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On the Negative Index Theorem for the Linearized Non-Linear Schrödinger Problem

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Abstract. A new and elementary proof is given of the recent result of Cuccagna, Pelinovsky, and Vougalter based on the variational principle for the quadratic form of a self-adjoint operator. It is the negative index theorem for a linearized NLS operator in three dimensions.

1 Introduction

In this article we present what we believe to be a simpler proof of the recent result of Cuccagna, Pelinovsky, and Vougalter [11]. It relates the spectrum of the linearized NLS equation to the negative spectrum of the energy operator. We use the notations of [11] and consider the linearized operator $\mathcal{L} = \sigma_3 \mathcal{H}$, where σ_3 is the standard Pauli matrix and $\mathcal{H}$ is the energy operator,

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix} -\Delta + \omega + f(x) & g(x) \\ g(x) & -\Delta + \omega + f(x) \end{pmatrix},$$

while $x \in \mathbb{R}^3$, $\omega > 0$, and $f, g \colon \mathbb{R}^3 \to \mathbb{R}$ are exponentially decaying C^∞ functions. The spectral problem for the operator $\mathcal{L}$ is considered on $L^2(\mathbb{R}^3, \mathbb{C}^2)$,

$$\mathcal{L}\psi = z\psi, \tag{1.1}$$

where $\psi = (\psi_1, \psi_2)^T$. It arises when we linearize the Non-Linear Schrödinger (NLS) equation on its special solution

$$\psi = \phi(x) e^{i\omega t}, \tag{1.2}$$

where $\phi \colon \mathbb{R}^3 \to \mathbb{R}$ is C^∞ and exponentially decreasing and solves the elliptic problem

$$-\Delta\phi + \omega\phi + U(x)\phi + F(\phi^2)\phi = 0.$$

The existence of such standing wave solutions was proven for a broad class of nonlinearities (see [1–4, 16, 21]), and the functions $f(x)$ and $g(x)$ involved in the energy operator are known explicitly (see [5, 11, 14, 23]):

$$f(x) = U(x) + F(\phi^2) + F'(\phi^2)\phi^2, \quad g(x) = F'(\phi^2)\phi^2.$$

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The eigenvalues z of the spectral problem (1.1) are called unstable if $\Im(z) > 0$, neutrally stable if $\Im(z) = 0$ and stable if $\Im(z) < 0$ (see [11, 22]). The studies of the spectral properties of the linearized operator $\mathcal{L}$ play a significant role in the proofs of the asymptotic stability of the NLS solitary waves (see [6, 8, 15, 18]).

For practical purposes, the system (1.1) can be conveniently rewritten in the new variables $\psi = (u + w, u - w)^T$:

$$\sigma_1 \mathrm{H}\mathbf{u} = z\mathbf{u}\,, \tag{1.3}$$

where $\mathbf{u} = (u, w)^T$, σ_1 is the standard Pauli matrix, and H is the new energy operator:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathrm{H} = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix},$$

where $L_\pm = -\Delta + \omega + f(x) \pm g(x)$. The numbers of the negative and the positive eigenvalues of the operator H on $L^2(\mathbb{R}^3, \mathbb{C}^2)$ are called the negative and the positive indices of H respectively and denoted as $n(\mathrm{H})$ and $p(\mathrm{H})$.

Apparently, there are several remarkable observations concerning the symmetries of the spectrum of problem (1.3) with respect to both x and y axes. While the essential spectrum consists of two intervals: $(-\infty, \omega]$ and $[\omega, \infty)$, the eigenvalues could be located anywhere in the complex plane. Their number and algebraic multiplicities are finite (see [11, Proposition 2.2]).

If z is an eigenvalue with eigenvector $\mathbf{u}$, then $-z$ is another eigenvalue of problem (1.3) with the corresponding eigenvector $\sigma_3\mathbf{u}$. We denote positive real and positive imaginary eigenvalues and the corresponding eigenvectors as z_r^j and $\mathbf{u}_r^j$, $1 \leq j \leq N_r$ and z_{im}^k and $\mathbf{u}_{im}^k$, $1 \leq k \leq N_{im}$. If z is a complex eigenvalue with nonzero real and imaginary parts, problem (1.3) has two additional eigenvalues $\bar{z}$, and $-\bar{z}$ and the eigenvectors are $\bar{\mathbf{u}}$ and $\sigma_3\bar{\mathbf{u}}$, respectively. Let us denote the complex eigenvalues of the linearized NLS problem located in the first open quadrant and the corresponding eigenvectors as z_c^l and $\mathbf{u}_c^l$, $1 \leq l \leq N_c$.

