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Extensions of finite nilpotent groups

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If G is a finite group and P is a group-theoretic property, G will be called P-max-core if for every maximal subgroup M of G, M/M_G has property P where $M_G = \bigcap_{x \in G} (x^{-1}Mx)$ is the core of M in G. In a joint paper with John D. Dixon and A.H. Rhemtulla, we showed that if p is an odd prime and G is (p-nilpotent)-max-core, then G is p-solvable, and then using the techniques of the theory of solvable groups, we characterized nilpotent-max-core groups as finite nilpotent-by-nilpotent groups. The proof of the first result used John G. Thompson's p-nilpotency criterion and hence required $p > 2$. In this paper I show that supersolvable-max-core groups (and hence (2-nilpotent)-max-core groups) need not be 2-solvable (that is, solvable). Also I generalize the second result, among others, and characterize (p-nilpotent)-max-core groups (for p an odd prime) as finite nilpotent-by-(p-nilpotent) groups.

1. Introduction

If H is a subgroup of a group G then the core H_G of H is defined to be $\bigcap_{x \in G} (x^{-1}Hx)$. I will call H/H_G the core factor of H. In a joint paper [4] with John D. Dixon and A.H. Rhemtulla, a finite group G was termed X-max-core, where X is a property (or class) of groups, if the core factors of the maximal subgroups of G all had property X

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(or lay in X). In particular, the trivial-max-core groups are just the finite nilpotent groups. Thus the concept of X-max-core extends that of finite nilpotent; more precisely, if $\{1\} \in X$, the class of X-max-core groups is an extension of the class of all finite nilpotent groups. But there is another sense in which X-max-core groups may be considered as extensions of nilpotent groups. In our previous paper [4], we obtained two main results (both of which I wish to examine here).

RESULT A. If p is an odd prime and G is (p-nilpotent)-max-core, then G is p-solvable.

RESULT B. If F is a formation of nilpotent groups, then a finite group G is F-max-core if and only if G is a nilpotent-by-F group. (Recall that a formation F is a class of groups satisfying

(a) if $G \in F$ and $N \triangleleft G$ then $G/N \in F$, and

(b) if $G/N_1, G/N_2 \in F$ then $G/(N_1 \cap N_2) \in F$.

By an X-by-Y group we mean a group G having a normal subgroup N in X with G/N in Y). Consequently, for some classes X, all X-max-core groups are extensions of nilpotent groups by X-groups.

It is a well-known result of O.J. Schmidt (see [6], p. 280) that if a finite group has only nilpotent maximal subgroups then it is solvable. N. Itô has shown that the condition "nilpotent" may be weakened to "p-nilpotent", and B. Huppert has shown that "supersolvable" may be substituted for "nilpotent" too (see [6], pp. 434, 718). It follows directly from Result A above that nilpotent-max-core groups are solvable, which generalizes Schmidt's theorem. Two questions arise immediately: does Result A hold for $p = 2$, and can the condition "p-nilpotent" be replaced by "supersolvable"? I will answer both of these negatively here. (In [4] we showed "p-solvable" could not be replaced by "solvable".)

Result B was deduced from A by using the techniques of the theory of solvable groups - for, nilpotent-max-core groups are solvable. It is natural to ask whether, having A, B could be extended to formations of p-nilpotent groups (for p odd) in the usual manner in which many properties of solvable groups carry over to p-solvable groups. I will prove that the (p-nilpotent)-max-core groups are indeed precisely the finite nilpotent-by-(p-nilpotent) groups, but that there are formations F

of p-nilpotent groups for which the concepts of F-max-core and finite nilpotent-by-F do not coincide.

Throughout this paper, all groups considered will be finite. As in [4], sX and QX will denote respectively the classes of subgroups and quotients of X-groups.

2. The counterexample

Let p be a prime satisfying $p^2 \equiv 1$ (modulo 16) - for example, $p = 7$. Let $G = PGL(2, p)$ be the projective linear group, which is the central factor of the general linear group of all 2×2 nonsingular matrices over the field of p elements, and let $S = PSL(2, p)$, the projective special linear group, be its (unique) subgroup of index 2.

