat{w}) = 0, \quad M(\hat{u}, \hat{w}) = 0$$

so that immediate particular integrals of equations (2.27) are

$$u_1 = \eta(\tau)Lu_0, \quad w_1 = \eta(\tau)Lw_0.$$

It follows that complete solutions of equations (2.27) may be written in the form

$$u_1 = U+\eta(\tau)Lu_0, \quad w_1 = W+\eta(\tau)Lw_0 \tag{3.1}$$

where (U, W) satisfy the usual homogeneous wave equations of axisymmetric elastodynamics

$$N(U, W) = \ddot{U}, \quad M(U, W) = \ddot{W}. \tag{3.2}$$

To find the boundary conditions satisfied by (U, W) we compare the known behaviour of (u_0, w_0) and (Lu_0, Lw_0) with the imposed conditions (2.15), (2.16) and (2.17). From (2.15), the known behaviour of $\tau_{rz}^{(0)}$ (equation (2.30)) and the properties of the solution Lu_0, Lw_0 discussed above, we have from $\tau_{rz} = 0$ on $z = 0$

$$\frac{\partial U}{\partial z} + \frac{\partial W}{\partial r} = 0, \ z = 0, \ (\text{all } r). \tag{3.3}$$

Similarly we derive from the condition that σ_{zz} vanishes on $z = 0$, $r>1$ (i.e. equation (2.17))

$$\nu\left(\frac{\partial U}{\partial r} + \frac{U}{r}\right) +(1-\nu)\frac{\partial W}{\partial z} = 0. \tag{3.4}$$

Finally, using (2.19), (2.20) and (2.24) the boundary condition (2.16) for the normal displacement under the punch becomes for $r<1$, $z = 0$

$$w_0+\varepsilon w_1+\ldots = q_0[1+\varepsilon\theta_1(\tau)+\ldots]-\frac{a_0^2r^2}{2B}(1+2\varepsilon\eta(\tau)+\ldots).$$

Now by definition on $z = 0$, $r<1$, $w_0 = q_0-\frac{1}{2}(a_0^2/B)r^2$ so that w_1 satisfies

$$w_1 = q_0\theta_1-a_0^2\eta r^2/B, \quad 0<r<1, \ z=0. \tag{3.5}$$

However on $z = 0$, the displacement component $\eta L(w_0)$ takes the value

$$\eta[L(w_0)]_{z=0} = \eta r \frac{\partial}{\partial r}\{q_0 - \tfrac{1}{2}a_0^2 r^2/B\}$$

$$= -a_0^2 \eta r^2/B. \tag{3.6}$$

Comparing the second of (3.1) with equations (3.5) and (3.6) leads finally to the boundary condition

$$W = q_0 \theta_1(\tau), \quad 0<r<1,\ z = 0. \tag{3.7}$$

The problem defined by equations (3.2) and the boundary conditions (3.3), (3.4) and (3.7) is that of a smooth flat circular punch indenting dynamically a half-space; the radius of the punch is unity and the penetration history $q_0\theta_1(\tau)$. This problem has been considered recently by Robertson (**5**) whose results are used in what follows.

So far it has been possible to proceed without specifying either $\eta(\tau)$ or equivalently $\theta_1(\tau)$. The relation between these quantities is derived from consideration of the normal stress σ_{zz} on $z = 0$. From (2.29) and equation (A.13) of the appendix, the normal stress on $z = 0$ is given by

$$\sigma_{zz} = -\frac{4\mu a_0^2}{\pi(1-\nu)Ba}\{(1+2\eta)(1-r^2)^{\frac{1}{2}} - \eta(\tau)(1-r^2)^{-\frac{1}{2}}\} + \Sigma_{zz} \tag{3.8}$$

where

$$\Sigma_{zz} = \frac{2\mu}{(1-2\nu)a}\left(\nu\left(\frac{\partial U}{\partial r} + \frac{U}{r}\right) + (1-\nu)\frac{\partial W}{\partial z}\right)_{z=0} \tag{3.9}$$

