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CORRIGENDUM

Rayleigh–Taylor problem for a liquid–liquid phase interface

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We thank Konstantin I. Ilin for discovering an oversight in our treatment of the inviscid Rayleigh–Taylor problem for a liquid–liquid phase interface. This oversight occurs in the case $J \neq 0$ for which mass transport is present in the base state. In this case, the ordinary differential equation (3.6) of the paper has characteristic roots $\pm k$ and $\varrho^{\pm} L\omega/J_0|At|T$. We discarded the latter root on the incorrect grounds that it is of 'indeterminate sign.' This root actually gives a decaying (at infinity) solution on one or other side of the interface depending on the sign of the real part of ω/J_0. If the root $\varrho^{\pm} L\omega/J_0|At|T$ is not discarded, then another interface condition is needed. We show below that accounting properly for the existence of this root does not alter the stability results presented in the paper.

Equation (1.7) of the paper assumes that the kinematical condition (1.11) holds. Indeed, without recourse to (1.11), the interfacial linear momentum balance should read

$$[\![\boldsymbol{S}]\!]\boldsymbol{n} - [\![p_e]\!]\boldsymbol{n} + J[\![\boldsymbol{u}]\!] = -\gamma K\boldsymbol{n} - \mathrm{div}_{\mathscr{S}}\mathbb{S}. \tag{1}$$

Since $\boldsymbol{n}$ and V are interfacial fields, it follows from the interfacial mass balance (1.6) that

$$[\![\boldsymbol{u}]\!] = [\![\boldsymbol{u} - V\boldsymbol{n}]\!] = [\![\boldsymbol{u}\cdot\boldsymbol{n} - V]\!]\boldsymbol{n} + \mathbb{P}[\![\boldsymbol{u} - V\boldsymbol{n}]\!] = -J[\![1/\varrho]\!]\boldsymbol{n} + \mathbb{P}[\![\boldsymbol{u}]\!] \tag{2}$$

and (1) reduces to (1.7) only when $\mathbb{P}[\![\boldsymbol{u}]\!] = \boldsymbol{0}$ (i.e. when (1.11) holds). Since we imposed both (1.7) and (1.11) in treating the viscous case of the Rayleigh–Taylor problem for a liquid–liquid phase interface, further comment is required only for the inviscid case.

For the inviscid case, $\boldsymbol{S} = \boldsymbol{0}$, $\mathbb{S} = \boldsymbol{0}$, and (1) reduces to

$$[\![p_e]\!]\boldsymbol{n} + J^2[\![1/\varrho]\!]\boldsymbol{n} - \gamma K\boldsymbol{n} = J\mathbb{P}[\![\boldsymbol{u}]\!]. \tag{3}$$

While the normal component of (3) is (3.2a) of the original paper, the tangential component of (3) requires that $J\mathbb{P}[\![\boldsymbol{u}]\!] = \boldsymbol{0}$. The complete system of interface conditions for the inviscid version of the problem is therefore

$$[\![\varrho(V - \boldsymbol{u}\cdot\boldsymbol{n})]\!] = 0, \tag{4a}$$

$$[\![p_e]\!] + J^2[\![1/\varrho]\!] = \gamma K, \tag{4b}$$

$$J\mathbb{P}[\![\boldsymbol{u}]\!] = \boldsymbol{0}, \tag{4c}$$

$$\Psi + [\![p_e/\varrho]\!] + \tfrac{1}{2}J^2[\![1/\varrho^2]\!] = 0. \tag{4d}$$

Equations (4a) and (4d) are, respectively, the interfacial mass balance (1.6) and the interfacial configurational momentum balance (3.2b) of the paper.

For the classical Rayleigh–Taylor problem, $J = 0$ and (3) reduces to (4b). It is therefore noteworthy that, in the inviscid version of classical Rayleigh–Taylor problem, the normal and tangential components of the interfacial momentum balance (3) are both satisfied regardless of whether the tangential component of the velocity is continuous across the interface.