PROPOSITION 1. *If p is a prime satisfying $p^2 \equiv 1$ (modulo 16) then $PGL(2, p)$ is supersolvable-max-core and (2-nilpotent)-max-core but $PGL(2, p)$ is not 2-solvable.*

Proof. Since S is of even order and is simple ([6], p. 182), G is not 2-solvable, or in fact p-solvable for any prime p dividing the order of G. And since supersolvable groups are 2-nilpotent ([6], p. 716), the Proposition reduces to showing only that G is supersolvable-max-core. To do this, it will be shown that the subgroups H of S are of two types:

(type A) $1 = H_0 < H_1 < \ldots < H_k = H$ is a characteristic series of H in which H_{i+1}/H_i is cyclic for $i = 0, 1, \ldots, k-1$,

(type B) $N_G(H) = N_S(H)$.

Then if M is a maximal subgroup of G, either $M = S$ (so $M/M_G = 1$) or M contains, as a normal subgroup, $M \cap S$, of index $|M : M \cap S| = |MS : S| = |G : S| = 2$. But $M \cap S$ must be of one of the two types above, and because $M \cap S \triangleleft M$, it cannot be of type B. On the other hand if $M \cap S$ is of type A, then M is supersolvable and hence M/M_G is too.

Now the subgroups of $PSL(2, p)$ are well-known ([6], p. 213), and when $p^2 \equiv 1$ (mod 16), the subgroups are either cyclic, dihedral, or a semidirect product of a (normal) p-cycle and a cyclic group of order dividing $p - 1$ (all of these are type A), or isomorphic to A_4, A_5 or S_4. L.E. Dickson has shown ([3], pp. 282, 285) that these latter subgroups are of type B, under the restrictions on p. Hence we are done.

A characterization of those (2-nilpotent)-max-core groups and supersolvable-max-core groups which are 2-solvable (and hence solvable by the Feit-Thompson theorem) is given in Theorem (4) of [4].

3. Some generalizations

In this Section I will generalize the results of Section 4 of [4] to π-soluble groups. Briefly (see [2] and [7]) a group G is π-soluble if, for a given set π of primes, the composition factors of G are either p-groups for $p \in \pi$, or π'-groups. G is π-nilpotent if it has a nilpotent Hall π-subgroup and a normal π-complement. The largest normal π-nilpotent subgroup of a group G is denoted $F^{\pi}(G)$. $F^{\pi}(G) = \bigcap_{p \in \pi} F^{p}(G)$ so ([6], p. 686) $F^{\pi}(G)$ may also be defined as $\bigcap C_G(H/K)$, the intersection taken for all composition factors which are p-groups for primes $p \in \pi$. A maximal subgroup of G of index a power of a prime $p \in \pi$ is called a p-maximal or π-maximal subgroup of G. In a π-soluble group every maximal subgroup has either π'-index or is π-maximal; the intersection of all the π-maximal subgroups of G is denoted $\Phi_{\pi}(G)$, the π-Frattini subgroup[1]. If $0_{\pi'}(G)$ denotes the largest normal π'-subgroup of a π-soluble group G then $0_{\pi'}(G) \leq \Phi_{\pi}(G) \leq F^{\pi}(G)$, $F^{\pi}(G)/0_{\pi'}(G) = F\left(G/0_{\pi'}(G)\right)$, $\Phi_{\pi'}(G)/0_{\pi'}(G) = \Phi\left(G/0_{\pi'}(G)\right)$, and $F^{\pi}(G/N) = F^{\pi}(G)/N$ for all $N \leq \Phi_{\pi}(G)$, $N \trianglelefteq G$.

In this context define the class $M(\pi, X)$ of all π-soluble groups G having $M/M_G \in X$ for all π-maximal subgroups M of G. It is now possible to state the generalizations of (5), and of (4), (6) and (8), of

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[1] I thank Dr Graham Chambers for his suggestion to use $\Phi_{\pi}(G)$ in place of $\Phi(G)$.