As in [11], the eigenvalues of the operator $\mathcal{L}$ are assumed to be simple, which has a trivial generalization to the case of semisimple eigenvalues and multiple eigenvalues in the limiting case.

The energy functional for the problem is defined on $H^1(\mathbb{R}^3, \mathbb{C}^2)$ as $h := \langle \mathbf{u}, \mathrm{H}\mathbf{u}\rangle$. Henceforth, the notation $\langle \mathbf{f}, \mathbf{g}\rangle$ stands for the inner product of $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^3, \mathbb{C}^2)$. The inner product of $f, g \in L^2(\mathbb{R}^3)$ is denoted as (f, g). We simplify the analysis with the following assumptions on the spectrum of problem (1.3), analogous to those of [11].

Assumption 1.1 *The endpoints $\pm\omega$ of the essential spectrum of the operator $\mathcal{L}$ are neither resonances nor eigenvalues.*

Assumption 1.2 *The kernel of the operator $\sigma_1 H$ is one-dimensional, while its generalized kernel is two-dimensional, and $\ker(\sigma_1 H) = \phi_0$, $N_g(\sigma_1 H) = \{\phi_0,\ \phi_1\}$, where*

$$\phi_0 = (0,\ \phi)^T, \quad \phi_1 = (-\partial_\omega \phi,\ 0)^T, \quad \textit{and} \quad \sigma_1 H\phi_1 = \phi_0. \tag{1.4}$$

Assumption 1.3 *No real eigenvalues z of $\sigma_1 H$ exist such that $\langle \mathbf{u},\ H\mathbf{u}\rangle = 0$, where $\mathbf{u}$ is the corresponding eigenvector of $\sigma_1 H$.*

The situation where resonances or eigenvalues occur at the endpoints of the essential spectrum has been studied in recent works, [10]. It was shown that they are structurally unstable under generic perturbations. Assumption 1.2 is natural, due to the fact that the additional zero modes $(\nabla\phi,\ 0)^T$ of the operator $\sigma_1 \mathrm{H}$ disappear in the presence of a coordinate-dependent potential $U(x)$ in the NLS equation. The case of higher algebraic multiplicity for the generalized kernel is the blow-up situation and was treated recently in [9]. The impact of real eigenvalues of zero energy on the negative index of the operator H and their bifurcations under generic perturbations was considered in [22]. The eigenvalues embedded in the interior of the essential spectrum $(-\infty,\ -\omega) \cup (\omega,\ \infty)$ are also nongeneric, such that the eigenvalues of positive energies disappear and those of negative energies produce the bound states in both upper and lower half-planes when a perturbation is applied (see [11]). Thus we make the assumption of their nonexistence. Such an assumption plays the significant role in proving the dispersive estimates for Schrödinger operators (see [12]), the existence of stable manifolds for an orbitally unstable NLS (see [20]). The nonexistence of resonances in the interior of the essential spectrum was proven in Proposition 2.3 of [11].

Assumption 1.4 *There are no eigenvalues embedded in the interior of the essential spectrum $(-\infty,\ -\omega) \cup (\omega,\ \infty)$ of the operator $\mathcal{L}$.*

Assumption 1.2 and the symmetry properties of the spectral problem imply that the dimension of the pure point spectrum of the linearized operator is

$$\dim(\sigma_p(\sigma_1 \mathrm{H})) = 2 + 2N_r + 2N_{im} + 4N_c.$$

Since real eigenvalues with zero energy are excluded, positive real eigenvalues correspond to eigenvectors of either positive or negative energy, such that $N_r = N_r^+ + N_r^-$.