For the inviscid Rayleigh–Taylor problem with a phase transformation, we considered two cases: $\jmath = 0$ and $\jmath \neq 0$. When $\jmath = 0$, the mass flux J_0 across the interface in the base state vanishes and the root $\varrho^{\pm} L\omega / J_0 |At| T$ does not exist. Moreover, the tangential component (4*c*) of the interfacial momentum balance yields an amplitude equation identical, at first order in ϵ, to the amplitude equation arising from the configurational momentum balance (4*d*). Specifically, the amplitude equations are

$$\left.\begin{aligned} \varrho^+ A^+ - \varrho^- A^- &= \frac{[\![\varrho]\!]\omega}{|At|T} C, \\ \frac{\varrho^+ L\omega}{|At|T} A^+ + \frac{\varrho^- L\omega}{|At|T} A^- &= \left\{ [\![\varrho]\!] g - \frac{\gamma k^2}{L^2} \right\} kC, \\ A^+ + A^- &= 0, \end{aligned}\right\} \tag{5}$$

and these equations lead to the dispersion relation (3.19) discussed in the paper for the case $\jmath = 0$. The stability results obtained for this case are therefore error-free.

For the case $\jmath \neq 0$, there are two possible subcases: $\mathrm{Re}\{\omega/J_0\} > 0$ and $\mathrm{Re}\{\omega/J_0\} < 0$.

Suppose, first, that $\mathrm{Re}\{\omega/J_0\} > 0$. For this subcase, there are two alternatives depending upon whether

$$\omega \neq \frac{\jmath |At| k}{1 - At} \qquad \text{or} \qquad \omega = \frac{\jmath |At| k}{1 - At}. \tag{6}$$

When the first alternative in (6) holds, it follows from (3.6) of the paper that v_1 takes the form

$$v_1(y) = \begin{cases} A^+ \exp(-ky), & y > 0, \\ A^- \exp(ky) + B^- \exp\left(\dfrac{\varrho^- L\omega}{J_0 |At| T} y \right), & y < 0. \end{cases} \tag{7}$$

Importantly, when $\omega = \jmath |At| k/(1 - At)$, (7) becomes

$$v_1(y) = \begin{cases} A^+ \exp(-ky), & y > 0, \\ (A^- + B^-) \exp(ky), & y < 0, \end{cases} \tag{8}$$

which is equivalent to (3.7) of the paper. We then have only three unknown amplitudes A^+, $A^- + B^-$, and C (from the interfacial disturbance) and four distinct interfacial equations arising from (4) to be satisfied. For this reason, (7) is incompatible with the second alternative in (6). When that alternative holds, the characteristic equation arising from (3.6) of the paper has a double root and v_1 takes the form

$$v_1(y) = \begin{cases} A^+ \exp(-ky), & y > 0, \\ (A^- + B^- ky) \exp(ky), & y < 0. \end{cases} \tag{9}$$

For $\jmath \neq 0$ and $\mathrm{Re}\{\omega/J_0\} > 0$, (3.7) of the paper should be replaced by either (7) or (9), corresponding to whether $\omega \neq \jmath |At| k/(1 - At)$ or $\omega = \jmath |At| k/(1 - At)$, respectively.

If $\omega \neq \jmath |At| k/(1 - At)$, so that v_1 is given by (7), then the corresponding expressions for u_1 and p_1 are

$$u_1(y) = \begin{cases} -\mathrm{i} A^+ \exp(-ky), & y > 0, \\ \mathrm{i} A^- \exp(ky) + \dfrac{\mathrm{i} \varrho^- L\omega}{J_0 |At| Tk} B^- \exp\left(\dfrac{\varrho^- L\omega}{J_0 |At| T} y \right), & y < 0, \end{cases} \tag{10}$$

