Can. J. Math., Vol. XXXVII, No. 6, 1985, pp. 1176-1200

ESTIMATES FOR SOLUTIONS OF WAVE EQUATIONS WITH VANISHING CURVATURE

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1. Introduction. The solution of the Cauchy problem for a hyperbolic partial differential equation leads to a linear combination of operators T_t of the form

$$\widehat{T_t f}(\xi) = m(\xi)\exp(it\lambda(\xi))\hat{f}(\xi).$$

For example, the solution of the initial value problem

$$u_{tt} - \Delta_x u = 0 \quad (x, t) \in \mathbf{R}^n \times (0, \infty)$$

$$u(x, 0) = 0 \quad u_t(x, 0) = f(x)$$

is given by $u(x, t) = T_t f(x)$ where

$$\widehat{T_t f}(\xi) = |\xi|^{-1}\sin(t|\xi|)\hat{f}(\xi).$$

Peral proved in [**11**] that T_t is bounded from $L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ if and only if

$$1/2 - 1/(n-1) \leqq 1/p \leqq 1/2 + 1/(n-1) \quad (1 \leqq p \leqq \infty).$$

From the homogeneity, the operator norm satisfies $||T_t|| \leqq Ct$ for all $t > 0$. If $\lambda(\xi)$ is positively homogeneous of degree one then the same result is true for the multiplier $\sin(t\lambda(\xi))/\lambda(\xi)$ as long as the Gaussian curvature of

$$\Sigma = \{\xi:|\lambda(\xi)| = 1\}$$

does not vanish and L^1 and L^∞ are replaced by H^1 and BMO.

When there are lower order terms present the decay rate of the operator norm $||T_t||$ changes significantly. For the Klein-Gordon equation,

$$u_{tt} - \Delta_x u + u = 0,$$

the Fourier multiplier is $\sin(t\sqrt{1 + |\xi|^2})(1 + |\xi|^2)^{-1/2}$ and

$$||T_t|| \leqq Ct^{-n|p-2|/2p} \quad (t \geqq 1).$$

This result appears in [**8**] and the nonradial case is in [**9**].

The purpose of this paper is to prove results like these for the case when the curvature of the surface Σ vanishes. Estimates will also be obtained for T_t as an operator from L^p to $L^{p'}$.

Received June 25, 1984. This research was supported by NSERC Grant U0074.

At the heart of these results are estimates obtained for the Fourier transform of measures $d\mu$ supported on Σ. In **[10]** two types of estimates are obtained in $\widehat{d\mu}$. The first type concerns the behavior of the spherical averages of $\widehat{d\mu}(x)$:

(1) $\quad \widehat{d\mu}(x) = |x|^{-(n-1)/2}\mathscr{P}(x') + h(x), \quad x = |x|x'.$

The function $\mathscr{P}$ is integrable over the unit sphere and the averages of h over the spheres $\{ |x| = R\}$ decrease faster than $CR^{-(n-1)/2}$. The second type of estimate is one of the form

(2) $\quad |\widehat{d\mu}(x)| \leqq C(1 + |x|)^{-\nu}.$

When the curvature of Σ does not vanish the constant ν in (2) equals $(n - 1)/2$ but when the curvature does vanish $\nu < (n - 1)/2$.

The estimates of T_t from L^p to L^p involve showing that an operator $(I - \Delta)^{-z/2}T_t$ is bounded on H^1. Since this means calculating the L^1 norm of its kernel, estimate (1) is well-suited to this situation. The advantage of estimate (1) is that at least in an average sense $\widehat{d\mu}$ decays as rapidly as the case where the curvature of Σ does not vanish. In fact the obstruction to better results is the lower order term $h(x)$ and not the main term. For the L^p to $L^{p'}$ estimates it is necessary to calculate the L^∞ or BMO norm of the kernel for $(I - \Delta)^{-z/2}T_t$. In this case, inequality (2) seems more natural.

2. The estimates for $\widehat{d\mu}$. A function f on $\mathbf{R}^{n-1}$ will be said to be of type τ if it satisfies the following conditions:

(a) $f(0) = 0$, $\nabla f(0) = 0$, and $f(y) = P(y) + h_*(y)$ for y in a neighborhood of the origin.

(b) there is a direct sum of orthogonal subspaces $V_1, \dots, V_s$ and polynomials $P_1, \dots, P_s$ homogeneous of degree $k_1, \dots, k_s$ respectively such that $V_1 \oplus \dots \oplus V_s = \mathbf{R}^{n-1}$ and

$$P(y) = P(y_1, \dots, y_s) = \sum_{j=1}^{s} P_j(y_j) \quad y_j \in V_j, j = 1, \dots, s.$$

(c) for every $j = 1, \dots, s$, $\det d^2P_j(y_j) = 0$ implies $y_j = 0$.

(d) the function h_* contains only higher order terms y^β such that for every $j = 1, \dots, s$, y^β is either independent of y_j or in the variables of V_j, y^β has homogeneity $\equiv \beta_j \geqq k_j$. Also $\Sigma' \beta_j > \Sigma' k_j$ where the sums are over those j where y^β is not independent of y_j.

Define

$$\tau = \min\{ (\dim V_j)/(k_j - 1) : k_j \neq 2\} \quad \text{if } \det d^2f(0) = 0,$$

and

$$\tau = 2 \quad \text{if } \det d^2f(0) \neq 0.$$

https://doi.org/10.4153/CJM-1985-064-9 Published online by Cambridge University Press

For example, the function

$$f(y) = y_1^{k_1} + y_2^{k_2}$$

is of type $\tau = 1/(k_2 - 1)$ if $2 < k_1 \leqq k_2$.

A point ξ' on the surface Σ is of type $\tau = \tau(\xi')$ if after a translation and an orthogonal change of coordinates in $\mathbf{R}^n$ the surface near ξ' can be put in the form $y_n = f(y)$ where f is a function of type τ on $\mathbf{R}^{n-1}$. The surface Σ will be of type τ_0 if every point ξ' on Σ is of type $\tau = \tau(\xi')$ for some τ and

$$\tau_0 = \inf\{\tau(\xi'):\xi' \in \Sigma\} > 0.$$

Let $\kappa(\xi')$ be the Gaussian curvature of Σ at ξ' and define

$$A(x) = \{\xi' \in \Sigma: \text{the tangent plane at } \xi' \text{ is perpendicular to } x\}.$$

Suppose that the surface near ξ' is transformed into $y_n = f(y)$ in such a way that the unit normal vector at ξ' pointing in the direction of x is mapped into $(0, -1) \in \mathbf{R}^{n-1} \times \mathbf{R}$. Let $\gamma = \gamma(\xi')$ be the number of positive eigenvalues of the matrix $d^2f(0)$ minus the number of negative eigenvalues. Let $d\omega$ be surface area on Σ, $g \in C^\infty(\Sigma)$. For any such function $g \in C^\infty(\Sigma)$ set

$$\mathscr{P}(g)(x) = \sum_{\xi' \in A(x)} g(\xi')e^{i\gamma\pi/4}e^{-ix\cdot\xi'}(2\pi)^{(n-1)/2}|\kappa(\xi')|^{-1/2}$$

where $\xi' \in \Sigma$.

Define $\tau_1 = \frac{1}{2}\min(\tau_0, 1)$ if Σ is not convex, and $\tau_1 = 1/2$ if Σ is convex.

THEOREM 1. **[10]** *Suppose that Σ is a compact $(n - 1)$-dimensional C^∞ submanifold of $\mathbf{R}^n$ of type $\tau_0 > 0$, $d\omega$ is surface area on Σ, $g \in C^\infty(\Sigma)$, $d\mu = gd\omega$, and for every $x \in \mathbf{R}^n$, $A(x)$ is a finite subset of Σ. Then for every $\tau < \tau_1$ there exist positive constants C_1 and C_2 such that*

$$R^{-(n-1)}\int_{|x|=R} |\widehat{d\mu}(x) - R^{-(n-1)/2}\mathscr{P}(g)(x)|dx \leqq C_1R^{-(n-1+2\tau)/2}$$

for all $R > 0$ and

$$R^{-(n-1)}\int_{|x|=R} |\mathscr{P}(g)(x)|dx \leqq C_2 \quad \textit{for all } R > 0.$$

If $\tau_0 > 1/2$ then this theorem holds for $\tau = 1/2$.

If the curvature of Σ does not vanish then the theorem holds for $\tau = 1$ (**[5]** or **[6]**). If Σ is not convex it seems unlikely that the theorem would hold for every $\tau < 1$. Near an inflection point of Σ in $\mathbf{R}^2$, $\widehat{d\mu}$ has a significant secondary term. For example, if the surface is given locally by

$\xi_2 = \xi_1^3$ then $A((\epsilon, 1)) = \emptyset$ for every $\epsilon > 0$ even though $(\epsilon, 1)$ is close to being perpendicular to the surface. It is the possibility of this type of situation that is reflected in the parameter τ_1 in Theorem 1. If Σ is convex there are no inflection points and $\tau_1 = 1/2$. It may be possible to improve this to $\tau_1 = 1$.