The main result of the paper is the closure relation between the eigenvalues of the linearized NLS problem and the negative index of the energy operator, which we prove differently than in [11]. The case of coupled nonlinear Schrödinger equations was studied in [17].

Theorem 1.5 *Let Assumptions* 1.1, 1.2, 1.3, *and* 1.4 *be satisfied. Then $Q'(\omega) \neq 0$, where $Q(\omega) = \int_{\mathbb{R}^3} \phi^2(x)dx$ is the squared L^2 norm of the standing wave solution* (1.2), *and the following closure relation is true for problem* (1.3) *on $L^2(\mathbb{R}^3, \mathbb{C}^2)$:*

$$n(H) = p(Q') + N_{im} + 2N_c + 2N_r^-. \tag{1.5}$$

Here $p(Q') = 1$ if $Q'(\omega) > 0$, and $p(Q') = 0$ if $Q'(\omega) < 0$.

2 Proof of Theorem 1.5

As was shown in [11], the eigenvectors $\mathbf{u}_i,\ \mathbf{u}_j$ corresponding to distinct eigenvalues z_i and z_j, such that $z_i \neq \pm z_j,\ z_i \neq \pm \bar{z}_j$ of problem (1.3) are skew-orthogonal, *i.e.*, $\langle \mathbf{u}_i,\ \sigma_1 \mathbf{u}_j \rangle = 0$. The following elementary lemma extends these relations to the generalized kernel of the linearized operator.

Lemma 2.1 *Let $\phi_i \in N_g(\sigma_1 H),\ i = 0, 1$ and let $\mathbf{u}$ be the eigenvector of the operator $\sigma_1 H$ corresponding to an eigenvalue $z,\ z \neq 0$. Then*

$$\langle \phi_i, \sigma_1 \mathbf{u} \rangle = 0,\ i = 0, 1 \qquad \textit{and} \qquad \langle \phi_1, \sigma_1 \phi_0 \rangle = -\frac{1}{2} Q'(\omega) \neq 0.$$

Proof Via (1.3), Assumption 1.2 and (1.4),

$$\langle \phi_0, \sigma_1 \mathbf{u} \rangle = \frac{1}{z} \langle \mathrm{H}\phi_0, \mathbf{u} \rangle = 0,$$

$$\langle \phi_1, \sigma_1 \mathbf{u} \rangle = \frac{1}{z} \langle \phi_0, \sigma_1 \mathbf{u} \rangle = 0,$$

and

$$\langle \phi_1, \sigma_1 \phi_0 \rangle = (-\partial_\omega \phi, \phi) = -\frac{1}{2} Q'(\omega).$$

Thus $Q'(\omega)$ does not vanish. Otherwise, by the Fredholm alternative theorem, there exists a second generalized eigenvector in $N_g(\sigma_1 \mathrm{H})$, which contradicts Assumption 1.2. ∎

The remaining orthogonality relations are between the eigenvectors corresponding to the eigenvalues of the same kind: positive real, positive pure imaginary and the complex located in the first open quadrant.

Lemma 2.2 *It is true that*

$$\langle \mathbf{u}_r^j,\ \sigma_1 \mathbf{u}_r^j \rangle \neq 0, \qquad \langle \mathbf{u}_r^j,\ \sigma_1 \sigma_3 \mathbf{u}_r^j \rangle = 0, \qquad 1 \leq j \leq N_r, \tag{2.1}$$

$$\langle \mathbf{u}_{im}^k,\ \sigma_1 \mathbf{u}_{im}^k \rangle = 0, \qquad \langle \mathbf{u}_{im}^k,\ \sigma_1 \bar{\mathbf{u}}_{im}^k \rangle \neq 0, \qquad 1 \leq k \leq N_{im}, \tag{2.2}$$

$$\langle \mathbf{u}_c^l,\ \sigma_1 \mathbf{u}_c^l \rangle = \langle \mathbf{u}_c^l,\ \sigma_1 \sigma_3 \mathbf{u}_c^l \rangle = \langle \mathbf{u}_c^l,\ \sigma_1 \sigma_3 \bar{\mathbf{u}}_c^l \rangle = 0, \tag{2.3}$$

$$\langle \mathbf{u}_c^l,\ \sigma_1 \bar{\mathbf{u}}_c^l \rangle \neq 0, \qquad 1 \leq l \leq N_c.$$

Proof By means of (1.3) and Assumption 1.3 we obtain

$$\langle \mathbf{u}_r^j,\ \sigma_1 \mathbf{u}_r^j \rangle = \frac{1}{z_r^j} \langle \mathrm{H}\mathbf{u}_r^j,\ \mathbf{u}_r^j \rangle \neq 0.$$

A straightforward computation and the fact that both components of $\mathbf{u}_r^j$ are real valued yield the second identity in (2.1).