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and

$$p_1(y) = \begin{cases} A^+\left\{ J_0 + \dfrac{\varrho^+ L\omega}{|At|Tk} \right\} \exp(-ky), & y > 0, \\ A^-\left\{ J_0 - \dfrac{\varrho^- L\omega}{|At|Tk} \right\} \exp(ky), & y < 0. \end{cases} \tag{11}$$

For $\jmath \neq 0$, $\mathrm{Re}\{\omega/J_0\} > 0$, and $\omega \neq \jmath|At|k/(1-At)$, (3.8) and (3.9) of the paper should be replaced by (10) and (11), respectively. The amplitude equations arising from (7), (10), and (11) are

$$\left.\begin{aligned} &\varrho^+A^+ - \varrho^-A^- - \varrho^-B^- = \frac{[\![\varrho]\!]\omega}{|At|T}C, \\ &\varrho^+\left\{ \frac{J_0 k}{\varrho^-} + \frac{L\omega}{|At|T} \right\}A^+ - \varrho^-\left\{ \frac{J_0 k}{\varrho^+} - \frac{L\omega}{|At|T} \right\}A^- + J_0\frac{[\![\varrho]\!]}{\varrho^+}kB^- \\ &\quad = \left\{ [\![\varrho]\!]g - \frac{\gamma k^2}{L^2} + \frac{2J_0\langle\!\langle\varrho\rangle\!\rangle[\![\varrho]\!]\omega}{\varrho^+\varrho^-|At|T} \right\}kC, \\ &A^+ + A^- + \frac{\varrho^- L\omega}{J_0|At|Tk}B^- = \frac{J_0 k[\![\varrho]\!]}{\varrho^+\varrho^- L}C, \\ &\left\{ \varrho^+ J_0\langle\!\langle 1/\varrho^2\rangle\!\rangle k + \frac{L\omega}{|At|T} \right\}A^+ - \left\{ \varrho^- J_0\langle\!\langle 1/\varrho^2\rangle\!\rangle k - \frac{L\omega}{|At|T} \right\}A^- - \frac{1}{2}\varrho^- J_0[\![1/\varrho^2]\!]kB^- \\ &\quad = \frac{2J_0\langle\!\langle\varrho\rangle\!\rangle^2[\![\varrho]\!]k\omega}{(\varrho^+\varrho^-)^2|At|T}C. \end{aligned}\right\} \tag{12}$$

For $\jmath \neq 0$, $\mathrm{Re}\{\omega/J_0\} > 0$, and $\omega \neq \jmath|At|k/(1-At)$, (3.12)–(3.14) of the paper should be replaced by (12). Necessary and sufficient for the system (12) to possess a non-trivial solution is the requirement that its determinant vanish, which yields

$$\left(\omega^2 - \frac{\jmath^2|At|^2k^2}{(1-At)^2} \right)\left\{ \omega^2 - \left(At - \frac{k^2}{2We} \right)\frac{k}{Fr} + \frac{\jmath^2 At^2 k^2}{1-At^2} \right\} = 0. \tag{13}$$

Bearing in mind that $\mathrm{Re}\{\omega/J_0\} > 0$ and that $\omega \neq \jmath|At|k/(1-At)$, (13) holds if and only if

$$\omega^2 = \left(At - \frac{k^2}{2We} \right)\frac{k}{Fr} - \frac{\jmath^2 At^2 k^2}{1-At^2}, \tag{14}$$

which is identical to the dispersion relation (3.18) discussed in the paper.

If $\omega = \jmath|At|k/(1-At)$, so that v_1 is given by (9), then the corresponding expressions for u_1 and p_1 are

$$u_1(y) = \begin{cases} -\mathrm{i}A^+ \exp(-ky), & y > 0, \\ \mathrm{i}(A^- + B^- + B^- ky)\exp(ky), & y < 0, \end{cases} \tag{15}$$

and

$$p_1(y) = \begin{cases} 2A^+\dfrac{J_0\langle\!\langle\varrho\rangle\!\rangle}{\varrho^-}\exp(-ky), & y > 0, \\ J_0 B^- \exp(ky), & y < 0. \end{cases} \tag{16}$$