Let S be the unit sphere in $\mathbf{R}^n$. The proof of Theorem 1 in **[10]** generalizes easily to the case where $g \in C^\infty(S \times \Sigma)$. If $x = rx'$, $(x', \xi') \in S \times \Sigma$, then the type of function encountered in Theorems 3 and 4 is of the form

$$g(x', \xi') = (x' \cdot \xi')^k \tilde{g}(\xi')$$

where $\tilde{g} \in C^\infty(\Sigma)$.

We will describe now the phase function $\lambda(\xi)$.

(i) λ_* is a real-valued function, positively homogeneous of degree one, $\lambda_* \in C^\infty(\mathbf{R}^n - \{0\})$, λ_* has no zeros in $\mathbf{R}^n - \{0\}$. For convenience we assume $\lambda_* \geqq 0$. Let $0 < \tau_2 \leqq 1$. Assume that $\Sigma = \{\xi : \lambda_*(\xi) = 1\}$ is a surface for which

$$R^{-(n-1)} \int_{|x|=R} |\widehat{d\mu}(x) - R^{-(n-1)/2}\mathscr{P}(g)(x)|dx \leqq C_1 R^{-(n-1-2\tau_2)/2}$$

for all $R > 0$, $g \in C^\infty(S \times \Sigma)$.

its leading term at infinity and

$$|\lambda_*(\xi)| \leqq C|\lambda(\xi)| \quad \text{for all } \xi \in \mathbf{R}^n.$$

Also

$$|D^\beta(\lambda - \lambda_*)(\xi)| \leqq C_\beta |\xi|^{-|\beta|}$$

and

$$|D^\beta \lambda(\xi)| \leqq C_\beta |\lambda(\xi)|\, |\xi|^{-|\beta|}$$

for every multi-index β.

(iii) Assume that there is a smooth nonnegative function σ on $\mathbf{R}$ and a constant $L \geqq 1$ such that $\lambda(\xi) = \sigma(\lambda_*(\xi))$, $\sigma(r) - r \to 0$ as $r \to \infty$,

$$C_1(1 + r)^{-L-2} \leqq \left|\frac{d^2\sigma}{dr^2}\right| \leqq C_2(1 + r)^{-L-2}$$

$$\left|\frac{d^k\sigma}{dr^k}(r)\right| \leqq C_k(1 + r)^{-L-2} \; r > 0, k \geqq 2.$$

$d\sigma/dr$ has a zero of order at most one at the origin and has no other zeros.

The assumptions that λ and λ_* be positive are for convenience only. The same proofs hold for negative phase functions. Condition (iii) implies that the level surfaces of λ are all dilates of Σ. This is not strictly necessary but it greatly simplifies the assumptions and proofs.

For the Klein-Gordon equation,

$$\lambda(\xi) = \sqrt{1 + |\xi|^2}, \lambda_*(\xi) = |\xi|, \sigma(r) = \sqrt{1 + r^2}, \text{ and } L = 1.$$

An example of an equation where the surface Σ is no longer convex is given by the homogeneous operator

$$(D_t^2 - 4D_{x_1}^2 - D_{x_2}^2)(D_t^2 - D_{x_1}^2 - 4D_{x_2}^2) - \epsilon(D_{x_1}^2 + D_{x_2}^2)^2$$

where the constant $\epsilon > 0$ is chosen small enough that the four roots of the characteristic equation

$$(\tau^2 - 4\xi_1^2 - \xi_2^2)(\tau^2 - \xi_1^2 - 4\xi_2^2) - \epsilon(\xi_1^2 + \xi_2^2)^2 = 0$$

are distinct for every $\xi = (\xi_1, \xi_2)$. The two positive roots are given by homogeneous functions $\tau = \lambda_1(\xi)$ and $\tau = \lambda_2(\xi)$. The graphs of

$$\Sigma_j = \{\xi:\lambda_j(\xi) = 1\} \quad (j = 1, 2)$$

are given in Figure 1. The shapes of the corresponding wave surfaces are drawn in Figure 2.

Examples arise more naturally in the case of elastic waves in $\mathbf{R}^3$ (See [**1**], [**3**]). In this case the characteristic equation has six roots. The three positive roots lead to surfaces Σ_1, Σ_2 and Σ_3. The estimates of this paper deal with the "regularly hyperbolic" equations, in which these surfaces are disjoint. In [**3**], Duff uses a perturbed equation similar to the one in Figure 1 to examine the singular case where the surfaces intersect. It is not clear however what such a perturbation will do to the L^p estimates of this paper.

3. The L^p estimates. Let $V^p(\mathbf{R}^n) = L^p(\mathbf{R}^n)$ if $1 < p < \infty$, $V^1 = H^1$, and $V^\infty = \text{BMO}$.

THEOREM 2. *Let T_t be the transformation with Fourier multiplier*

$$m(\xi) = \sin(t\lambda(\xi))/\lambda(\xi)$$

where $\lambda(\xi)$ satisfies (i) *and* (ii). *Then T_t is a bounded linear operator from $V^p(\mathbf{R}^n)$ to $V^p(\mathbf{R}^n)$ if*

$$\frac{1}{2} - \frac{1}{n + 1 - 2\tau_2} < \frac{1}{p} < \frac{1}{2} + \frac{1}{n + 1 - 2\tau_2} \tag{3}$$

where $\tau_2 < 1$ is the constant in (i). *The operator norm of T_t satisfies*

$$\|T_t\| \leqq Ct \quad \textit{for all } 0 < t \leqq 1.$$

From (3) it is evident that Theorem 1 with $\tau_2 = 1$ would give the same range of p as when the curvature of Σ does not vanish, except that the endpoints would be missing.

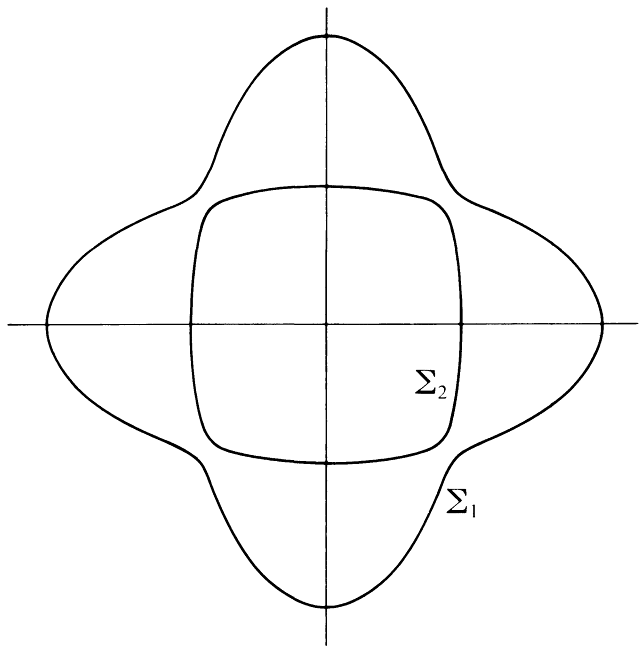

Figure 1

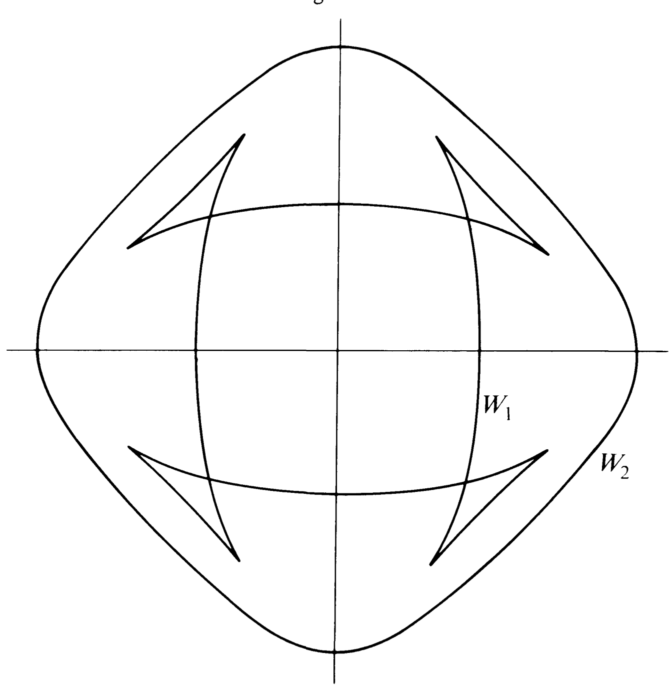

Figure 2

Proof. We will first show that the inhomogeneous case follows from the estimates where λ is homogeneous. Suppose λ^* is the homogeneous part of λ. Then

$$\frac{\sin(t\lambda)}{\lambda} = \frac{\sin(t\lambda^*)}{\lambda^*}\left\{\cos(t(\lambda - \lambda^*))\frac{\lambda^*}{\lambda}\right\} + \left\{\frac{\cos(t\lambda^*)\sin(t(\lambda - \lambda^*))}{\lambda}\right\}$$

$$\equiv \frac{\sin(t\lambda^*)}{\lambda^*}m_1(\xi) + m_2(\xi).$$

Since m_1 satisfies

(4) $\quad |D_\xi^\beta m_1(\xi)| \leqq C_\beta|\xi|^{-|\beta|}$

where C_β is independent of t, then m_1 is a bounded multiplier on L^p, $1 < p < \infty$, and on H^1 and BMO. Similarly,

$$|D_\xi^\beta m_2(\xi)| \leqq C_\beta t|\xi|^{-|\beta|} \quad 0 < t \leqq 1.$$

Therefore to prove Theorem 2 it suffices to consider a homogeneous phase function $\lambda(\xi)$. In fact, if $\lambda(\xi)$ is homogeneous then we may assume that $t = 1$.