The equality in (2.2) is an elementary consequence of the fact that the first component of $\mathbf{u}_{im}^k,\ 1 \leq k \leq N_{im}$ is real and the second one is pure imaginary (see [11]). Now, $\langle \mathbf{u}_{im}^k, \sigma_1 \bar{\mathbf{u}}_{im}^k \rangle,\ 1 \leq k \leq N_{im}$ and $\langle \mathbf{u}_c^l,\ \sigma_1 \bar{\mathbf{u}}_c^l \rangle,\ 1 \leq l \leq N_c$ do not vanish, otherwise by the Fredholm alternative theorem there would exist generalized eigenvectors $\mathbf{v}_{im}^k$ and $\mathbf{v}_c^l$ satisfying

$$\sigma_1 \mathrm{H}\mathbf{v}_{im}^k = z_{im}^k \mathbf{v}_{im}^k + \mathbf{u}_{im}^k, \qquad \sigma_1 \mathrm{H}\mathbf{v}_c^l = z_c^l \mathbf{v}_c^l + \mathbf{u}_c^l,$$

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which contradicts the assumption that the eigenvalues are simple.

In the case of complex eigenvalues we express $\mathbf{u}_c^l = (u_R^l + iu_I^l,\ w_R^l + iw_I^l)^T$, $1 \le l \le N_c$. According to [11], the real and imaginary parts satisfy the following relations

$$(u_R^l,\ w_R^l) = -(u_I^l,\ w_I^l), \qquad (u_R^l,\ w_I^l) = (u_I^l,\ w_R^l), \qquad 1 \le l \le N_c.$$

An elementary computation using these identities yields (2.3). ■

Having established the orthogonality relations, we explicitly define the skew-orthogonal projection operator P_d onto the subspace X_d spanned by the eigenvectors corresponding to the elements of the pure point spectrum of the linearized operator, *i.e.*,

$$X_d := \operatorname{span}\left\{ \{\phi_i\}_{i=0}^1,\ \{\mathbf{u}_r^j,\ \sigma_3\mathbf{u}_r^j\}_{j=1}^{N_r},\ \{\mathbf{u}_{im}^k,\ \bar{\mathbf{u}}_{im}^k\}_{k=1}^{N_{im}},\ \{\mathbf{u}_c^l,\ \bar{\mathbf{u}}_c^l,\ \sigma_3\mathbf{u}_c^l,\ \sigma_3\bar{\mathbf{u}}_c^l\}_{l=1}^{N_c} \right\}$$

Lemmas 2.1 and 2.2 yield the following corollary, which can be verified via a straightforward computation.

Corollary 2.3 *Any* $\mathbf{f} \in L^2(\mathbb{R}^3,\ \mathbb{C}^2)$ *can be decomposed as follows:*

$$\mathbf{f} = \sum_{i=0}^{1} c_0^i\phi_i + \sum_{j=1}^{N_r}(c_r^j\mathbf{u}_r^j + d_r^j\sigma_3\mathbf{u}_r^j) + \sum_{k=1}^{N_{im}}(c_{im}^k\mathbf{u}_{im}^k + d_{im}^k\bar{\mathbf{u}}_{im}^k) + \sum_{l=1}^{N_c}(c_c^l\mathbf{u}_c^l + d_c^l\bar{\mathbf{u}}_c^l + \alpha_c^l\sigma_3\mathbf{u}_c^l + \beta_c^l\sigma_3\bar{\mathbf{u}}_c^l) + \mathbf{f}_c,$$