For $\jmath \neq 0$, $\mathrm{Re}\{\omega/J_0\} > 0$, and $\omega \neq \jmath|At|k/(1-At)$, (3.8) and (3.9) of the paper should be replaced by (15) and (16), respectively. The amplitude equations arising from (9), (15), and (16) are

$$
\left.\begin{aligned}
\varrho^+A^+ - \varrho^-A^- &= \frac{J_0[\![\varrho]\!]k}{\varrho^-L}C,\\
2J_0\frac{\varrho^+}{\varrho^-}A^+ + J_0\frac{[\![\varrho]\!]}{\varrho^+}A^- - J_0B^- &= \left\{[\![\varrho]\!]g - \frac{\gamma k^2}{L^2} + \frac{J_0^2[\![\varrho^2]\!]k}{\varrho^+\varrho^{-2}L}\right\}C,\\
A^+ + A^- + B^- &= \frac{J_0[\![\varrho]\!]k}{\varrho^+\varrho^-L}C,\\
4\frac{\langle\!\langle\varrho\rangle\!\rangle^2}{\varrho^+\varrho^{-2}}A^+ - \varrho^-[\![1/\varrho^2]\!]A^- - \frac{2}{\varrho^-}B^- &= \frac{4J_0\langle\!\langle\varrho\rangle\!\rangle^2[\![\varrho]\!]k}{\varrho^{+2}\varrho^{-3}L}C,
\end{aligned}\right\} \tag{17}
$$

where the relation $\omega = \jmath|At|k/(1-At)$ has been used to eliminate dependence upon ω. For $\jmath \neq 0$, $\mathrm{Re}\{\omega/J_0\} > 0$, and $\omega = \jmath|At|k/(1-At)$, (3.12)–(3.14) of the paper should be replaced by (17). Necessary and sufficient for the system (17) to possess a non-trivial solution is the requirement that its determinant vanish, which yields

$$
k^2 + \frac{4WeFrAt^2\jmath^2}{(1-At^2)(1-At)}k - 2AtWe = 0. \tag{18}
$$

The quadratic (18) has two real roots

$$
k_\pm = \pm\sqrt{\left(\frac{2WeFrAt^2\jmath^2}{(1-At^2)(1-At)}\right)^2 + 2AtWe} - \frac{2WeFrAt^2\jmath^2}{(1-At^2)(1-At)} \tag{19}
$$

and the growth rate $\omega_+ = \jmath|At|k_+/(1-At)$ corresponding to the positive root k_+ is unstable. Recall that the cutoff wavenumber k_c determined by the dispersion relation (14) (cf. (3.20) of the paper) has the form

$$
k_c = \sqrt{\left(\frac{WeFrAt^2\jmath^2}{1-At^2}\right)^2 + 2AtWe} - \frac{WeFrAt^2\jmath^2}{1-At^2}. \tag{20}
$$

Since $\sqrt{b_1^2+a} - b_1 < \sqrt{b_2^2+a} - b_2$, provided that a, b_1, and b_2 are real numbers satisfying $a > 0$ and $0 < b_2 < b_1$, and since $2/(1-At) > 1$, it follows that $k_+ < k_c$. Moreover, setting $k = k_+$ in (14) gives $\omega = \jmath|At|k_+/(1-At) = \omega_+$. Hence, the degenerate alternative $\omega = \jmath|At|k/(1-At)$ yields no growing modes beyond those determined by the dispersion relation (14) for $\omega \neq \jmath|At|k/(1-At)$.

The analysis of the remaining subcase $\mathrm{Re}\{\omega/J_0\} < 0$ is completely analogous to that of the subcase $\mathrm{Re}\{\omega/J_0\} > 0$. In particular, two alternatives,

$$
\omega \neq -\frac{\jmath|At|k}{1+At} \qquad \text{or} \qquad \omega = -\frac{\jmath|At|k}{1+At}, \tag{21}
$$

arise.