By the Plancherel Theorem, $(I - \Delta)^{1/2}T_1$ is bounded from $L^2(\mathbf{R}^n)$ to itself. We will prove that $(I - \Delta)^{-(n-1-2\tau)/4}T_1$ is bounded on $H^1(\mathbf{R}^n)$ for every $\tau < \tau_2$. Since $(I - \Delta)^{iy}$ is a bounded linear operator on $H^1(\mathbf{R}^n)$ for $y \in R$, an interpolation using the analytic family of operators $(I - \Delta)^z T_1$ shows that T_1 is a bounded linear operator from $L^p(\mathbf{R}^n)$ to itself where

$$\frac{1-\sigma}{2} + \frac{\sigma}{1} = \frac{1}{p} \quad \text{and} \quad (1 - \sigma) + \sigma(-1)(n - 1 - 2\tau)\frac{1}{2} = 0.$$

This is equivalent to

$$\frac{1}{p} = \frac{1}{2} + \frac{\sigma}{2} = \frac{1}{2} + \frac{1}{n + 1 - 2\tau}.$$

The corresponding estimates for $2 < p < \infty$ follow from duality.

If K_0 is the kernel for the transformation $(I - \Delta)^{-(n-1-2\tau)/4}T_1$ then

$$\hat{K}_0(\xi) = \frac{(1 + (\lambda(\xi))^2)^{-(n-1-2\tau)/4}\sin(\lambda(\xi))}{\lambda(\xi)} \times \left(\frac{1 + (\lambda(\xi))^2}{1 + |\xi|^2}\right)^{(n-1-2\tau)/4}$$

Since the last expression on the right satisfies (4) it suffices to consider $K(x)$ where

$$\hat{K}(\xi) = (1 + (\lambda(\xi))^2)^{-(n-1-2\tau)/4}\sin(\lambda(\xi))/\lambda(\xi).$$

It is natural to calculate the inverse Fourier transform of $\hat{K}$ by integrating first over the surfaces $\{\xi:\lambda(\xi) = r\}$. Let $\Sigma = \{\xi:\lambda(\xi) = 1\}$.

$$K(x) = \int_0^\infty \int_\Sigma e^{ix\cdot\xi}\frac{\sin r}{(1 + r^2)^{(n-1-2\tau)/4}r}r^{n-1}g(\xi')d\xi' dr$$

where $d\xi'$ is surface area on Σ and $r^{n-1}g(\xi')$ is the Jacobian for the change of coordinates.

Because λ is smooth in $\mathbf{R}^n - \{0\}$ and positively homogeneous of degree one then

$$\xi' \cdot \nabla\lambda(\xi') = \lambda(\xi') = 1 \quad \text{for any } \xi' \in \Sigma.$$

Since $\xi' \cdot \nabla\lambda(\xi')/|\xi'|\,|\nabla\lambda(\xi')|$ is the cosine of the angle between the radius that

$$g(\xi') = \xi' \cdot \nabla\lambda(\xi')/|\xi'|\,|\nabla\lambda(\xi')| = (|\xi'|\,|\nabla\lambda(\xi')|)^{-1}.$$

If $d\mu = gd\xi'$ then

$$\widehat{d\mu}(-rx) = \int_\Sigma e^{ix\cdot\xi}g(\xi')d\xi' \quad \xi = r\xi'$$

and

$$(5) \quad K(x) = \int_0^\infty \widehat{d\mu}(-rx)(\sin r)(1 + r^2)^{-(n-1-2\tau)/4}r^{n-2}dr.$$

By (i),

$$\widehat{d\mu}(-rx) = (r|x|)^{-(n-1)/2}\mathcal{P}(-rx) + h(-rx)$$

where

$$\frac{1}{R^{n-1}}\int_{|x|=R}|h(-rx)|dx \leqq C(Rr)^{-(n-1)/2-\tau_2}.$$

Therefore

$$(6) \quad \int_{|x|=R}|K(x)|dx = \int_{|x|=R}\left|\int_0^\infty \mathcal{P}(-rx)\frac{(\sin r)r^{(n-3)/2}}{r_1^{(n-1-2\tau)/2}}dr\right|dx + H(R)$$

where $r_1 = (1 + r^2)^{1/2}$ and

$$(7) \quad |H(R)| \leqq C\int_0^\infty R^{n-1}\frac{(Rr)^{-(n-1)/2-\tau_2}}{r_1^{(n-1-2\tau)/2}}r^{n-2}dr$$

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$$\leqq CR^{(n-1)/2-\tau_2}$$

since $\tau < \tau_2$.

By Theorem 1,

$$\mathscr{P}(-rx) = \sum_{\xi' \in A(x)} e^{i\gamma\pi/4} e^{ix\cdot\xi'} (2\pi)^{(n-1)/2} |K(\xi')|^{-1/2} (|\xi'|\,|\nabla\lambda(\xi')|)^{-1}$$

where $A(x) = A(-x)$ is the set of points in $\Sigma = \{\xi:\lambda(\xi) = 1\}$ such that the normal to Σ at ξ is parallel to x. Thus the main term of (6) can be written as

$$(8)\quad \int_{|x|=R} \Bigg| \sum_{\xi' \in A(x)} \frac{e^{i\gamma\pi/4}(2\pi)^{(n-1)/2}}{R^{(n-1)/2}|\xi'|\,|\nabla\lambda(\xi')|\,|\kappa(\xi')|^{1/2}} \times \int_0^\infty e^{ix\cdot\xi'} \frac{(\sin r)r^{(n-3)/2}}{r_1^{(n-1-2\tau)/2}} dr \Bigg| dx$$

$$\leqq \frac{C}{R^{(n-1)/2}} \int_{|x|=R} \sum_{\xi' \in A(x)} |\kappa(\xi')|^{-1/2} \times \left| \int_0^\infty e^{ix\cdot\xi'} \left(\frac{r}{r_1}\right)^{(n-3)/2} \frac{\sin r}{r_1^{1-\tau}} dr \right| dx.$$

If $\tau < 1$ then an integration by parts shows that

$$(9)\quad \left| \int_0^\infty e^{ix\cdot\xi'} \left(\frac{r}{r_1}\right)^{(n-3)/2} \sin r \frac{dr}{r_1^{1-\tau}} \right| \leqq \left| \int_0^1 \right| + \left| \int_1^\infty \right| \leqq C.$$

It is a consequence of Theorem 1 that

$$\int_{|x|=R} \sum_{\xi' \in A(x)} |\kappa(\xi')|^{-1/2} dx \leqq CR^{n-1}.$$

This combined with (7) shows that

$$\int_{|x|=R} |K(x)|dx \leqq cR^{(n-1)/2} + cR^{(n-1)/2-\tau}.$$

Therefore

$$(10)\quad \int_{|x|\leqq 1} |K(x)|dx \leqq C.$$

The estimates obtained thus far take care of the region $\{|x| \leqq 1\}$. If $|x| > 1$ then we begin by integrating by parts in (5). To integrate

$$\widehat{d\mu}(-rx) = \int_\Sigma e^{ix\cdot\xi}(|\xi'|\,|\nabla\lambda(\xi')|)^{-1} d\xi'$$

with respect to r it is convenient to introduce a partition of unity on Σ. Suppose that x is restricted to a narrow cone Γ. The cone Γ is chosen so narrow that $|x_1 \cdot \xi_2'|/R$ is bounded away from zero for $\xi_2' \in A(x_2)$, $x_1 \in \Gamma$, $x_2 \in \Gamma$. This is possible since if $\xi' \in A(x)$ then

$$|x \cdot \xi'|/R \geqq C > 0.$$

Clearly $\mathbf{R}^n$ can be written as a finite union of such cones. Suppose that η is a C^∞ function on Σ that equals one in a neighborhood of $\{\xi' : \xi' \in A(x)$ for some $x \in \Gamma\}$ and η is supported away from the set $\{\xi : x \cdot \xi' = 0$ for some $x \in \Gamma\}$. Then since $(1 - \eta)$ is supported in the part of Σ that is transverse to planes where $x \cdot \xi$ is constant, we have

$$\left| \int_\Sigma e^{ix\cdot\xi}(1 - \eta)(\xi')(\,|\xi'|\,|\nabla\lambda(\xi')\,|\,)^{-1}d\xi' \right| \leqq C_N(1 + Rr)^{-N}$$

for any $N > 0$. Also, by integrating

$$(13) \quad \int_0^\infty \widehat{d\tilde{\mu}}(-rx)\sin(rx)\frac{r^{n-2}}{r_1^{(n-1-2\tau)/2}}dr$$
$$= \frac{1}{(x \cdot \xi)^k} \int_0^\infty \widehat{d\tilde{\mu}}(-rx)\frac{d^k}{dr^k}\left\{\frac{(\sin r)r^{n-2}}{r_1^{(n-1-2\tau)/2}}\right\}dr$$

where $d\tilde{\mu} = \eta d\mu$. The integral in (13) is similar to (5) except for the factor

$$|\,(x \cdot \xi')\,|^{-k} \leqq cR^{-k}.$$

Therefore, calculating as before and summing over the cones Γ gives

$$\int_{|x|=R} |K(x)\,|dx \leqq cR^{-k}\{cR^{(n-1)/2} + cR^{(n-1)/2-\tau}\}.$$

If k is chosen large enough this shows that K is integrable over the region $\{\,|x| \geqq 1\}$. This together with (10) completes the proof of Theorem 2.