is the stress distribution associated with the flat punch problem. Physically we expect σ_{zz} to be finite over $0 \leqq r \leqq 1$ for a spherical punch and this is only possible if the singularity of the type $(1-r^2)^{-\frac{1}{2}}$ given explicitly in (3.8) is cancelled by an identical singularity appearing in Σ_{zz}. Such a singularity is to be anticipated in Σ_{zz}, at least in the case of slowly varying $\theta(\tau)$, for then the flat punch problem is of a quasi-stationary nature; the singularity of the static flat circular punch problem is precisely of the required type and the choice of amplitude of θ_1 required to eradicate the infinite singularities in σ_{zz} then provides the relevant relation between θ and η. The argument for the dynamic case is of a similar nature and is given in the first instance for the particular case of periodic $\theta_1(\tau)$

$$\text{viz., } \theta_1(\tau) = \delta e^{-i\omega\tau} \tag{3.10}$$

where $\delta \ll 1$ is a constant amplitude and ω the dimensionless circular frequency. From Robertson's analysis we have for the corresponding Σ_{zz}

$$\Sigma_{zz} = \frac{2\mu q_0 \delta}{\pi(1-\nu)a} e^{-i\omega\tau}[-(1-r^2)^{-\frac{1}{2}}P(\omega) + (1-r^2)^{\frac{1}{2}}\{a_{12}\omega^2 + ia_{13}\omega^3 + a_{14}\omega^4 + \dots\}$$
$$+ (1-r^2)^{\frac{3}{2}}\{a_{34}\omega^4 + \dots\} + O\{(1-r^2)^{\frac{5}{2}}\omega^6\}]. \tag{3.11}$$

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In this expression $P(\omega)$ is the series

$$P(\omega) = 1 - i\frac{I_1}{\pi}\omega + \left(\frac{I_2}{\pi} - \frac{I_1^2}{\pi^2}\right)\omega^2 + i\left(\frac{2I_3}{3\pi} - \frac{5I_1I_2}{3\pi^2} + \frac{I_1^3}{\pi^3}\right)\omega^3$$
$$+\left(-\frac{I_4}{3\pi} + \frac{I_1I_3}{\pi^2} + \frac{2I_2^2}{3\pi^2} - \frac{7I_1^2I_2}{3\pi^3} + \frac{I_1^4}{\pi^4}\right)\omega^4 + \ldots \tag{3.12}$$

while a_{12}, a_{13}, a_{14} and a_{34} are given by

$$a_{12} = \frac{I_2}{\pi}, \quad a_{13} = \frac{I_3}{\pi} - \frac{I_1I_2}{\pi^2}$$
$$a_{14} = -\frac{2I_4}{3\pi} + \frac{I_1I_3}{\pi^2} + \frac{2I_2^2}{3\pi^2} - \frac{I_1^2I_2}{\pi^3}$$
$$a_{34} = \frac{I_4}{9\pi} - \frac{I_2^2}{9\pi^2}.$$

Finally the I_n are certain definite integrals which depend solely on Poisson's ratio; for $\nu = \frac{1}{3}$ Robertson finds †

$$I_1 = 2{\cdot}45791, \quad I_2 = 2{\cdot}28989, \quad I_3 = 2{\cdot}26230, \quad I_4 = 2{\cdot}29960.$$

Comparing (3.8) and (3.11) we see that in order to cancel the relevant singularities it is necessary to choose $\eta(\tau)$ periodic and of the form

$$\eta(\tau) = \tfrac{1}{2}\delta P(\omega)e^{-i\omega\tau}. \tag{3.13}$$

In deriving (3.13) we have used the result $q_0 = a_0^2/B$ given by the static problem. In general, for non-periodic motion, equations (3.10) and (3.13) provide a relation connecting the Fourier transforms of $\eta(\tau)$ and $\theta_1(\tau)$. For sufficiently slow variation of either $\eta(\tau)$ or $\theta_1(\tau)$ the series for $P(\omega)$ will be dominated by the first few terms of (3.12); in this approximation we find