where

$$c_0^i = \frac{-2\langle\sigma_1\phi_{1-i}, \mathbf{f}\rangle}{Q'(\omega)}, \quad i = 0, 1,$$

$$c_r^j = \frac{\langle\sigma_1\mathbf{u}_r^j, \mathbf{f}\rangle}{\langle\sigma_1\mathbf{u}_r^j, \mathbf{u}_r^j\rangle}, \qquad d_r^j = -\frac{\langle\sigma_1\sigma_3\mathbf{u}_r^j, \mathbf{f}\rangle}{\langle\sigma_1\mathbf{u}_r^j, \mathbf{u}_r^j\rangle}, \quad 1 \le j \le N_r,$$

$$c_{im}^k = \frac{\langle\sigma_1\bar{\mathbf{u}}_{im}^k, \mathbf{f}\rangle}{\langle\sigma_1\bar{\mathbf{u}}_{im}^k, \mathbf{u}_{im}^k\rangle}, \quad d_{im}^k = \frac{\langle\sigma_1\mathbf{u}_{im}^k, \mathbf{f}\rangle}{\langle\sigma_1\mathbf{u}_{im}^k, \bar{\mathbf{u}}_{im}^k\rangle}, \quad 1 \le k \le N_{im},$$

$$c_c^l = \frac{\langle\sigma_1\bar{\mathbf{u}}_c^l, \mathbf{f}\rangle}{\langle\sigma_1\bar{\mathbf{u}}_c^l, \mathbf{u}_c^l\rangle}, \qquad d_c^l = \frac{\langle\sigma_1\mathbf{u}_c^l, \mathbf{f}\rangle}{\langle\sigma_1\mathbf{u}_c^l, \bar{\mathbf{u}}_c^l\rangle},$$

$$\alpha_c^l = -\frac{\langle\sigma_1\sigma_3\bar{\mathbf{u}}_c^l, \mathbf{f}\rangle}{\langle\sigma_1\bar{\mathbf{u}}_c^l, \mathbf{u}_c^l\rangle}, \quad \beta_c^l = -\frac{\langle\sigma_1\sigma_3\mathbf{u}_c^l, \mathbf{f}\rangle}{\langle\sigma_1\mathbf{u}_c^l, \bar{\mathbf{u}}_c^l\rangle}, \quad 1 \le l \le N_c,$$

and $\langle\mathbf{f}_c, \sigma_1\mathbf{g}\rangle = 0$ *for any* $\mathbf{g} \in X_d$.

Definition 2.4 For an arbitrary function $\mathbf{f} \in L^2(\mathbb{R}^3, \mathbb{C}^2)$

$$P_d\mathbf{f} := \mathbf{f} - \mathbf{f}_c,$$

which is given explicitly in terms of the functions of the subspace X_d in Corollary 2.3.

Remark It is easy to check that the projection operator satisfies the relation $P_d^2 = P_d$.

We define the restriction of the quadratic form of the energy operator onto the subspace X_d as $\mathrm{H}|_{X_d} := P_d^* \mathrm{H} P_d$, where P_d^* is the projection onto the generalized eigenspaces of the adjoint operator $\mathrm{H}\sigma_1$ and draw the conclusion about the number of negative eigenvalues of the restricted operator, which enables us to prove the negative index theorem.

Proof of Theorem 1.5 Let us define the subspace $X_- \subset X_d$ as the space of vectors of the form

$$\mathbf{u} = a p(Q')\phi_1 + \sum_{j=1}^{N_r^-} (a_j \mathbf{u}_r^{j,-} + b_j \sigma_3 \mathbf{u}_r^{j,-}) + \sum_{k=1}^{N_{im}} c_k (\mathbf{u}_{im}^k + s_k \bar{\mathbf{u}}_{im}^k) \\ + \sum_{l=1}^{N_c} d_l (\mathbf{u}_c^l - e^{-i\varphi_l} \bar{\mathbf{u}}_c^l) + q_l \sigma_3 (\mathbf{u}_c^l - e^{-i\varphi_l} \bar{\mathbf{u}}_c^l)\,,$$

where $\mathbf{u}_r^{j,-}$ are the eigenvectors of $\sigma_1\mathrm{H}$ corresponding to real eigenvalues of negative energy,

