DENSITY AND REPRESENTATION THEOREMS FOR MULTIPLIERS OF TYPE (p, q)

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Let G be a locally compact Abelian Hausdorff group (abbreviated LCA group); let X be its character group and dx, $d\chi$ be the elements of the normalised Haar measures on G and X respectively. If $1 \leqq p, q < \infty$, and $L^p(G)$ and $L^q(G)$ are the usual Lebesgue spaces, of index p and q respectively, with respect to dx, a *multiplier* of type (p, q) is defined as a bounded linear operator T from $L^p(G)$ to $L^q(G)$ which commutes with translations, i.e. $\tau_x T = T\tau_x$ for all $x \in G$, where $\tau_x f(y) = f(x+y)$. The space of multipliers of type (p, q) will be denoted by L_p^q. Already, much attention has been devoted to this important class of operators (see, for example, [3], [4], [7]).

It is known that if G is non-compact and $p > q$, then $L_p^q = \{0\}$ (cf. [7], Theorem 1.1). The discussion therefore divides naturally into two parts: one for compact G with $p > q$, and one for general LCA G with $p \leqq q$. Observe that if $h \in C_c(G)$ (the space of continuous functions with compact supports) and $T_h f = h * f$ $(f \in L^p(G))$, it is easy to see that T_h is a multiplier of type (p, q): for if G is an LCA group and $1 \leqq p \leqq q < \infty$, then $h * f \in L^\infty \cap L^p \subset L^q$; if G is compact and p and q satisfy $1 \leqq q < p < \infty$, we have again that $T_h \in L_p^q$, this time because $h * f \in C(G)$ (the space of continuous functions) and $C(G) \subset L^q(G)$ for all q.

The main result of this note (Theorem 1) asserts that every element of L_p^q can be approximated, boundedly in the strong operator topology, by multipliers of the form T_h with $h \in C_c(G)$. For multipliers of type (p, p) with $p \neq 1$, this result is contained in [3]. In Theorem 2, we establish a representation theorem for multipliers of type (p, q); this is an analogue of Theorem 1 of [3].

THEOREM 1. *Suppose that $T \in L_p^q$ (with $p \leqq q$ if G is non-compact); then there exists a net (φ_α) in C_c such that $\lim_\alpha \varphi_\alpha * f = Tf$ in the norm of L^q for every $f \in L^p$, and*

$$\|\varphi_\alpha * f\|_q \leqq \|T\| \, \|f\|_p.$$

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PROOF. There are three cases to consider.

Case (i). $q \neq 1$. It suffices to show that there exists a net (φ_α) in C_c such that $\lim_\alpha \varphi_\alpha * f = Tf$ weakly in L^q and such that $||\varphi_\alpha * f||_q \leqq ||T|| \, ||f||_p$. It will follow then ([1], VI.1.5) that a net of convex combinations of the φ_α's will satisfy the conclusion of the theorem. Write (h_β) for an approximate identity in $L^1(G)$ with $h_\beta \in C_c * C_c$, $||h_\beta||_1 = 1$ and h_β vanishing outside some fixed compact set for all β. Let (k_γ) be an approximate identity in $L^1(X)$ satisfying $\hat{k}_\gamma \in C_c$ and $||k_\gamma||_1 = 1$ ($\hat{k}_\gamma$ is the Fourier transform of k_γ). Now T is continuous and commutes with translations; since $1 \leqq p$, $q < \infty$, it is easy to see that T commutes with convolution by functions in C_c; hence Th_β is continuous for all β. Write $\varphi_\alpha = \varphi_{(\beta,\gamma)} = \hat{k}_\gamma Th_\beta$ and give $\alpha = (\beta, \gamma)$ the usual product ordering; then (φ_α) is a net of functions in C_c. We prove first that $||\varphi_\alpha * f||_q \leqq ||T|| \, ||f||_p$, and then use this to establish the remaining assertion of the theorem. It suffices to show that for $f, g \in C_c$,

$$|\varphi_\alpha * f * g(0)| \leqq ||T|| \, ||f||_p ||g||_{q'},$$

(Throughout this note, we write r' for the usual conjugate index of r when $1 \leqq r \leqq \infty$.) Since $k_\gamma \in L^1(X)$, we have that

$$\begin{aligned} \hat{k}_\gamma Th_\beta * f * g(0) &= \int_G (\hat{k}_\gamma Th_\beta)(-y) f * g(y) dy \\ &= \int_G \int_G \int_X k_\gamma(\chi) Th_\beta(-y)\chi(-y) f(y-t) g(t) d\chi \, dt \, dy \end{aligned}$$