The obstacle to getting a bounded operator on a large range of p is the error term $h(x)$ in Theorem 1. To further illustrate this we will calculate the kernel $K(x)$ if $\tau_2 = 1$ in a simple case: $n = 3$ and Σ is convex and symmetric with respect to the origin. That is, $x \in \Sigma$ implies $-x \in \Sigma$. Since the integral in (8) does not make sense when $\tau_2 = 1$, K must be considered as the distributional inverse Fourier transform of the function

$$m(\xi) = \sin(\lambda(\xi)\,)/\lambda(\xi).$$

If φ is any C^∞ function in the Schwartz class $\mathcal{S}$,

$$(14) \quad \int_{\mathbf{R}^n} K(x)\varphi(x)dx = (2\pi)^{-n}\int_{\mathbf{R}^n} m(\xi)\hat{\varphi}(\xi)d\xi$$
$$= (2\pi)^{-3}\int_0^\infty \frac{\sin r}{r}\int_{\mathbf{R}^3}\varphi(x)$$

$$\times \int_\Sigma e^{-ix\cdot\xi} \frac{d\xi'}{|\xi'|\,|\nabla\lambda(\xi')|} dx r^2 dr.$$

According to Theorem 1 the principal part of the integral over Σ is

$$(15) \quad \mathscr{P}(x) = \sum_{\xi' \in A(x)} \frac{2\pi e^{i\pi\gamma/4} e^{-ix\cdot\xi}}{r|x|\,|\xi'|\,|\nabla\lambda(\xi')|\,|K(\xi')|^{1/2}}.$$

Since the multiplier is even we may assume that $\lambda \geqq 0$. We will show that this part of $K(x)$ is a measure supported on the wave surface corresponding to

$$\Sigma = \{\xi : \lambda(\xi) = 1\}.$$

The wave surface is the set

$$W = \{x = \nabla\lambda(\xi) : \lambda(\xi) = 1\}.$$

Except where the curvature of Σ at ξ vanishes the corresponding part of W is a smooth $(n - 1)$-dimensional manifold in $\mathbf{R}^n$ that crosses each ray from the origin at most once. The points of zero curvature in Σ correspond to cusps in W, as in Figures 1 and 2. See also [**1**] and [**3**]. In the present calculation, since Σ is convex, W is star-shaped with respect to the origin. Therefore there is a function $\lambda^{\#}$ that is positively homogeneous of degree one in $\mathbf{R}^n$ such that

$$W = \{x : \lambda^{\#}(x) = 1\}.$$

The wave surface W is dual to the "slowness surface" Σ in the sense that

$$\Sigma = \{\xi = \nabla\lambda^{\#}(x) : \lambda^{\#}(x) = 1\}.$$

Consider the Gauss map

$$\xi' \to \theta = \nabla\lambda(\xi')/|\nabla\lambda(\xi')|$$

from Σ to the unit sphere. The Gaussian curvature measures the change of area: $\kappa(\xi')d\xi' = d\theta$ where $d\theta$ is surface area on the unit sphere. Because λ is homogeneous of degree one,

$$\xi' \cdot \nabla\lambda(\xi') = \lambda(\xi') = 1.$$

Since the cosine of the angle between ξ' and $x' = \nabla\lambda(\xi')$ is

$$\xi' \cdot \nabla\lambda(\xi')/|\xi'|\,|\nabla\lambda(\xi')| = 1/|\xi'|\,|\nabla\lambda\xi')|$$

then $dx' = |x'|\,|\xi'|d\theta$. Therefore

$$dx' = \kappa(\xi')\,|x'|\,|\xi'|d\xi$$

where $d\xi'$, dx' are surface area on Σ and W respectively. Similarly using the function $\lambda^{\#}$ associated to the wave surface,

$$d\xi' = \kappa^{\#}(x')\,|x'|\,|\xi'|dx'$$

where $\kappa^{\#}$ is the curvature on W. Hence

(16) $\kappa(\xi')\kappa^{\#}(x')\,|x'|^2|\xi'|^2 = 1.$

This argument leading to (16) is taken from **[3]**. Because of the duality between Σ and W and the fact that Σ is convex, $\nabla\lambda$ and $\nabla\lambda^{\#}$ are inverses. Therefore it follows from the definition of $A(x)$ that

$$\begin{aligned} A(x) &= \{\xi' \in \Sigma : \nabla\lambda(\xi') \text{ is parallel to } x\} \\ &= \{\nabla\lambda^{\#}(x'), \nabla\lambda^{\#}((-x)')\}. \end{aligned}$$

Also the convexity of Σ implies that

$$\gamma(\nabla\lambda^{\#}((\pm x)')) = \pm 2.$$

Finally,

$$x \cdot \xi' = \pm\lambda^{\#}(\pm x)(\pm x)' \cdot \nabla\lambda^{\#}((\pm x)') = \pm\lambda^{\#}(\pm x).$$

Since Σ is symmetric this last expression is

$$\pm\lambda^{\#}(x) \equiv \pm\rho.$$

Putting all this information into (15) shows that

$$\begin{aligned} \mathscr{P}(x) &= \frac{2\pi|\kappa(x')|^{1/2}}{r|x|}\{e^{-ir\rho+i\pi/2} + e^{ir\rho-i\pi/2}\} \\ &= \frac{4\pi|\kappa(x')|^{1/2}}{r\rho|x'|}\sin(r\rho). \end{aligned}$$

Let

$$\Phi(\rho) = \frac{\rho}{2\pi^2}\int_W \varphi(\rho x')\frac{|\kappa(x')|^{1/2}dx'}{|x'|^2|\nabla\lambda^{\#}(x')|} \quad \text{if } \rho > 0$$

and $\Phi(\rho) = 0$ if $\rho \leqq 0$. Then the part of (14) that is associated with $\mathscr{P}$ is

$$\begin{aligned} &(2\pi)^{-3}\int_0^\infty \sin r \int_{\mathbf{R}^3} \varphi(x)\frac{4\pi|\kappa(x')|^{1/2}}{\rho|x'|}\sin(r\rho)dxdr \\ &= \int_0^\infty \sin r \int_0^\infty \Phi(\rho)\sin(r\rho)d\rho dr \\ &= -\frac{1}{2i}\int_{-\infty}^\infty \sin r\,\hat{\Phi}(r)dr = \frac{\pi}{2}(\Phi(1) - \Phi(-1)) = \frac{\pi}{2}\Phi(1) \\ &= \frac{1}{4\pi}\int_W \varphi(x')\frac{|\kappa(x')|^{1/2}dx'}{|x'|^2(\nabla\lambda^{\#}(x')|}. \end{aligned}$$

This shows that the part of K associated with $\mathscr{P}$ is a measure on W and hence is a bounded operator on $L^1(\mathbf{R}^3)$. This seems to suggest that T_t might be bounded for the full range $1 \leqq p \leqq \infty$. The problem is in knowing how to take care of h.

THEOREM 3. *Let T_t be the transformation with Fourier multiplier*

$$m(\xi) = e^{it\lambda(\xi)}m_1(\xi)$$

where $\lambda(\xi)$ satisfies (i), (ii), (iii) *and $m_1(\xi)$ is such that for every β,*

$$(17) \quad |D_\xi^\beta m_1(\xi)| \leqq C_\beta(1 + |\xi|)^{-1-|\beta|}.$$

Then T_t is a bounded linear operator from $V^p(\mathbf{R}^n)$ to $V^p(\mathbf{R}^n)$ if p satisfies (3). *The operator norm of T_t is*

$$(18) \quad \|T_t\| \leqq C_\alpha t^{\alpha|1/p-1/2|} \quad \textit{for } 1 \leqq t < \infty$$

where $\alpha = n$ if $\tau_2 > 1/2$ and $\alpha > n + 1 - 2\tau_2$ if $\tau_2 \leqq 1/2$.