$$e^{i\varphi_l} = \frac{\langle z_c^l \mathbf{u}_c^l,\ \sigma_1 \bar{\mathbf{u}}_c^l \rangle}{|\langle z_c^l \mathbf{u}_c^l,\ \sigma_1 \bar{\mathbf{u}}_c^l \rangle|},\ s_k = \operatorname{sign}(z_{im}^k \langle \mathbf{u}_{im}^k,\ \sigma_1 \bar{\mathbf{u}}_{im}^k \rangle)$$

and $a,\ a_j,\ b_j,\ c_k,\ d_l,\ q_l \in \mathbb{C}$ are arbitrary constants. Clearly $\dim X_- = p(Q') + 2N_r^- + N_{im} + 2N_c$.

The orthogonality relations stated in Lemmas 2.1 and 2.2 along with equation (1.3) enable us to show that X_- is a negative subspace for the energy operator H and for its restriction $\mathrm{H}|_{X_d}$. Thus

$$\langle \mathrm{H}\mathbf{u},\ \mathbf{u} \rangle = \langle \mathrm{H}|_{X_d}\mathbf{u},\ \mathbf{u} \rangle = -\frac{|a|^2}{2} p(Q') Q'(\omega) + \sum_{j=1}^{N_r^-} (|a_j|^2 + |b_j|^2) \langle \mathrm{H}\mathbf{u}_r^{j,-},\ \mathbf{u}_r^{j,-} \rangle \\ - \sum_{k=1}^{N_{im}} 2|c_k|^2 |\langle z_{im}^k \mathbf{u}_{im}^k,\ \sigma_1 \bar{\mathbf{u}}_{im}^k \rangle| - \sum_{l=1}^{N_c} 2(|d_l|^2 + |q_l|^2) |\langle z_c^l \mathbf{u}_c^l,\ \sigma_1 \bar{\mathbf{u}}_c^l \rangle| < 0$$

for all $\mathbf{u} \in X_-,\ \mathbf{u} \neq \mathbf{0}$. Therefore by the Rayleigh–Ritz theorem (see [19, Theorem XIII.3]),

$$(2.4)\quad n(\mathrm{H}) \geq p(Q') + 2N_r^- + N_{im} + 2N_c, \quad n(\mathrm{H}|_{X_d}) \geq p(Q') + 2N_r^- + N_{im} + 2N_c.$$

To derive the lower bound on the positive index of $\mathrm{H}|_{X_d}$, we introduce another auxiliary subspace $X_+ \subset X_d$ of vectors of the form

$$\mathbf{v} = a(1 - p(Q'))\phi_1 + \sum_{j=1}^{N_r^+}(a_j \mathbf{u}_r^{j,+} + b_j \sigma_3 \mathbf{u}_r^{j,+}) + \sum_{k=1}^{N_{im}} c_k(\mathbf{u}_{im}^k - s_k \bar{\mathbf{u}}_{im}^k) + \sum_{l=1}^{N_c} d_l(\mathbf{u}_c^l + e^{-i\varphi_l}\bar{\mathbf{u}}_c^l) + q_l \sigma_3(\mathbf{u}_c^l + e^{-i\varphi_l}\bar{\mathbf{u}}_c^l) ,$$

where the eigenvectors $\mathbf{u}_r^{j,+}$ correspond to real eigenvalues of positive energy of the linearized operator and $a, a_j, b_j, c_k, d_l, q_l \in \mathbb{C}$ are arbitrary, such that $\dim X_+ = 1 - p(Q') + 2N_r^+ + N_{im} + 2N_c$. A straightforward computation analogous to the one above yields the negativity of the quadratic form of the operator $-\mathrm{H}|_{X_d}$ on any vector $\mathbf{v} \in X_+$, $\mathbf{v} \neq \mathbf{0}$. Hence via the Rayleigh–Ritz theorem

$$p(\mathrm{H}|_{X_d}) \geq 1 - p(Q') + 2N_r^+ + N_{im} + 2N_c \tag{2.5}$$

The following orthogonal decomposition is the completeness of spectrum, and was proven in Proposition 4.1 of [11] using the method of wave operators. The absence of the residual spectrum was shown in [7] via the theory of Pontryagin spaces. Hence