$$\eta(\tau) = \tfrac{1}{2}\left\{\theta_1(\tau) + \frac{I_1}{\pi}\dot\theta_1(\tau) - \left(\frac{I_2}{\pi} - \frac{I_1^2}{\pi^2}\right)\ddot\theta_1(\tau) + \left(\frac{2I_3}{3\pi} - \frac{5I_1I_2}{\pi^2} + \frac{I_1^3}{\pi^3}\right)\dddot\theta_1(\tau)\right.$$
$$\left. + \left(-\frac{I_4}{3\pi} + \frac{I_1I_3}{\pi^2} + \frac{2I_2^2}{3\pi^2} - \frac{7I_1^2I_2}{3\pi^3} + \frac{I_1^4}{\pi^4}\right)\ddddot\theta_1(\tau) + \ldots\right\}$$
$$= \tfrac{1}{2}\{\theta_1(\tau) + 0{\cdot}7824\dot\theta_1(\tau) - 0{\cdot}1168\ddot\theta_1(\tau) + 0{\cdot}0085\dddot\theta_1(\tau) + 0{\cdot}0072\ddddot\theta_1(\tau) + \ldots\} \tag{3.14}$$

when $\nu = \frac{1}{3}$, where now $\eta(\tau)$ and $\theta_1(\tau)$ are both real and we have assumed the validity of differentiating the Fourier integrals.

In the same approximation, and when $\nu = \frac{1}{3}$, the stress distribution under the punch is given by

$$\sigma_{zz} = -\frac{6\mu a_0}{\pi B}[(1-r^2)^{\frac{1}{2}}\{1 + \theta_1(\tau) + 0{\cdot}7824\dot\theta_1(\tau) - 0{\cdot}2477\ddot\theta_1(\tau) - 0{\cdot}0664\dddot\theta_1(\tau)$$
$$-0{\cdot}0010\ddddot\theta_1(\tau) + \ldots\} + (1-r^2)^{\frac{3}{2}}\{0{\cdot}0112\ddddot\theta_1(\tau) + \ldots]$$

† In Robertson's paper the first eight I_n are tabulated for nine values of ν.

and the total load is given by

$$L = -\int_0^1 2\pi a^2 \sigma_{zz} r dr$$

$$= \frac{4\mu a_0^3}{B}[1+\theta_1(\tau)+0{\cdot}7824\dot{\theta}_1(\tau)+0{\cdot}2477\ddot{\theta}_1(\tau)-0{\cdot}0664\dddot{\theta}_1(\tau)-0{\cdot}0077\ddddot{\theta}_1(\tau)+\ldots].$$

These formulae are adequate for sufficiently slow variations in $\theta_1(\tau)$. Sufficiently rapid variations in $\theta_1(\tau)$ render the above results invalid; in these circumstances there is no explicit form for the solution of the dynamic flat punch problem and the present problem could be solved only by numerical procedures. In practice the formulae given here are satisfactory for most purposes, even for example in the extreme case of the generation of high frequency ultrasonic waves at 100 Kc/s frequency across a contact surface of 1 mm radius.

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Appendix

Properties of the operator

$$L = r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z}.$$

This Appendix is concerned with the proposition that if u, w are elastostatic components of an axisymmetric displacement field for which $\tau_{rz} = 0$ on $z = 0$ then the displacement field

$$\hat{u} = Lu, \quad \hat{w} = Lw$$

also satisfies the elastostatic equations and yields $\tau_{rz} = 0$ on $z = 0$.

It is well known that axisymmetric half-space problems in elastostatics with $\tau_{rz} = 0$ on $z = 0$ are formally solvable in terms of a single harmonic function ϕ which satisfies Laplace's equation. We adopt the representation given by Galin (4)

$$u = (1-2\nu)\int_z^\infty \frac{\partial \phi}{\partial r} dz - z\frac{\partial \phi}{\partial z} \tag{A1}$$

$$w = 2(1-\nu)\phi - z\frac{\partial \phi}{\partial z} \tag{A2}$$

where

$$\nabla^2 \phi \equiv \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0. \tag{A3}$$

Straightforward calculations yield

$$\sigma_{zz} = 2\mu\left(\frac{\partial\phi}{\partial z} - z\frac{\partial^2\phi}{\partial z^2}\right) \tag{A4}$$

$$\tau_{rz} = -2\mu z\frac{\partial^2\phi}{\partial r\partial z} \tag{A5}$$

so that the result $(\tau_{rz})_{z=0} = 0$ is immediately verified. Consider now the function

$$\psi = L\phi. \tag{A6}$$

It is easily shown that if ϕ is harmonic then ψ is also harmonic, so that the substitution of ψ for ϕ in equations $(A1)$ and $(A2)$ yields an elastostatic displacement field for which $\tau_{rz} = 0$ on $z = 0$. We now show further that the displacement Lu derives from formula $(A1)$ in which ϕ is replaced by $L\phi$.