If $\omega \neq -\jmath|At|k/(1+At)$, then it follows from (3.6) of the paper that v_1 takes the form

$$
v_1(y) = \begin{cases} A^+\exp(-ky) + B^+\exp\left(\dfrac{\varrho^+L\omega}{J_0|At|T}y\right), & y > 0,\\ A^-\exp(ky), & y < 0. \end{cases} \tag{22}
$$

Further, the corresponding expressions for u_1 and p_1 are

$$u_1(y) = \begin{cases} -\mathrm{i}A^+ \exp(-ky) + \dfrac{\mathrm{i}\varrho^+ L\omega}{J_0|At|Tk} B^+ \exp\left(\dfrac{\varrho^+ L\omega}{J_0|At|T} y\right), & y > 0, \\ \mathrm{i}A^- \exp(ky), & y < 0, \end{cases} \tag{23}$$

and

$$p_1(y) = \begin{cases} A^+ \left\{ J_0 + \dfrac{\varrho^+ L\omega}{|At|Tk} \right\} \exp(-ky), & y > 0, \\ A^- \left\{ J_0 - \dfrac{\varrho^- L\omega}{|At|Tk} \right\} \exp(ky), & y < 0. \end{cases} \tag{24}$$

For $\jmath \neq 0$, $\mathrm{Re}\{\omega/J_0\} < 0$, and $\omega \neq -\jmath|At|k/(1+At)$, (3.7), (3.8) and (3.9) of the paper should be replaced by (22), (23) and (24), respectively. The amplitude equations arising from (22)–(24) are

$$\left.\begin{aligned} &\varrho^+ A^+ + \varrho^+ B^+ - \varrho^- A^- = \frac{[\![\varrho]\!]\omega}{|At|T} C, \\ &\varrho^+ \left\{ \frac{J_0 k}{\varrho^-} + \frac{L\omega}{|At|T} \right\} A^+ + J_0 \frac{[\![\varrho]\!]}{\varrho^-} k B^+ - \varrho^- \left\{ \frac{J_0 k}{\varrho^+} - \frac{L\omega}{|At|T} \right\} A^- \\ &\quad = \left\{ [\![\varrho]\!] g - \frac{\gamma k^2}{L^2} + \frac{2J_0 \langle\!\langle \varrho \rangle\!\rangle [\![\varrho]\!]\omega}{\varrho^+\varrho^-|At|T} \right\} kC, \\ &A^+ - \frac{\varrho^+ L\omega}{J_0|At|Tk} B^+ + A^- = \frac{J_0 k [\![\varrho]\!]}{\varrho^+\varrho^- L} C, \\ &\left\{ \varrho^+ J_0 \langle\!\langle 1/\varrho^2 \rangle\!\rangle k + \frac{L\omega}{|At|T} \right\} A^+ - \frac{1}{2} \varrho^+ J_0 [\![1/\varrho^2]\!] k B^+ - \left\{ \varrho^- J_0 \langle\!\langle 1/\varrho^2 \rangle\!\rangle k - \frac{L\omega}{|At|T} \right\} A^- \\ &\quad = \frac{2J_0 \langle\!\langle \varrho \rangle\!\rangle^2 [\![\varrho]\!] k\omega}{(\varrho^+\varrho^-)^2 |At|T} C. \end{aligned}\right\} \tag{25}$$

For $\jmath \neq 0$, $\mathrm{Re}\{\omega/J_0\} < 0$, and $\omega \neq -\jmath|At|k/(1+At)$, (3.12)–(3.14) of the paper should be replaced by (25). Necessary and sufficient for the system (25) to possess a non-trivial solution is the requirement that its determinant vanish. Bearing in mind that $\mathrm{Re}\{\omega/J_0\} < 0$ and that $\omega \neq -\jmath|At|k/(1+At)$, the solvability criterion for this system is

$$\omega^2 = \left(At - \frac{k^2}{2We} \right) \frac{k}{Fr} + \frac{\jmath^2 At^2 k^2}{1 - At^2}, \tag{26}$$

which coincides, again, with the dispersion relation (3.18) discussed in the paper.