Applying Fubini's Theorem and recalling that $\overline{\chi(x)} = \chi(-x)$, we have

$$\begin{aligned} |\varphi_\alpha * f * g(0)| &\leqq \int_X |k_\gamma(\chi)| \left| \int_G \int_G Th_\beta(-y) f(y-t) g(t) \bar{\chi}(y) dt \, dy \right| d\chi \\ &\leqq ||k_\gamma||_1 \operatorname{Sup}_{\chi \in X} \left| \int_G Th_\beta(-y) (\bar{\chi} f * \bar{\chi} g)(y) dy \right| \\ &= \operatorname{Sup}_{\chi \in X} |Th_\beta * (\bar{\chi} f * \bar{\chi} g)(0)| \\ &= \operatorname{Sup}_{\chi \in X} |T(h_\beta * \bar{\chi} f) * \bar{\chi} g(0)| \\ &\leqq ||T|| \operatorname{Sup}_{\chi \in X} ||h_\beta * \bar{\chi} f||_p \, ||\bar{\chi} g||_{q'} \\ &\leqq ||T|| \, ||f||_p \, ||g||_{q'}. \end{aligned}$$

Therefore

$$||\varphi_\alpha * f||_q \leqq ||T|| \, ||f||_p,$$

and the operators T_{φ_α} satisfy

$$||T_{\varphi_\alpha}|| \leqq ||T||.$$

Since $q \neq 1$, each closed ball of L_p^q is compact in the weak operator topology. The net (T_{φ_α}) therefore has a limiting point $U \in L_p^q$ (for this same topology) with $||U|| \leqq ||T||$. Without loss of generality, suppose that $\lim_\alpha T_{\varphi_\alpha} = U$ in the weak operator topology. But it is easy to see that

$$\lim_\beta \lim_\gamma (\hat{k}_\gamma T h_\beta) * f * g(0) = Tf * g(0)$$

for $f, g \in C_c$ since $\hat{k}_\gamma \to 1$ locally uniformly, (h_β) is an approximate identity, and T commutes with convolution by functions in C_c. Hence $T = U$, and the theorem is proved whenever $q \neq 1$.

Case (ii). G is non-compact, and $p = q = 1$. In this case it is known that $L_1^1 = M_{bd}$ (isometrically and isomorphically) where M_{bd} is the space of bounded Radon measures on G ([7], Theorem 1.4). Suppose then that $\mu \in M_{bd}$, that (h_β) is an approximate identity as in (i), and that (μ_K) is the net of measures in M_{bd} defined by $\mu_K = \xi_K \mu$ where ξ_K is the characteristic function of the compact subset K of G, and the compact sets (K) are directed by set inclusion. Define $\varphi_\alpha = \varphi_{(K,\beta)} = \mu_K * h_\beta$. Clearly, $\varphi_\alpha \in C_c$ for all α. If $f \in L^1$,

$$\begin{aligned} ||\varphi_\alpha * f - \mu * f||_1 &\leqq ||\varphi_\alpha * f - \mu * h_\beta * f||_1 + ||\mu * h_\beta * f - \mu * f||_1 \\ &\leqq ||\mu - \mu_K||\,||f||_1 + ||\mu||\,||h_\beta * f - f||_1. \end{aligned}$$

Here $||\mu - \mu_K||$ and $||h_\beta * f - f||_1$ can be made arbitrarily small by taking K and β "sufficiently large". Finally, $||\mu_K * h_\beta||_{M_{bd}} \leqq ||\mu_K|| \leqq ||\mu||$.

Case (iii). G compact, $1 \leqq p < \infty$, $q = 1$. Let (h_β) be an approximate identity composed of trigonometric polynomials h_β such that $||h_\beta||_1 = 1$. Define $\varphi_\alpha \equiv \varphi_\beta = Th_\beta$, also a trigonometric polynomial. (Observe that $\chi * \chi = \chi$ and $T(\chi * \chi) = T\chi * \chi = (T\chi)^\wedge(\chi) \cdot \chi$ for every $\chi \in X$ quà function on G.) Then if $f \in L^p$,

$$\begin{aligned} ||Th_\beta * f||_1 &= ||h_\beta * Tf||_1 \\ &\leqq ||Tf||_1 \leqq ||T||\,||f||_p \end{aligned}$$

for every β. Further, $\varphi_\beta * f = h_\beta * Tf \to Tf$ in L^1 since (h_β) is an approximate identity. The proof is now complete.

We shall in a moment prove the analogue, for multipliers of type (p, q), of [3], Theorem 1; for this we need the following definition.

DEFINITION. (i) Suppose that G is an LCA group and that $1 \leqq p \leqq q < \infty$. Write $1/r = 1/p - 1/q$. For $p \neq q$, the space A_p^q is defined as the subset of $L^r(G)$ consisting of those functions u which can be written $u = \sum_{i=1}^\infty f_i * g_i$ a.e. with $f_i, g_i \in C_c$ and $\sum ||f_i||_p ||g_i||_{q'} < \infty$.