Operating on both sides of $(A1)$ with $L = r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z}$ we have

$$Lu = (1-2\nu)\int_z^\infty r\frac{\partial^2\phi}{\partial r^2}dz - (1-2\nu)z\frac{\partial\phi}{\partial r} - z\frac{\partial}{\partial z}(L\phi) \tag{A7}$$

where in the last term we have used the result that L commutes with $z\frac{\partial}{\partial z}$.

Consider now

$$\int_z^\infty \frac{\partial}{\partial r}(L\phi)dz \equiv \int_z^\infty\left[r\frac{\partial^2\phi}{\partial r^2} + \frac{\partial\phi}{\partial r} + z\frac{\partial^2\phi}{\partial r\partial z}\right]dz. \tag{A8}$$

Integrating by parts the last term in the integral on the right of $(A8)$ leads to

$$\int_z^\infty z\frac{\partial^2\phi}{\partial r\partial z}dz = \left[z\frac{\partial\phi}{\partial r}\right]_z^\infty - \int_z^\infty \frac{\partial\phi}{\partial r}dz$$

so that $(A8)$ becomes

$$\int_z^\infty \frac{\partial}{\partial r}(L\phi)dz \equiv \int_z^\infty r\frac{\partial^2\phi}{\partial r^2}dz - z\frac{\partial\phi}{\partial r}, \tag{A9}$$

on assuming that ϕ and its derivatives vanish sufficiently rapidly as $z\to\infty$. Comparing $(A7)$, $(A8)$ and $(A9)$ now yields the result

$$Lu = (1-2\nu)\int_z^\infty \frac{\partial}{\partial r}(L\phi)dz - z\frac{\partial}{\partial z}(L\phi) \tag{A10}$$

which is obtainable by the substitution $u\to Lu$, $\phi\to L\phi$ in $(A1)$. A similar result for $(A2)$ follows immediately since

$$Lw = 2(1-\nu)L\phi - L\left(z\frac{\partial\phi}{\partial z}\right)$$

$$= 2(1-\nu)L\phi - z\frac{\partial}{\partial z}(L\phi)$$

so that Lu, Lw form an admissible elastostatic displacement field.

Given explicit formulae for w_0 and $\sigma_{zz}^{(0)}$ on $z = 0$ the above theorems are useful for evaluating properties of the solution Lu_0, Lw_0 on $z = 0$. For the spherical punch we have from equations (2.29) and (2.31)

$$w_0 = \frac{a_0^2}{B}(1-\tfrac{1}{2}r^2), \qquad z=0, \quad r<1 \tag{A11}$$

$$\sigma_{zz}^{(0)} = -\frac{4\mu a_0^2}{\pi(1-\nu)Ba}(1-r^2)^{\frac{1}{2}}, \quad z=0, \quad r<1. \tag{A12}$$

The result

$$(Lw_0)_{z=0} = -\frac{a_0^2 r^2}{B}$$

follows immediately from (A11). To find the surface stress $\sigma_{zz}^{(1)}$ due to the displacement field Lu_0, Lw_0 we make use of the fact that the associated potential function is $L\phi$ so that from (A4) at $z = 0$

$$\begin{aligned}
\sigma_{zz}^{(1)} &= 2\mu\left(\frac{\partial}{\partial z}(L\phi)\right)_{z=0} \\
&= 2\mu\left(\frac{\partial\phi}{\partial z} + r\frac{\partial^2\phi}{\partial r\partial z}\right)_{z=0} \\
&= \left(1 + r\frac{\partial}{\partial r}\right)(\sigma_{zz}^{(0)})_{z=0} \\
&= -\frac{4\mu a_0^2}{\pi(1-\nu)Ba}\{2(1-r^2)^{\frac{1}{2}} - (1-r^2)^{-\frac{1}{2}}\}
\end{aligned} \tag{A13}$$

This result together with (A12) is used to obtain equation (3.8) of the text.

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