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[4] required. As the proofs follow precisely the lines laid out in [4], except for the obvious substitutions from the concepts and results summarized above, I omit them.

LEMMA 2. *Let* $\{N_1, \ldots, N_k\}$ *be a set of normal subgroups of a* π*-soluble group* G *satisfying* $\bigcap_{i=1}^{k} N_i \le \Phi_\pi(G)$, *and define* F_i^π *by* $F_i^\pi(G/N_i)$. *Then* $\bigcap_{i=1}^{k} F_i^\pi = F^\pi(G)$.

THEOREM 3. *Let* $\mathfrak{X} = Q\mathfrak{X}$ *and let* G *be a* π*-soluble group. Then*

(i) if $G/F^\pi(G) \in \mathfrak{X}$ *then* $G \in \mathfrak{M}(\pi, \mathfrak{X})$, *and*

(ii) if $G \in \mathfrak{M}(\pi, \mathfrak{X})$ *and* $\mathfrak{X}$ *is a formation, then* $G/F^\pi(G) \in \mathfrak{X}$.

R. Baer has obtained a similar result ([1], p. 147): if $\mathfrak{X}$ is a formation then $G/F^p(G)$ is a p-solvable $\mathfrak{X}$-group if and only if whenever M is a maximal subgroup of G and G/M_G has a monolith of order divisible by p , then $M/M_G \in \mathfrak{X}$ and M is p-maximal.

4. (p-nilpotent)-max-core

If $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are classes of groups, denote by $\mathfrak{X}_1\mathfrak{X}_2$ the class of groups G having a normal subgroup $N \in \mathfrak{X}_1$ with $G/N \in \mathfrak{X}_2$; that is, the class of $\mathfrak{X}_1$-by-$\mathfrak{X}_2$ groups. The following Proposition (a second generalization of (8) of [4]) provides, together with Theorem 3 above, sufficient information for a characterization of (p-nilpotent)-max-core groups.

PROPOSITION 4. *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be classes of groups,* $\mathfrak{X} = Q\mathfrak{X} = S\mathfrak{X}$, *and* $\mathfrak{Y} = Q\mathfrak{Y}$. *Suppose that whenever* $G/F(G) \in \mathfrak{Y}$ *then* G *is* $\mathfrak{Y}$*-max-core. Then whenever* $G/F(G) \in \mathfrak{X}\mathfrak{Y}$, G *is* $\mathfrak{X}\mathfrak{Y}$*-max-core.*

Proof. Suppose $G/F(G) \in \mathfrak{X}\mathfrak{Y}$, and let M be a maximal subgroup of G . Since $Q\mathfrak{X} = \mathfrak{X}$ and $Q\mathfrak{Y} = \mathfrak{Y}$ then $Q(\mathfrak{X}\mathfrak{Y}) = \mathfrak{X}\mathfrak{Y}$ by an elementary application of the homomorphism theorems; therefore we may assume, without loss of generality, that $M_G = 1$. If $F(G) \ne 1$ then $F(G)$

complements M ([5], p. 219) so $M/M_G = M \simeq G/F(G) \in \mathfrak{X}\mathcal{Y}$, and we are done. On the other hand when $F(G) = 1$ then $G \in \mathfrak{X}\mathcal{Y}$. By our assumptions on $\mathcal{Y}$, we may suppose that G contains a non-trivial normal subgroup N with $N \in \mathfrak{X}$ and $G/N \in \mathcal{Y}$. As $N \neq 1$ but $M_G = 1$ then $G = MN$. Then $M/M \cap N \simeq MN/M = G/N \in \mathcal{Y}$ and $M \cap N \in s\mathfrak{X} = \mathfrak{X}$, so $M/M_G = M \in \mathfrak{X}\mathcal{Y}$.