If on the other hand $\lambda(\xi)$ is positively homogeneous of degree one and satisfies (i) *then T_t is bounded on V^p for p in the interval of* (3) *and*

$$\|T_t\| \leqq Ct \quad 1 \leqq t < \infty.$$

Proof. If $n = 1$, the problem of the curvature of Σ does not arise. We will therefore assume that $n \geqq 2$.

Suppose that $\lambda(\xi)$ is homogeneous. By composing with a multiplier satisfying (4) we see that it suffices to consider the multiplier

$$m(\xi) = e^{it\lambda(\xi)}(1 + |\lambda(\xi)|^2)^{-1/2}.$$

But

$$m(\xi) = \frac{\cos(t\lambda(\xi))}{(1 + (\lambda(\xi))^2)^{1/2}} + \frac{i\sin(t\lambda(\xi))}{\lambda(\xi)}\frac{\lambda(\xi)}{(1 + |\lambda(\xi)|^2)^{1/2}}.$$

From Theorem 2 and (4) it is clear that the second expression on the right is a bounded multiplier on $V^p(\mathbf{R}^n)$. For the first expression it is necessary to show that

$$\cos(\lambda(\xi))(\lambda(\xi)^2 + t^2)^{-1/2}$$

is bounded on V^p with norm independent of t for $t \geqq 1$. This calculation is similar to the one carried out for $\sin(\lambda(\xi))/\lambda(\xi)$ in the proof of Theorem 2.

Now suppose that $\lambda(\xi)$ is not homogeneous. Let $\varphi(s)$ be a C^∞ function of compact support on $\mathbf{R}$ such that φ is identically equal to one in a neighborhood of the origin. Then

$$(19) \quad \frac{e^{it\lambda}}{(1 + |\xi|^2)^{1/2}} = \varphi\left(\frac{\lambda_*}{t}\right)\frac{e^{it\lambda}}{(1 + |\xi|^2)^{1/2}}$$

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$$+ \left(1 - \varphi\left(\frac{\lambda_*}{t}\right)\right) e^{it(\lambda - \lambda_*)} \frac{e^{it\lambda_*}}{(1 + |\xi|^2)^{1/2}}.$$

The multiplier $\left\{1 - \varphi\left(\frac{\lambda_*}{t}\right)\right\} \exp(it(\lambda - \lambda_*))$ satisfies (4). Therefore an application of the homogeneous part of this theorem shows that the second part of (19) is a bounded multiplier on L^p with operator norm $\leqq Ct \leqq Ct^{n/2}$. This leaves just the first multiplier in (19). The rest of the proof follows from the next lemmas.

LEMMA 1. *Suppose that $z > (n + 1)/2 - \tau_2$ and $\tau_2 > 1/2$. The operator with Fourier multiplier $\varphi(\lambda_*/t)e^{it\lambda}(1 + \lambda_*^2)^{-z/2}$ is bounded from $H^1(\mathbf{R}^n)$ to $H^1(\mathbf{R}^n)$ with norm $\leqq Ct^{n/2}$ for $t \geqq 1$.*

If $\tau_2 \leqq 1/2$ then for every $\tau < \tau_2$ the operator is bounded on $H^1(\mathbf{R}^n)$ with norm $\leqq C_\tau t^{(n+1)/2-\tau}$ for $t \geqq 1$.

LEMMA 2. *If $\tau_2 > 1/2$ and $n = 2$ then $\varphi(\lambda_*/t)e^{it\lambda}(1 + \lambda_*^2)^{-1/2}$ is bounded on $H^1(\mathbf{R}^n)$ with norm $\leqq Ct$ for $t \geqq 1$.*

Lemma 2 completes the lower dimensional case $n = 2$. The operator in Lemma 1 is bounded on $L^2(\mathbf{R}^n)$ if $\operatorname{Re} z = 0$. Therefore an interpolation proves that

$$\varphi\left(\frac{\lambda_*}{t}\right)\exp(it\lambda)(1 + \lambda^2)^{-1/2}$$

is a bounded multiplier on $L^p(\mathbf{R}^n)$ with norm $\leqq Ct^\gamma$ if

$$\frac{1}{p} < \frac{1}{2} + \frac{1}{n + 1 - 2\tau_2} \quad \text{and} \quad \gamma > \max(1, n/(n + 1 - 2\tau_2)).$$

If $\tau_2 > 1/2$ we may take $\gamma = 1$.

Thus the proof of Theorem 3 will be complete when we prove Lemmas 1 and 2.

LEMMA 3. *If $|g(r)| \leqq C(1 + r)^{-5/2}$, $h(r) = t\sigma(r) + \rho r$ or $h(r) = t\sigma(r) - \rho r$ and $H(s) = \int_0^s \exp(ih(r))dr$ then*

$$\left|\int_0^t H(r)g(r)dr\right| \leqq Ct^{-1/2} \quad \text{for } t \geqq 1$$

where C is a constant independent of t and ρ.

Proof. Since $|d^2h/dr^2| \geqq Ct(1 + r)^{-L-2}$ then by van der Corput's lemma ([**13**], p. 197)

$$|H(r)| \leqq C(1 + r)^{(L+2)/2}t^{-1/2} \quad \text{and}$$

$$\left|\int_0^1 H(r)g(r)dr\right| \leqq Ct^{-1/2}$$

we may assume that $\sigma \geqq 0$. If $h(r) = t\sigma(r) + \rho r$ then $h'(r) \geqq Ct$. Since

$$|H(r) - H(1)| \leqq C/t$$

then

$$\left| \int_1^t H(r)g(r)dr \right| \leqq Ct^{-1/2}.$$

If $h(r) = t\sigma(r) - \rho r$ then $h'(r)$ can have at most one zero. Call this zero r_0. Let

$$[a, b] = [1, t] \cap [r_0/2, 2r_0]$$

where $a = t$ if $r_0 \geqq 2t$ and $b = 1$ if $2r_0 \leqq 1$. If r_0 does not exist then $a = t$ or $b = 1$ depending on which of $|h'(\infty)|$ and $|h'(0)|$ is the smaller. If $r \in (1, a)$ and $s \in (1, r)$ then

$$|h'(s)| \geqq |h'(2s) - h'(s)| \geqq C\int_s^{2s} \frac{tdy}{(1 + y)^{L+2}} \geqq Ct/s^2 > Ct/r^2.$$

This means that $|H(r)| \leqq Cr^2/t$ and

$$\left| \int_1^a H(r)g(r)dr \right| \leqq Ct^{-1/2}. \tag{20}$$

If $s \in (b, r)$ then

$$\begin{aligned} |h'(s)| &> |h'(s) - h'(s/2)| \\ &\geqq C\int_{s/2}^{s} \frac{tdy}{(1 + y)^{L+2}} \geqq Ct/s^2 > Ct/r^2. \end{aligned}$$

In this case $|H(r) - H(b)| \leqq Cr^2/t$ and

$$\left| \int_b^t (H(r) - H(b))g(r)dr \right| \leqq Ct^{-1/2}. \tag{21}$$

Also

$$\left| \int_b^t H(b)g(r)dr \right| \leqq Ct^{-1/2}(1 + b)^{3/2}b^{-3/2} \leqq Ct^{-1/2}. \tag{22}$$

Finally if $r \in [r_0/2, 2r_0]$ we use van der Corput's lemma

$$\left| \int_a^b H(b)g(r)dr \right| \leqq Ct^{-1/2} \int_{r_0/2}^{2r_0} \frac{dr}{r} = Ct^{-1/2}. \tag{23}$$

The combination of (20), (21), (22), and (23) completes the proof.

Proof of Lemma 1. Again, by using (4) it suffices to consider the multiplier

$$\varphi(\lambda_*/t)e^{it\lambda}(1 + \lambda_*^2)^{-z/2}.$$

The kernel for this transformation is

(24) $$K(x) = C \int_0^1 \int_\Sigma e^{it\lambda} e^{ix\cdot\xi}(1 + r^2)^{-z/2} \varphi\left(\frac{r}{t}\right) \frac{d\xi' r^{n-1} dr}{|\xi'|\,|\nabla\lambda_*(\xi')|}$$

we will show that $\|K\|_1 \leqq Ct^\beta$ where

$$\beta = \max(n/2, (n+1)/2 - \tau_2).$$

As in the proof of Theorem 2 we will consider two regions separately:

$$U_1 = \{x:|x| \leqq t\} \quad \text{and} \quad U_2 = \{x:|x| > t\} \quad t \geqq 1.$$

