

**NOTE ON THE BEHAVIOR OF SOLUTIONS
 OF PARABOLIC EQUATIONS
 WITH UNBOUNDED COEFFICIENTS***

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To Professor Katuzi Ono on the occasion of his 60th birthday

1. Let

$$L = \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial}{\partial x_i} + c - \frac{\partial}{\partial t}$$

be a parabolic differential operator defined in $\Omega = R^n \times (0, \infty)$, where R^n is the n -dimensional Euclidean space, the point $x \in R^n$ is represented by its coordinates (x_1, \dots, x_n) and $a_{ij}(=a_{ji})$, b_i and c are functions in $(x, t) \in \Omega$. We assume that there exist constants $k_1(> 0)$, K_1 , $K_2(\geq 0)$, $K_3(> 0)$ and K_4 such that

$$(1) \quad \begin{cases} k_1(|x|^2 + 1)^{1-\lambda} |\xi|^2 \leq \sum_{i,j=1}^n a_{ij} \xi_i \xi_j \leq K_1(|x|^2 + 1)^{1-\lambda} |\xi|^2, \\ |b_i| \leq K_2(|x|^2 + 1)^{1/2}, \quad (1 \leq i \leq n), \\ c \leq -K_3(|x|^2 + 1)^\lambda + K_4 \end{cases}$$

for some $\lambda \in (0, 1]$.

Consider the Cauchy problem

$$(2) \quad \begin{cases} Lu = 0 & \text{in } \Omega, \\ u(x, 0) = f(x). \end{cases}$$

2. Throughout this note, we shall say that $u(x, t)$ is a solution of the problem (2) when $u(x, t)$ is continuous in $\bar{\Omega} = R^n \times [0, \infty)$, twice continuously differentiable in Ω and satisfies (2).

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In this note we shall prove the following which is a general form of Krzyżański's theorem [2].

THEOREM. *Let $u(x, t)$ be a solution of the Cauchy problem (2) and $|u(x, t)| \leq K_5 e^{\mu(|x|^2+1)\lambda}$ for some constants K_5 and μ . Assume that the coefficients of L satisfy*

(1). *If the Cauchy data $f(x)$ is bounded in R^n and if*

$$(3) \quad \frac{1}{2K_1} [2K_1(1-\lambda) - k_1 n] (\sqrt{K_3^2 n^2 + 4K_1 K_3} - K_2 n) + K_4 < 0,$$

then $u(x, t)$ tends to zero uniformly in $x \in R^n$ as t tends to infinity.

In the case of the differential operator

$$L_0 = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + (-K_3|x|^2 + K_3') - \frac{\partial}{\partial t},$$

we may take $k_1 = K_1 = 1$, $K_2 = 0$, $K_4 = K_3' - K_3$ and $\lambda = 1$ in Theorem. So the solution $u(x, t)$ of the Cauchy problem

$$\begin{cases} L_0 u = 0 & \text{in } \Omega, \\ u(x, 0) = f(x) \end{cases}$$

for a bounded Cauchy data $f(x)$ tends to zero uniformly in $x \in R^n$ as t tends to infinity, if

$$\sqrt{K_3} n > K_3' - K_3.$$

3. To prove our theorem, we use the following.

LEMMA. *Let α be a positive root of the quadratic equation $AX^2 + BX + C = 0$, where $B \geq 0$ and $C < 0$. Then the function*

$$\varphi(t) = \alpha \tanh A\alpha t$$

satisfies the inequality

$$\varphi'(t) + A\varphi^2(t) + B\varphi(t) + C \leq 0.$$

The proof is given by the direct calculation, so we may omit it here. Now we shall give the proof of Theorem.

Let $\varphi(t)$ and $\psi(t)$ be functions twice continuously differentiable in $[0, \infty)$.

Putting

$$(4) \quad H(x, t) = \exp[-\varphi(t)(|x|^2 + 1)^\lambda + \psi(t)],$$

we see from (1) that

$$\begin{aligned} \frac{LH}{H} &= 4\lambda^2\varphi^2(t)(|x|^2 + 1)^{2\lambda-2} \sum_{i,j=1}^n a_{ij}x_i x_j + 4\lambda(1-\lambda)\varphi(t)(|x|^2 + 1)^{\lambda-2} \sum_{i,j=1}^n a_{ij}x_i x_j \\ &\quad - 2\lambda\varphi(t)(|x|^2 + 1)^{\lambda-1} \sum_{i=1}^n (a_{ii} + b_i x_i) + c + \varphi'(t)(|x|^2 + 1)^\lambda - \psi'(t) \\ &\leq (|x|^2 + 1)^\lambda [\varphi'(t) + 4K_1\lambda^2\varphi^2(t) + 2K_2n\lambda\varphi(t) - K_3] \\ &\quad + [4\lambda(1-\lambda)K_1\varphi(t) - 2\lambda k_1n\varphi(t) + K_4 - \psi'(t)]. \end{aligned}$$

So, if

$$(5) \quad \varphi(t) = \alpha \tanh 4K_1\lambda^2\alpha t$$

for the positive root

$$\alpha = \frac{-K_2n + \sqrt{K_2^2n^2 + 4K_1K_3}}{4K_1\lambda}$$

of the quadratic equation $4K_1\lambda^2X^2 + 2K_2n\lambda X - K_3 = 0$, then we see from Lemma that

$$\varphi'(t) + 4K_1\lambda^2\varphi^2(t) + 2K_2n\lambda\varphi(t) - K_3 \leq 0.$$

Further, it is easy to see that

$$(6) \quad \psi(t) = \frac{2K_1(1-\lambda) - k_1n}{2K_1\lambda} \log[\cosh 4K_1\lambda^2\alpha t] + K_4t$$

satisfies

$$4\lambda(1-\lambda)K_1\varphi(t) - 2\lambda k_1n\varphi(t) + K_4 - \psi'(t) = 0$$

for $\varphi(t)$ given by (5). Thus $H(x, t)$ given by (4) for $\varphi(t)$ in (5) and $\psi(t)$ in (6) satisfies

$$LH \leq 0$$

in Ω . It is evident that $H(x, 0) = 1$.

As the Cauchy data $f(x)$ in (2) is bounded, we may assume $|f(x)| < M$ in R^n . If we put

$$w_+(x, t) = MH(x, t) + u(x, t),$$

then $Lw_+ = MLH + Lu = MLH \leq 0$ in Ω and $w_+(x, 0) = M + f(x) \geq 0$. Moreover, we have clearly $|w_+(x, t)| \leq K'_4 e^{\mu'(|x|^2+1)\lambda}$ in Ω for some constants K'_4

and μ' . The maximum principle due to Bodanko [1] implies that $w_+(x, t) \geq 0$ in Ω , that is,

$$-MH(x, t) \leq u(x, t)$$

in Ω . We apply the same argument to $w_-(x, t) = MH(x, t) - u(x, t)$ as the above, we get

$$MH(x, t) \geq u(x, t).$$

Thus we obtain

$$\begin{aligned} |u(x, t)| &\leq MH(x, t) \\ &\leq M \exp\{[4K_1\lambda(1-\lambda)\alpha - 2\lambda k_1na + K_1]t\} \end{aligned}$$

for α in (5) throughout Ω . From the assumption (3), it is obvious that $u(x, t)$ tends to zero uniformly in $x \in R^n$ as t tends to infinity.

Remark. Krzyżański [2] considered the case $\lambda = 1$ in our Theorem and gave an analogous result.

REFERENCES

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