This result yields slightly more than a characterization of (p-nilpotent)-max-core groups:

THEOREM 5. *Let* π *be a non-empty set of odd primes. Denote the class of all* π'*-groups by* $\mathcal{Z}$ *, and let* $\mathcal{Y}$ *be a formation of nilpotent* π*-groups and* $\mathfrak{X}$ *be the* π*-nilpotent formation* $\mathcal{Z}\mathcal{Y}$ *. Then a group* G *is* $\mathfrak{X}$*-max-core if and only if* $G/F(G) \in \mathfrak{X}$ *.*

Proof. In the case that $G/F(G) \in \mathfrak{X}$, then by Result B, the conditions of Proposition 4 are satisfied and consequently G must be $\mathfrak{X}$-max-core.

Suppose now that G is $\mathfrak{X}$-max-core but $G/F(G) \notin \mathfrak{X}$, and let G be a minimal such counterexample.

(a) If $1 \neq N \triangleleft G$, then $(G/N)/F(G/N) \in \mathfrak{X}$ by the minimality of G and the fact that quotients of $\mathfrak{X}$-max-core groups are again $\mathfrak{X}$-max-core.

(b) G is π-soluble by Result B, and $G/F^{\pi}(G) \in \mathfrak{X}$ by Theorem 3 above.

(c) G has a nontrivial normal π'-subgroup, for otherwise $F^{\pi}(G) = F(G)$ and so by (b) $G/F(G) \in \mathfrak{X}$ - a contradiction.

(d) $\Phi(G) = 1$, for $F\big(G/\Phi(G)\big) = F(G)/\Phi(G)$ ([6], p. 277), and so if $\Phi(G) \neq 1$ they by (a) $G/F(G) \in \mathfrak{X}$ – a contradiction.

(e) G has a monolith K (a unique minimal normal subgroup): for, if $1 \neq K_i \triangleleft G$ $(i = 1, 2)$, $K_1 \cap K_2 = 1 = \Phi(G)$, then define $F_i \supseteq K_i$ by $F_i/K_i = F\big(G/K_i\big)$; by (a), $G/F_i \in \mathfrak{X}$ and by Lemma (5) of [4], $F_1 \cap F_2 = F(G)$; because $\mathfrak{X}$ is a formation, then $G/F(G) \in \mathfrak{X}$ – a contradiction. $\big(G$ is not simple, by (c).$\big)$

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(f) K is a π'-group by (c).

(g) G contains a subgroup $M \in X$ such that $G = MK$: for, if $K \subseteq M$ for all maximal subgroups M of G then $K \subseteq \Phi(G)$, contradicting (d); hence there exists a maximal subgroup M of G such that $K \nsubseteq M$ and so $G = MK$; now if $M_G \neq 1$ then $K \subseteq M_G \subseteq M$, a contradiction, and so $M_G = 1$, and $M = M/M_G \in X$.

(h) Hence $G/K = MK/K \simeq M/M \cap K \in QX = X$. By (f) K is a π'-group, and so $G \in ZY = X$. Then $G/F(G) \in QX = X$ - a contradiction. The Theorem is proved.

Theorem 5 above immediately raises the conjecture that if X is a formation of p-nilpotent groups for some odd prime p, then G is X-max-core if and only if $G/F(G) \in X$. But this is not the case: let $p > 3$ and let X be the class of solvable p-nilpotent groups. The group $PSL\left(2, 3^p\right)$ has p'-order, has a non-trivial automorphism of order p induced from the automorphism of $GF\left(3^p\right)$, and has every subgroup solvable; its extension E by this automorphism is p-nilpotent and is obviously (solvable p-nilpotent)-max-core, but is not solvable. For $p = 3$ a similar construction using the Suzuki group $S_z(2^3)$ holds (see [8]). Similarly for the converse: let $p > 3$, let Y be the formation of (finite) direct products of $PSL\left(2, 3^p\right)$, and let X be the p-nilpotent formation YP where P is the class of all p-groups. The p-nilpotent group E constructed above is in S but E is not S-max-core. Again, for $p = 3$ use $S_z(2^3)$.

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