Case 1. ($x \in U_1$) Integrate by parts in (14):

$$K(x) = \frac{C}{t} \int_0^\infty \int_\Sigma \frac{1}{i\lambda'} e^{it\lambda} \frac{d}{dr}\left\{e^{ix\cdot\xi}(1 + r^2)^{-z/2} r^{n-1} \varphi\left(\frac{r}{t}\right)\right\}$$
$$\times \frac{d\xi' dr}{|\xi'|\,|\nabla\lambda_*(\xi')|}$$
$$= \frac{C}{t} \int_0^\infty \int_\Sigma \left(\frac{x\cdot\xi'}{\lambda'}\right) e^{it\lambda + ix\xi} \frac{r^{n-1}}{(1 + r^2)^{z/2}} \varphi\left(\frac{r}{t}\right)$$
$$\times \frac{d\xi' dr}{|\xi'|\,|\nabla\lambda_*(\xi')|} + E_1(x)$$

where $\lambda' = d\lambda/dr$. $E_1(x)$ is the term arising from

$$\frac{d}{dr}\left\{\frac{r^{n-1}}{(1 + r^2)^{z/2}} \varphi\left(\frac{r}{t}\right)\right\}.$$

Because of (i)

$$\int_{|x|=R} |E_1(x)|\,dx \leqq \frac{C}{t}(\log t) R^{(n-1)/2}$$

and

(25) $$\int_{|x|\leqq t} |E_1(x)|\,dx \leqq \frac{C(\log t)}{t} t^{(n+1)/2} \leqq Ct^{n/2}.$$

Similarly we may use (i) to write

$$K(x) = \frac{C}{t} \int_0^\infty \mathscr{P}(g)(-rx) \frac{e^{it\sigma}}{\sigma'} \frac{r^{(n-1)/2}}{|x|^{(n-1)/2} r_1^z} \varphi\left(\frac{r}{t}\right) dr + E_2(x)$$

where

$$\sigma(r) = \lambda(\xi) \quad \text{and} \quad g(x, \xi') = (x\cdot\xi')(\,|\xi'|^2|\nabla\lambda_*(\xi')|\,)^{-1}$$

since $\lambda' = \sigma'|\xi'|$. Also

$$\int_{|x|=R} |E_2(x)|\,dx \leqq \frac{CR^n}{tR^{(n-1)/2}} \int_0^{Ct} \frac{r^{(n-1)/2}}{(Rr)^{\tau_2} r_1^z} dr.$$

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The integral is bounded either by $R^{-\tau_2}\log t$ or $R^{-\tau_2}$ depending on whether $\tau_2 > 1/2$ or $\tau_2 \leqq 1/2$. Thus

$$(26)\quad \int_{|x|\leqq t} |E_2(x)|dx \leqq \begin{cases} Ct^{(n+1)/2-\tau_2} & \tau_2 \leqq 1/2 \\ Ct^{(n+1)/2-\tau_2}\log t & \tau_2 > 1/2 \end{cases}$$
$$\leqq Ct^{\beta}.$$

The wave surface is the union of finitely many smooth surfaces W_j with corresponding functions $\lambda_j^{\#}$. If Γ_j is the cone generated by W_j then

$$W_j = \{x \in \Gamma_j : \lambda_j^{\#}(x) = 1\}.$$

Each point ξ' in $A(x)$ is of the form $\nabla\lambda_j^{\#}(\pm x)$ for some j. Thus

$$x \cdot \xi' = \pm(\pm x) \cdot \nabla\lambda_j^{\#}(\pm x) = \pm\lambda_j^{\#}(\pm x).$$

Let $\{\rho_k\}$ be an ordering of the set

$$\bigcup_j \{\lambda_j^{\#}(x)\} \cup \bigcup_j \{-\lambda_j^{\#}(-x)\}.$$

Then group the terms of $\mathscr{P}(g)$ accordingly:

$$\mathscr{P}(g)(-rx) = \sum_{\xi' \in A(x)} \frac{x \cdot \xi' e^{i\gamma\pi/4}}{|\xi'|\,|\nabla\lambda_*(\xi')|} e^{irx\cdot\xi'}(2\pi)^{(n-1)/2}|\kappa(\xi')|^{-1/2}$$
$$\equiv \sum_{k=1}^{M(x)} G_k(x)e^{i\rho_k r}$$

where

$$\int_{|x|=R} |G_k(x)|\chi_{\Gamma_k}(x)dx \leqq CR^n.$$

For example, for the wave surface of Figure 2 many directions have a sum over six terms since a straight line through the origin can intersect W at six points.

Let $h(r) = t\sigma(r) + \rho_k r$. We are now left to consider the integrals

$$(27)\quad K_k(x) = \frac{CG_k(x)}{tR^{(n-1)/2}} \int_0^{\infty} e^{ih(r)}\frac{r^{(n-1)/2}}{\sigma' r_1^z}\varphi\left(\frac{r}{t}\right)dr.$$

We will first estimate these integrals assuming $z = (n-1)/2$. Define $H(r)$ to be the primitive of $\exp(ih(r))$ with $H(0) = 0$ (as in Lemma 3). Another integration by parts shows that

$$K_k(x) = \frac{CG_k(x)}{tR^{(n-1)/2}} \lim_{\substack{N\to\infty \\ \epsilon\to 0}} \left\{ H(r)\frac{r^{(n-1)/2}}{\sigma' r_1^z}\varphi\left(\frac{r}{t}\right)\Big|_{\epsilon}^{N} - \int_{\epsilon}^{N} H(r)\frac{d}{dr}\left\{\frac{r^{(n-1)/2}}{\sigma' r_1^z}\varphi\left(\frac{r}{t}\right)\right\}dr\right\}.$$

The integrals over $\{ |x| = R\}$ of the boundary term at $r = \epsilon$ go to zero as $\epsilon \to 0$. The boundary term as $N \to \infty$ is zero since φ has compact support. We split the integral into two parts according to

$$(28)\quad \frac{d}{dr}\{\ \} = \frac{d}{dr}\left\{\frac{r^{(n-1)/2}}{\sigma' r_1^z}\right\}\varphi\left(\frac{r}{t}\right) + \frac{r^{(n-1)/2}}{\sigma' r_1^z}\frac{1}{t}\varphi'\left(\frac{r}{t}\right).$$

For the first term in (28) we use Lemma 3. The derivative is

$$\left|\frac{d}{dr}\left\{\frac{1}{\sigma'}\left(\frac{r}{r_1}\right)^{(n-1)/2}\right\}\right| \leqq C(1 + r)^{-3}.$$

Therefore Lemma 3 shows that

$$(29)\quad \left|\int_0^{Ct} H(r)\varphi\left(\frac{r}{t}\right)\frac{d}{dr}\left\{\frac{1}{\sigma'}\left(\frac{r}{r_1}\right)^{(n-1)/2}\right\}dr\right| \leqq Ct^{-1/2}.$$

The second term of (28) is supported in an interval of the form $C_1 t \leqq r \leqq C_2 t$. If $\rho_k \geqq 0$ then $h'(r) \geqq Ct$ and $|H(r)| \leqq Ct$. In this case

$$(30)\quad \left|\int_{C_1 t}^{C_2 t}\frac{1}{t}\varphi'\left(\frac{r}{t}\right)H(r)\frac{r^{(n-1)/2}}{\sigma' r_1^z}dr\right|$$
$$\leqq \frac{C}{t^2}\int_{C_1 t}^{C_2 t}\frac{r^{(n-1)/2}}{r_1^z}dr \leqq Ct^{n-z-2}.$$

Suppose instead that $\rho_k < 0$. The integral to be considered is

$$J = C\int_0^\infty H(r)\frac{r^{(n-1)/2}}{\sigma' r_1^z}\frac{1}{t}\varphi'\left(\frac{r}{t}\right)dr$$
$$= C\int_0^\infty H(r)\frac{1}{t}\varphi'\left(\frac{r}{t}\right)dr + E_3$$

where $|E_3| \leqq Ct^{-1}$. This integral will be split into two parts according to

$$H(r) = \{H(r) - H(C_1 t)\} + H(C_1 t).$$

Call the corresponding integrals J_1 and J_2. By integration,

$$J_1 = \int_{C_1 t}^{C_2 t}\varphi\left(\frac{r}{t}\right)e^{ih(r)}dr.$$

Clearly $|J_1| \leqq Ct$. On the other hand we may integrate J_1:

$$J_1 = \int_0^\infty e^{i\rho_k r + itr}\left\{e^{it(\sigma - r)}\varphi\left(\frac{r}{t}\right)\right\}dr.$$

Hence

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$$|J_1| \leqq C \int_0^\infty \frac{1}{t}\left|\varphi'\left(\frac{r}{t}\right)\right| \frac{1}{|t + \rho_k|}$$
$$+ \left|\varphi\left(\frac{r}{t}\right)\right| \frac{t}{(1 + r)^{L+1}} \frac{1}{|t + \rho_k|} dr$$
$$\leqq C|t + \rho_k|^{-1}.$$

This shows that

$$|J_1| \leqq Ct^{1/4}|t + \rho_k|^{-3/4}.$$

Also

$$J_2 = \int_0^\infty H(C_1 t)\frac{1}{t}\varphi'\left(\frac{r}{t}\right)dr = \int_0^{C_1 t} e^{ih(r)}dr.$$