$$L^2(\mathbb{R}^3,\ \mathbb{C}^2) = \mathrm{Ran}(P_d^*) \oplus \mathrm{Ran}(I - P_d),$$

where $\mathrm{Ran}(I - P_d)$ coincides with the set of functions of the continuous spectrum of the operator $\sigma_1\mathrm{H}$, and the subspaces involved in the direct sum above are mutually orthogonal. Thus, we can decompose the zero mode ϕ_0 of the linearized operator as

$$\phi_0 = \phi_0^* + \phi_0^{**},$$

where $\phi_0^* \in \mathrm{Ran}(P_d^*)$, $\phi_0^{**} \in \mathrm{Ran}(I - P_d)$, and $\langle \phi_0^*, \phi_0^{**} \rangle = 0$. Since $\phi_0 \notin \mathrm{Ran}(I - P_d)$, we have $\phi_0^* \neq \mathbf{0}$ and it is the zero mode of the restricted operator $\mathrm{H}|_{X_d}$. We choose the basis out of the eigenvectors corresponding to negative and positive eigenvalues of the operator $\mathrm{H}|_{X_d}$, which numbers are estimated from below in (2.4) and (2.5) respectively, and the zero mode ϕ_0^* in the subspace $\mathrm{Ran}(P_d^*)$, whose dimension equals $\dim(\mathrm{Ran}(P_d^*)) = 2 + 2N_r + 2N_{im} + 4N_c$. Thus we obtain

$$n(\mathrm{H}) \geq n(\mathrm{H}|_{X_d}) = p(Q') + 2N_r^- + N_{im} + 2N_c \tag{2.6}$$

for problem (1.3) on $L^2(\mathbb{R}^3,\ \mathbb{C}^2)$. It remains to show that this lower bound is the equality. An elementary computation yields the identity, and we estimate the quadratic forms of the operators involved in it:

$$\mathrm{H} = P_d^*\mathrm{H}P_d + (I - P_d^*)\mathrm{H}(I - P_d) + (I - P_d^*)\mathrm{H}P_d + P_d^*\mathrm{H}(I - P_d) \tag{2.7}$$

Let $\mathbf{w} \in L^2(\mathbb{R}^3,\ \mathbb{C}^2)$ be arbitrary. For the quadratic form of the third operator in the right side of (2.7), we have $\langle (I - P_d^*)\mathrm{H}P_d\mathbf{w},\ \mathbf{w}\rangle = \langle \sigma_1\mathrm{H}P_d\mathbf{w},\ \sigma_1(I - P_d)\mathbf{w}\rangle = 0$

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since $\sigma_1 \mathrm{H} P_d \mathbf{w} \in X_d$ and $\mathrm{Ran}(I - P_d)$ is skew-orthogonal to X_d (see Definition 2.4). Analogously, the quadratic form of the operator $P_d^* \mathrm{H}(I - P_d)$ vanishes for any $\mathbf{w} \in L^2(\mathbb{R}^3, \mathbb{C}^2)$. Since $\mathrm{Ran}(I - P_d)$ coincides with the set of functions of the continuous spectrum of the linearized operator on which the energy operator H under our assumptions is strictly positive (see [11, Theorem 2.11] and [7, Theorem 4]), we have $\langle (I - P_d^*)\mathrm{H}(I - P_d)\mathbf{w}, \mathbf{w}\rangle \geq 0$. Hence $\langle \mathrm{H}\mathbf{w}, \mathbf{w}\rangle \geq \langle \mathrm{H}|_{X_d}\mathbf{w}, \mathbf{w}\rangle$. Then by the min-max principle (see [19]) $n(\mathrm{H}) \leq n(\mathrm{H}|_{X_d})$. This inequality along with (2.6) yield identity (1.5). Note that a similar argument using the min-max principle can be adopted for proving the coercivity of the energy functional on the subspace of vector-functions skew-orthogonal to the generalized kernel of the linearized NLS operator, which plays a crucial role in the study of dynamics of NLS solitary waves in an external potential (see [13]). ■

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