The space A_p^q will be endowed with the norm

$$||u|| = \operatorname{Inf} \sum ||f_i||_p ||g_i||_{q'},$$

the infimum being taken with respect to all representations $u = \sum f_i * g_i$ a.e. of u with $f_i, g_i \in C_c$ and $\sum ||f_i||_p ||g_i||_{q'} < \infty$.

For $p = q$, A_p^p is defined in an exactly analogous way to that in which A_p^q is defined for $p \neq q$ except that (a) the equality $u = \sum f_i * g_i$ is assumed

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to hold pointwise everywhere; (b) A_p^p is defined as a subset of $C_0(G)$. ($C_0(G)$ is the space of continuous functions on G which vanish at infinity.)

(ii) Suppose that G is compact and that $1 \leqq q < p < \infty$. The space A_p^q is defined as the subset of $C(G)$ consisting of those functions u which can be written $u = \sum_{i=1}^{\infty} f_i * g_i$ with $f_i, g_i \in C$ and $\sum ||f_i||_p ||g_i||_{q'} < \infty$. The norm on A_p^q is defined as in (i).

Note that in case (i), if $u \in A_p^q$ and $u = \sum f_i * g_i$ with $\sum ||f_i||_p ||g_i||_{q'} < \infty$, the series converges in the norm of L^r (resp. uniformly if $p = q$) to u. Indeed,

$$||f_i * g_i||_r \leqq ||f_i||_p ||g_i||_{q'}$$

if $f_i, g_i \in C_c$, $1/r = 1/p - 1/q$, and $p \leqq q$ ([2], Theorem 9.5.1).

In case (ii), the series corresponding to $u \in A_p^q$ converges uniformly to u since $L^p \subset L^q$ and $||f||_q \leqq ||f||_p$ for $f \in L^p(G)$ if $p > q$. Observe also that in both cases A_p^q is a Banach space under the prescribed norm (cf. the proof of [5], Theorem 2.4).

THEOREM 2. *The space L_p^q is isometrically isomorphic to $(A_p^q)'$, the (topological) dual of A_p^q ($1 \leqq p < \infty$, $1 \leqq q < \infty$).*

PROOF. Suppose that $T \in L_p^q$ and define the linear form t on A_p^q by

$$t(u) = \sum T f_i * g_i(0)$$

where $u = \sum f_i * g_i$ is a representation of u as an element of A_p^q. t is well-defined, i.e. $t(u)$ is independent of the particular representation of u chosen. For suppose that $\sum f_i * g_i = 0$ is a representation of 0 as an element of A_p^q. Choose a net (T_{φ_α}) satisfying the conditions of Theorem 1. Then

$$\sum T f_i * g_i(0) = \lim_\alpha \sum \varphi_\alpha * f_i * g_i(0)$$

since the series $s_\alpha = \sum_i \varphi_\alpha * f_i * g_i(0)$ are convergent, uniformly with respect to α, and $\varphi_\alpha * f_i \to T f_i$ in L^q for each i. Again, $\varphi_\alpha \in C_c$ and $\sum f_i * g_i$ converges in L^r (resp. C in case (ii)). Hence

$$\begin{aligned}\sum_i \varphi_\alpha * f_i * g_i(0) &= \sum \int \varphi_\alpha(-x) f_i * g_i(x) dx \\ &= \int \varphi_\alpha(-x) \sum f_i * g_i(x) dx \\ &= 0\end{aligned}$$

and t is well-defined.

Now t is evidently continuous on A_p^q with $||t|| \leqq ||T||$. Further,

$$\begin{aligned}||T|| &= \text{Sup}\,\{|Tf * g(0)| : f, g \in C_c, ||f||_p \leqq 1, ||g||_{q'} \leqq 1\}. \\ &\leqq ||t||\end{aligned}$$

Hence $||T|| = ||t||$.

In order to show that the map $T \to t$ is onto, suppose that $t \in (A_p^q)'$; for $f \in C_c$, define the linear form $g \to t(f * g)$ on C_c. Since $t \in (A_p^q)'$,

$$|t(f * g)| \leqq ||t|| \, ||f||_p ||g||_{q'}. \tag{1}$$

Suppose first that $q' \neq \infty$. Then, since L^q is the dual of $L^{q'}$, there exists a unique element of $L^q(G)$, say Tf, with $Tf * g(0) = t(f * g)$ for $f, g \in C_c$. Clearly, $||Tf||_q \leqq ||t|| \, ||f||_p$ and so T may be extended to map L^p linearly and continuously into L^q with $||T|| \leqq ||t||$; this extended T commutes with translations. For if $y \in G$ and $f, g \in C_c$,

$$\begin{aligned} T(\tau_y f) * g(0) &= t(\tau_y f * g) = t(f * \tau_y g) \\ &= Tf * \tau_y g(0) = \tau_y(Tf) * g(0) \end{aligned}$$

whence it follows that $T(\tau_y f) = \tau_y(Tf)$ for $f \in C_c$; by continuity and the denseness of C_c in L^p, the same equality holds for $f \in L^p$.