If $\omega = -\jmath|At|k/(1+At)$, then it follows from (3.5) of the paper that v_1 takes the form

$$v_1(y) = \begin{cases} (A^+ - B^+ ky) \exp(-ky), & y > 0, \\ A^- \exp(ky), & y < 0. \end{cases} \tag{27}$$

Further, the corresponding expressions for u_1 and p_1 are

$$u_1(y) = \begin{cases} -\mathrm{i}(A^+ + B^+ - B^+ ky) \exp(-ky), & y > 0, \\ \mathrm{i}A^- \exp(ky), & y < 0, \end{cases} \tag{28}$$

and

$$p_1(y) = \begin{cases} J_0 B^+ \exp(-ky), & y > 0, \\ 2A^- \dfrac{J_0 \langle\!\langle \varrho \rangle\!\rangle}{\varrho^+} \exp(ky), & y < 0. \end{cases} \tag{29}$$

The amplitude equations arising from (27)–(29) are

$$\left.\begin{aligned} \varrho^+ A^+ - \varrho^- A^- &= -\frac{J_0 [\![\varrho]\!] k}{\varrho^+ L} C, \\ J_0 \frac{[\![\varrho]\!]}{\varrho^-} A^+ + J_0 B^+ - 2J_0 \frac{\varrho^-}{\varrho^+} A^- &= \left\{ [\![\varrho]\!] g - \frac{\gamma k^2}{L^2} - \frac{J_0^2 [\![\varrho^2]\!] k}{\varrho^{+2}\varrho^- L} \right\} C, \\ A^+ + B^+ + A^- &= \frac{J_0 [\![\varrho]\!] k}{\varrho^+ \varrho^- L} C, \\ \varrho^+ [\![1/\varrho^2]\!] A^+ - \frac{2}{\varrho^+} B^+ + 4 \frac{\langle\!\langle \varrho \rangle\!\rangle^2}{\varrho^{+2}\varrho^-} A^- &= \frac{4 J_0 \langle\!\langle \varrho \rangle\!\rangle^2 [\![\varrho]\!] k}{\varrho^{+3}\varrho^{-2} L} C, \end{aligned}\right\} \tag{30}$$

where the relation $\omega = -\jmath |At| k/(1 + At)$ has been used to eliminate dependence upon ω. For $\jmath \neq 0$, $\mathrm{Re}\{\omega/J_0\} < 0$, and $\omega = -\jmath |At| k/(1 + At)$, (3.12)–(3.14) of the paper should be replaced by (30). Necessary and sufficient for the system (30) to possess a non-trivial solution is the requirement that its determinant vanish, which leads to the quadratic

$$k^2 + \frac{4WeFrAt^2\jmath^2}{(1 - At^2)(1 + At)} k - 2AtWe = 0. \tag{31}$$

The positive root,

$$k_+ = \sqrt{\left(\frac{2WeFrAt^2\jmath^2}{(1 - At^2)(1 + At)}\right)^2 + 2AtWe} - \frac{2WeFrAt^2\jmath^2}{(1 - At^2)(1 + At)}, \tag{32}$$

of (31) yields an unstable growth rate $\omega_+ = -\jmath |At| k_+/(1 + At)$ via (21). Inspection of this root shows that it belongs to the set of unstable modes determined when $\omega \neq -\jmath |At| k/(1 + At)$ by the dispersion relation (26).

In conclusion, we have shown that accounting correctly for the additional characteristic root $\varrho^{\pm} L\omega/J_0 |At| T$ of the differential equation (3.6) of the paper does not alter the stability results obtained for the inviscid Rayleigh–Taylor problem with a phase transformation. Specifically, in this case all unstable growth rates are determined by the dispersion relation (3.18) given in the paper.