If $R \leqq t$ then $|\rho_k/t| \leqq C$. Therefore r_0, the solution of

$$h'(r) = t(\sigma' + \frac{\rho_k}{t}) = 0,$$

is also bounded independent of R and t. Thus $r_0 \leqq C_3$. The part of the integral J_2 over the interval

$$I_0 = [r_0 - t^{-1/2}, r_0 + t^{-1/2}]$$

is clearly bounded by $Ct^{-1/2}$. On the other hand if $r \notin I_0$ then

$$|h'(r)| = |h'(r) - h'(r_0)| = \left|\int_{r_0}^r h''(s)ds\right|$$
$$\geqq Ct\left|\int_{r_0}^r \frac{ds}{(1 + s)^{L+2}}\right|.$$

If $r \leqq 2C_3$ then

$$|h'(r)| \geqq Ct\left|\int_{r_0}^r ds\right| \geqq C\sqrt{t}.$$

If $r \geqq 2C_3$ then

$$|h'(r)| \geqq Ct|(1 + r)^{-L-1} - (1 + r_0)^{-L-1}| \geqq Ct.$$

Therefore outside I_0, $|h'(r)| \geqq C\sqrt{t}$. This shows that

$$|J_2| \leqq Ct^{-1/2}.$$

This completes the analysis of the terms arising from (28). We have shown that

$$|K_k(x)| \leqq \frac{C|G_k(x)|}{tR^{(n-1)/2}} \{t^{-1/2} + t^{1/4}|t + \rho_k|^{-3/4}\}.$$

The calculations from (27) have been under the assumption that $z = (n - 1)/2$ so that Lemma 3 could be applied in (29). If however $z = \gamma + (n - 1)/2$ where $\gamma > 0$ then the integral in (27) equals

$$-\int_0^\infty \left\{ \int_0^s e^{ih(r)} \frac{1}{\sigma'} \left(\frac{r}{r_1}\right)^{(n-1)/2} \varphi\left(\frac{r}{t}\right) dr \right\} \frac{d}{ds}\left(\frac{1}{s_1^\gamma}\right) ds.$$

This is dominated by

$$\int_0^\infty \{t^{-1/2} + t^{1/4}|t + \rho_k|^{-3/4}\} \left|\frac{d}{ds}\left(\frac{1}{s_1^\gamma}\right)\right| ds$$

$$\leqq C\{t^{-1/2} + t^{1/4}|t + \rho_k|^{-3/4}\}.$$

Therefore from (25), (26), and (27),

$$(31) \quad \int_{|x| \leqq t} |K(x)|dx \leqq Ct^\beta + C \sum_k \int_{|x| \leqq t} \frac{|G_k(x)|}{tR^{(n-1)/2}} \chi_{\Gamma_k}(x) \times \{t^{-1/2} + t^{1/4}|t + \rho_k|^{-3/4}\}dx.$$

The first term can be approximated by using polar coordinates:

$$\leqq C \sum_k t^{-3/2} \int_0^t R^{-(n-1)/2} R^n dR = Ct^{n/2}.$$

For the second term it is more natural to integrate over the level sets of $\lambda_k^\#$. Suppose that

$$(32) \quad \int_{\lambda_k^\#(x)=s} |G_k(x)|\chi_{-\Gamma_k}(x) \frac{|\lambda_k^\#(x)|}{|x|\,|\nabla\lambda_k^\#(x)|} dx \leqq Cs^n.$$

Since $|\rho_k| \leqq cR$, the second term is bounded by

$$C \sum_k t^{-3/4} \int_0^{ct} s^{-(n-1)/2} s^n |t - s|^{-3/4} ds \leqq Ct^{n/2}.$$

To prove (32) it suffices to show that

$$\sum_k \left| \int_{\lambda_k^\#(x) \quad 1} \chi_{\Gamma_k}(x) \frac{dx}{|\kappa(\xi')|^{1/2}} \right| \leqq C$$

where $\xi' = \nabla\lambda_j^\#(x)$. By the argument leading to (16) this equals

$$\int_\Sigma \frac{|\kappa(\xi')|\,|\nabla\lambda(\xi')|\,|\xi'|}{|\kappa(\xi')|^{1/2}} d\xi' \leqq C \int_\Sigma |\kappa(\xi')|^{1/2} d\xi' \leqq C$$

since the curvature of Σ is bounded and Σ has finite area. This completes the proof that

$$\int_{|x| \leqq t} |K(x)|dx \leqq Ct^\beta + Ct^{n/2}.$$

Case 2. ($x \in U_2$). This calculation in this region is similar to that of Case 1. The kernel is given by

$$K(x) = C\int_0^\infty \int_\Sigma e^{it\lambda}e^{ix\cdot\xi}r^{n-1}r_1^{-z}\varphi\left(\frac{r}{t}\right)\frac{d\xi' dr}{|\xi'|\,|\nabla\lambda_*(\xi')|}.$$

As in the proof of Theorem 2 we will estimate $K(x)$ in a small conic neighborhood of a point x_0. Since $\mathbf{R}^n$ is covered by finitely many such neighborhoods this will be sufficient. Let η be a function in $C^\infty(\mathbf{R}^n - \{0\})$ that is homogeneous of degree zero such that $x_0 \cdot \xi/|x_0|\,|\xi|$ is bounded away from zero in the support of η and $\eta \equiv 1$ in a conic neighborhood Γ_0 of the set $A(x_0)$. Then

$$\left|\int_\Sigma e^{ix\cdot\xi}(1 - \eta(\xi))\frac{d\xi'}{|\xi'|\,|\nabla\lambda(\xi')|}\right| \leqq C_N(1 + Rr)^{-N}$$

for any $N > 0$. This part of $K(x)$ equals

$$(33)\quad C\int_0^{R^{\epsilon-1}} \int_\Sigma e^{it\lambda}e^{ix\cdot\xi}r^{n-1}r_1^{-z}(1 - \eta(\xi))\frac{d\xi' dr}{|\xi'|\,|\nabla\lambda_*(\xi')|}$$
$$+ O\left(\int_{R^{\epsilon-1}}^{t} r^{(n-1)/2}(1 + Rr)^{-N}dr\right).$$

If $\epsilon > 0$ this error term is integrable over the region $|x| \geqq t$ and the integral is less than $Ct^{(n-1)/2}$ when N is sufficiently large. In the main term of (33) it is possible to replace r_1 and $\exp(it\lambda(r))$ by 1 and $\exp(it\lambda(0))$ leaving errors that are less than $Ct^{n/2}$ if $(n + 2)\epsilon < 2$ and $(n + 1)\epsilon < 1$ respectively. But

$$\int_0^{R^{\epsilon-1}} \int_\Sigma e^{ix\cdot\xi}(1 - \eta(\xi))\frac{r^{n-1}d\xi' dr}{|\xi'|\,|\nabla\lambda_*(\xi')|}$$

is, except for an error less than $CR^{-\epsilon N}$, the inverse Fourier transform of $1 - \eta(\xi)$. Clearly η can also be chosen so that

$$\int_{|\xi|=1} (1 - \eta(\xi))d\xi = 0.$$

Then $K_0(x) = (1 - \eta)(-x)$ is the kernel of a Calderon-Zygmund singular integral operator, which is bounded on $H^1(\mathbf{R}^n)$.

This now leaves

$$K_*(x) = C\int_0^\infty \int_\Sigma e^{it\lambda}e^{ix\cdot\xi}r^{n-1}r_1^{-z}\eta(\xi')\varphi\left(\frac{r}{t}\right)\frac{d\xi' dr}{|\xi'|\,|\nabla\lambda_*(\xi')|}$$

for $x \in \Gamma$. Integration by parts k times gives

$$(34)\quad K_*(x) = C\int_0^\infty \int_\Sigma \frac{e^{ix\cdot\xi}}{(-ix\cdot\xi')^k}\frac{d^k}{dr^k}\left\{\frac{r^{n-1}}{r_1^z}\varphi\left(\frac{r}{t}\right)e^{it\lambda}\right\}\frac{\eta(\xi')d\xi' dr}{|\xi'|\,|\nabla\lambda_*(\xi')|}$$

where $k > (n + 3)/2$. The main part of (34) is

$$K_1(x) = Ct^k \int_0^\infty \int_\Sigma e^{ix\cdot\xi + it\lambda}(x \cdot \xi')^{-k}\varphi\left(\frac{r}{t}\right)\frac{d^k}{dr^k}\left\{\frac{r^{n-1}}{r_1^z}\right\}d\tilde{\mu}(\xi')dr.$$

This integral is similar to (24) except for the factor $t^k(x \cdot \xi')^{-k}$. Since $(x \cdot \xi')^{-k}\eta(\xi')$ is a smooth function on Σ bounded by CR^{-k} the calculations of Case 1 lead to an estimate similar to (31) except that the factor $(t/R)^k$ will make it integrable over the region $|x| \geqq t$. Therefore as in Case 1,

$$\int_{|x|\geqq t} |K_1(x)|\chi_\Gamma dx \leqq Ct^\beta + Ct^{n/2}.$$

The terms of

$$g(r) = \frac{d^k}{dr^k}\left\{\frac{r^{n-1}}{r_1^z}\varphi\left(\frac{r}{t}\right)e^{it\lambda}\right\}$$

in which $\varphi\left(\frac{r}{t}\right)$ is differentiated at least once satisfy

$$|g_2(r)| \leqq Ct^{k-2}r^{n/2}r_1^{-1/2}\left|\varphi\left(\frac{r}{t}\right)\right|.$$