On the other hand, if $q' = \infty$ $(q = 1)$, (1) may be written

$$|t(f * g)| \leqq ||t|| \, ||f||_p ||g||_\infty \tag{1'}$$

and it then follows that there exists a unique bounded measure, say Tf, with $Tf * g(0) = t(f * g)$ for $f, g \in C_c$. From (1′), we see that $||Tf|| \leqq ||t|| \, ||f||_p$ for $f \in C_c$, so that T may be extended linearly and continuously to map L^p into M_{bd}. As before, T commutes with translations. We can now show that if $f \in L^p$, then $Tf \in L^1$. For if $y \to 0$ in G, $T(\tau_y f) = \tau_y(Tf) \to Tf$ in M_{bd} and the mapping $y \to \tau_y(Tf)$ is continuous from G into M_{bd}: but this can happen only if Tf is absolutely continuous with respect to Haar measure on G ([6], (19.27)). Hence T maps L^p into L^1 continuously, and commutes with translations. That is, $T \in L_p^1$.

Whatever the value of q in $[1, \infty)$, we have therefore established the existence of $T \in L_p^q$ for which $Tf * g(0) = t(f * g)$ for $f, g \in C_c$. It is easy to see that

$$t(u) = \sum Tf_i * g_i(0)$$

whenever $u = \sum f_i * g_i$ is an element of A_p^q.

The mapping $T \to t$ is thus a norm-preserving linear map of L_p^q onto $(A_p^q)'$. Since norms are preserved, the mapping is one-to-one. The proof of the therem is therefore complete.

Remarks. It is known (cf. [7], Theorem 1.4) that $L_1^1 = M_{bd}$ and $L_1^q = L^q$ if $1 < q < \infty$, the isomorphisms here expressed being isometric. As a corollary of Theorem 2, we deduce that $(A_1^1)' = M_{bd}$ and $(A_1^q)' = L^q$ if $1 < q < \infty$. However, it is possible to prove *directly* that $A_1^1 = C_0$ and that $A_1^q = L^{q'}$ for $1 < q < \infty$, and so to deduce that $(A_1^1)' = M_{bd}$ and that $(A_1^q)' = L^q$ for $1 < q < \infty$.

The starting point for proving that $A_1^1 = C_0$ is the result that

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$C_0 = L^1 * C_0$ (see Hewitt [8]). Suppose that $f \in C_0$ and that $f = g * h$ where $g \in L^1$ and $h \in C_0$. Since C_c is dense in L^1 and in C_0, we may write $g = \sum g_i$ with $\sum ||g_i||_1 < \infty$ and $h = \sum h_i$ with $\sum ||h_i||_\infty < \infty$. Consider the series $g_1 * h_1 + g_1 * h_2 + g_2 * h_2 + g_2 * h_1 + \cdots = \alpha_1 * \beta_1 + \alpha_2 * \beta_2 + \cdots$ say. Clearly,

$$\sum ||\alpha_i||_1 \, ||\beta_i||_\infty \leqq (\sum ||g_i||_1)(\sum ||h_i||_\infty) < \infty.$$

The series therefore converges in C_0 to sum f_1 say. But $(\sum_1^N g_i) * (\sum_1^N h_i) = \sum_1^{N^2} \alpha_i * \beta_i$, and as $N \to \infty$, $(\sum_1^N g_i) * (\sum_1^N h_i) \to g * h = f$. Hence $f = f_1$. In a similar way, one uses the result $L^{q'} = L^1 * L^{q'}$ (see [8]) to prove that $A_1^q = L^{q'}$ if $1 < q < \infty$.

As an alternative approach to proving that $A_1^1 = C_0$, one may use the fact that $(A_1^1)' = L_1^1 = M_{bd}$ in the following way: A_1^1 is a dense vector subspace of C_0 and therefore has the same dual (viz. M_{bd}) when endowed with the uniform norm as it does with its usual norm. It then follows from a result of Fichtenholtz ([2], Exercise 8.9) that the uniform norm and the usual norm on A_1^1 are equivalent; but A_1^1 is complete under its usual norm, so $A_1^1 = C_0$. A similar argument applies to A_1^q with $1 < q < \infty$.

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