If the corresponding integral is $K_2(x)$ then

$$\begin{aligned}\int_{|x|=R} |K_2(x)|\chi_\Gamma dx &\leqq Ct^{k-2}R^{-k}\int_0^t (R/r)^{(n-1)/2}r^{n/2}r_1^{-1/2}dr \\ &\leqq Ct^{k-1}R^{(n-1)/2-k}.\end{aligned}$$

The other terms of $g(r)$ are less than

$$|g_3(r)| \leqq Ct^{k-1}r^{n/2}r_1^{-3/2}\left|\varphi\left(\frac{r}{t}\right)\right|.$$

Then

$$\begin{aligned}\int_{|x|=R} |K_3(x)|\chi_\Gamma dx &\leqq Ct^{k-1}R^{-k}\int_0^t (R/r)^{(n-1)/2}r^{n/2}r_1^{-3/2}dr \\ &\leqq Ct^{k-1}R^{(n-1)/2-k}\log t.\end{aligned}$$

This shows that

$$\int_{|x|\geqq t} |K_2 + K_3|\chi_\Gamma dx \leqq Ct^{(n-1)/2}(1 + \log t) \leqq Ct^{n/2}.$$

The proof of Lemma 1 is now complete.

Lemma 2 is a consequence of Lemma 1 since $(1 + \lambda_*^2)^{-\epsilon/2}$ is a bounded multiplier on $H^1(\mathbf{R}^n)$ for every $\epsilon \geqq 0$.

4. The $L^p - L^{p'}$ estimates. Showing that T_t is bounded from L^p to $L^{p'}$ involves studying an operator $(I - \Delta)^{-z/2}T_t$ from H^1 to L^∞. This means calculating the L^∞ or possibly BMO norm of its kernel. Therefore instead of assuming that the slowness surface

$$\Sigma = \{\xi:\lambda_*(\xi) = 1\}$$

satisfies Theorem 1 we will assume that for any C^∞ function g on $S \times \Sigma = \{ (x', \xi'):|x'| = 1, \xi' \in \Sigma\}$,

$$\left| \int_\Sigma e^{-ix\cdot\xi'} g(x', \xi')d\xi' \right| \leqq C(1 + |x|)^{-\nu}. \tag{35}$$

If the curvature of Σ does not vanish then $\nu = (n - 1)/2$.

THEOREM 4. *Let T_t be the operator with multiplier* $\exp(it\lambda(\xi))m_1(\xi)$ *where* $\lambda(\xi)$ *satisfies* (i), (ii), (iii) *and $m_1(\xi)$ is as in* (17). *Then T_t is a bounded linear operator from $L^p(\mathbf{R}^n)$ to $L^{p'}(\mathbf{R}^n)$ if*

$$\frac{1}{2} \leqq \frac{1}{p} < \frac{1}{2} + \frac{1}{2(n - \nu)}. \tag{36}$$

Also

$$\|T_t\| \leqq C_\beta t^{-\beta} \quad t \geqq 1$$

for every $\beta > 2\nu\left(\frac{1}{p} - \frac{1}{2}\right)$.

If λ is homogeneous and the multiplier is $\sin(t\lambda(\xi))/\lambda(\xi)$ then clearly by homogeneity we may take $\beta = (n - 1)(1/p - 1/2)$.

Proof. The proof of this theorem is similar to the case where the curvature of Σ does not vanish. This proof appears in **[9]**. We therefore give only an outline. We will show that $(I - \Delta)^{(1-z)/2}T_t f$ is a bounded operator from H^1 to L^∞ for every $z > n - \nu$ with operator norm $\leqq Ct^{-\nu}$. The statements of the theorem then follow by interpolation between this operator and $(I - \Delta)^{1/2}T_t$, which is bounded on $L^2(\mathbf{R}^n)$ by the Plancherel Theorem.

By composing with multipliers satisfying (4) it suffices to consider

$$m(\xi) = e^{it\lambda(\xi)}(1 + (\lambda_*(\xi))^2)^{-z/2}(|\xi'|\,|\nabla\lambda_*(\xi')|)$$

where $\xi = \lambda_*(\xi)\xi'$. The kernel associated with this multiplier is

$$K(x) = \int_{\mathbf{R}^n} e^{ix\cdot\xi}m(\xi)d\xi = \int_0^\infty \int_\Sigma e^{ix\cdot\xi + it\sigma(r)}d\xi'(1 + r^2)^{-z/2}r^{n-1}dr.$$

Let $k = [\nu]$, the integral part of ν. After integrating by parts k times

$$K(x) = \left(\frac{iR}{t}\right)^k \int_0^\infty \int_\Sigma e^{it\sigma + ix\cdot\xi}(ix'\cdot\xi')^k d\xi' \frac{r^{n-1}dr}{r_1^z(\sigma')^k} + E(x) \tag{37}$$

where $\|E\|_\infty \leqq Ct^{-\nu-1/2}$. By (35), the integrand of (37) is bounded by

$$Cr_1^{n-1-\nu-z}R^{-\nu} = Cr_1^{-1-\epsilon}R^{-\nu} \quad \text{where } \epsilon > 0.$$

Therefore

$$(38) \quad |K_1(x)| \leqq C\left(\frac{R}{t}\right)^k \int_0^\infty r_1^{-1-\epsilon}R^{-\nu}dr = C\left(\frac{R}{t}\right)^k R^{-\nu}.$$

If $\nu > k$ then integrate by parts again in (37):

$$K_1(x) = \frac{CR^k}{t^{k+1}} \int_0^\infty \int_\Sigma e^{it\sigma + ix\cdot\xi} g(x', \xi) d\xi' dr$$

where $|g| \leqq Cr_1^{n-1-z}(R + 1/r)$. Therefore as in (38),

$$(39) \quad |K_1(x)| \leqq C\left(\frac{R}{t}\right)^{k+1} R^{-\nu}.$$

Since $k \leqq \nu < k + 1$, (38) and (39) together show that $\|K\|_\infty \leqq Ct^{-\nu}$. This proves Theorem 4.

When the curvature does not vanish a better approximation for the integral in (37) can be obtained by splitting the integral into parts

$$\int_0^\infty = \int_0^{t^{1/L}} + \int_{t^{1/L}}^\infty.$$

The interval from $t^{1/L}$ to ∞ can be still approximated using (35) so long as $z > n - \nu$, and

$$\left| \int_{t^{1/L}}^\infty \right| \leqq Ct^{-\nu-\alpha}$$

if $z = n - \nu + \alpha L$ and $0 < \alpha \leqq 1/2$. The interval $[0, t^{1/L}]$ however presents a problem because in approximating

$$\int_0^{t^{1/L}} \int_\Sigma e^{it\sigma + ix\cdot\xi}(ix' \cdot \xi')^k d\xi' \frac{r^{n-1}dr}{r_1^z(\sigma')^k}$$

it is necessary to use (35) for the oscillation over Σ and van der Corput's lemma for the cancellation in r. To do both, as in **[8]** and **[9]**, it appears necessary to get an expression for the leading term in (35).

REFERENCES

1. V. T. Buchwald, *Elastic waves in anisotropic media*, Proc. Roy. Soc. London, Ser. A, *253* (1959), 563-580.

2. R. Courant and D. Hilbert, *Methods of mathematical physics. Vol. II* (Interscience Publishers, New York, 1962).

3. G. F. D. Duff, *The Cauchy problem for elastic waves in an anisotropic medium*, Philos. Trans. Roy. Soc. London, Ser. A, *252* (1960), 249-273.

4. A. Greenleaf, *Principal curvature and harmonic analysis*, Indiana Math. J. *30* (1981), 519-537.

5. C. Herz, *Fourier transforms related to convex sets*, Ann. of Math. *75* (1962), 81-92.
6. E. Hlawka, *Über Integrale auf konvexen Körpern I*, Monatsh. Math. *54* (1950), 1-36; *II*, ibid. *54* (1950), 81-99.
7. W. Littmann, *Fourier transforms of surface-carried measures and differentiability of surface averages*, Bull. Amer. Math. Soc. *69* (1963), 766-770.
8. B. Marshall, W. Strauss and S. Wainger, *$L^p - L^q$ estimates for the Klein-Gordon equation*, J. Math. pures et appl. *59* (1980), 417-440.
9. B. Marshall, *$L^p - L^q$ multipliers of anisotropic wave equations*, Indiana Math. J. *33* (1984), 435-457.
10. ——— *The Fourier transforms of smooth measures on hypersurfaces of* $\mathbf{R}^{n+1}$, Can. J. Math.
11. J. Peral, *L^p estimates for the wave equation*, J. Funct. Anal. *36* (1980), 114-145.
12. R. S. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. *44* (1977), 705-714.
13. A. Zygmund, *Trigonometric series* (Cambridge U. Press, 1959).

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