PART 3

DYNAMO THEORY
AND MAGNETIC DISSIPATION

MEAN-FIELD MAGNETOHYDRODYNAMICS OF THE SOLAR CONVECTION ZONE

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1. Introduction

1.1. Observations of the solar surface show that some of the physical quantities, especially the velocity field and the magnetic field, show random character.

Their time and space variations are irregular and neither can be predicted with certainty. However, at least the existence of the 22-year solar cycle indicates that these quantities show in the average a regular, periodic behavior.

The idea of constructing a theory of the mean quantities was conceived by M. Steenbeck *et al.* (1963) in connection with an explanation of the solar cycle. They suggested to develop a theory of the mean electromagnetic fields where the action of the turbulent motion is regarded by constitutive equations.

1.2. There are previous suggestions to explain some large-scale effects out of the action of small scale effects. In this connection may be noted a paper of Gurewitch and Lebedinskii (1945) where the convective motions of the plasma in the convection zone of the Sun are considered responsible for the build-up of the magnetic fields of the sunspots.

Also of interest is a paper of Biermann (1951) where the influence of an anisotropic convection on the rotation law of a rotating body is discussed. His finding that the large-scale action of the convection makes a rigid rotation impossible, can be regarded as the discovery of the basic phenomena responsible for the differential rotation.

As a next step we note the papers of Sweet (1950) and Csada (1951), who suggested that turbulence may destroy large-scale magnetic fields. The problem originates mainly in the observation of sunspots: A sunspot magnetic field of a diameter $\bar{L} \approx 10^4$ km should have a decay time $\bar{T} = \mu\sigma\bar{L}^2$ of about 3000 yr ($\mu \sim$ permeability, $\sigma \sim$ conductivity). But it is observed to decay in about 3 or 4 months. From this discrepancy it was concluded that turbulence may provide for an enhancement of the magnetic diffusivity or, what comes out to be the same, for a diminution of the conductivity with respect to the large-scale field. The same was expected to be the case for the permeability (Zeldovich, 1956).

Without doubt the most important result in this regard is due to Parker (1955). He discovered that small-scale cyclonic motions on a rotating body can produce a poloidal magnetic field out of a toroidal field. By combining this effect with a constant shear Parker was able to prove the existence of migrating dynamo waves for homogeneous models.

As a last example we mention the random walk migration of magnetic regions as described by Leighton (1964). This is a large-scale behaviour which is caused by

Bumba and Kleczek (eds.), Basic Mechanisms of Solar Activity, 305–321.

turbulent diffusion under the presence of the large-scale shear due to differential rotation.

2. Basic Ideas of Mean-Field Magnetohydrodynamics

2.1. Mean-field magnetohydrodynamics, or – as the first step – mean-field electrodynamics intends to explain those different phenomena in the frame of one theory. It starts from the assumption that the basic laws describing those phenomena are well known, namely the Maxwell equations

$$\operatorname{rot} \mathbf{E} = -\dot{\mathbf{B}}, \qquad \operatorname{rot} \mathbf{H} = \mathbf{j}, \qquad \operatorname{div} \mathbf{B} = 0, \tag{1}$$

combined with the constitutive equations

$$\mathbf{B} = \mu \mathbf{H}, \qquad \mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}), \tag{2}$$

and the Navier–Stokes equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \operatorname{grad})\mathbf{u}\right) = -\operatorname{grad} p + \mathbf{j} \times \mathbf{B} + \cdots, \tag{3}$$

the equation of continuity

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{u} = 0, \tag{4}$$

and equations of state.

Here $\mathbf{E}$ is the electric field strength, $\mathbf{H}$ is the magnetic field strength, $\mathbf{B}$ the magnetic induction, $\mathbf{j}$ the current density, $\mathbf{u}$ the velocity field, p the pressure and ρ the density. By the suspension points in the Navier–Stokes equation we indicate that there are additional quantities which need not be specified in our consideration.

By taking the average we can derive equations for the mean quantities, in this way following the proposal of O. Reynolds (1883) made for the Navier–Stokes equation in connection with his investigations of hydrodynamic turbulence.

2.2. It is very convenient to use statistical means for the theoretical (mathematical) investigations. We adopt the point of view of Gibbs that there are one million or more Suns and we take the average over this set of Suns (Figure 1). This average can be exchanged with all linear operations, especially with integration and differentiation operation, thus simplifying the deductions.

However, from the observational data of the Sun we can only derive averages over the space of the time coordinates. The question arises whether we can apply the deduced results to the mean quantities derived from the observational data.

For a quantity F we denote the average by $\bar{F}$, and the fluctuation $F - \bar{F}$ by F'. For the statistical average we have the rule

$$\bar{\bar{F}} = \bar{F} \tag{5}$$

or, equivalently,

$$\overline{F'} = 0. \tag{6}$$

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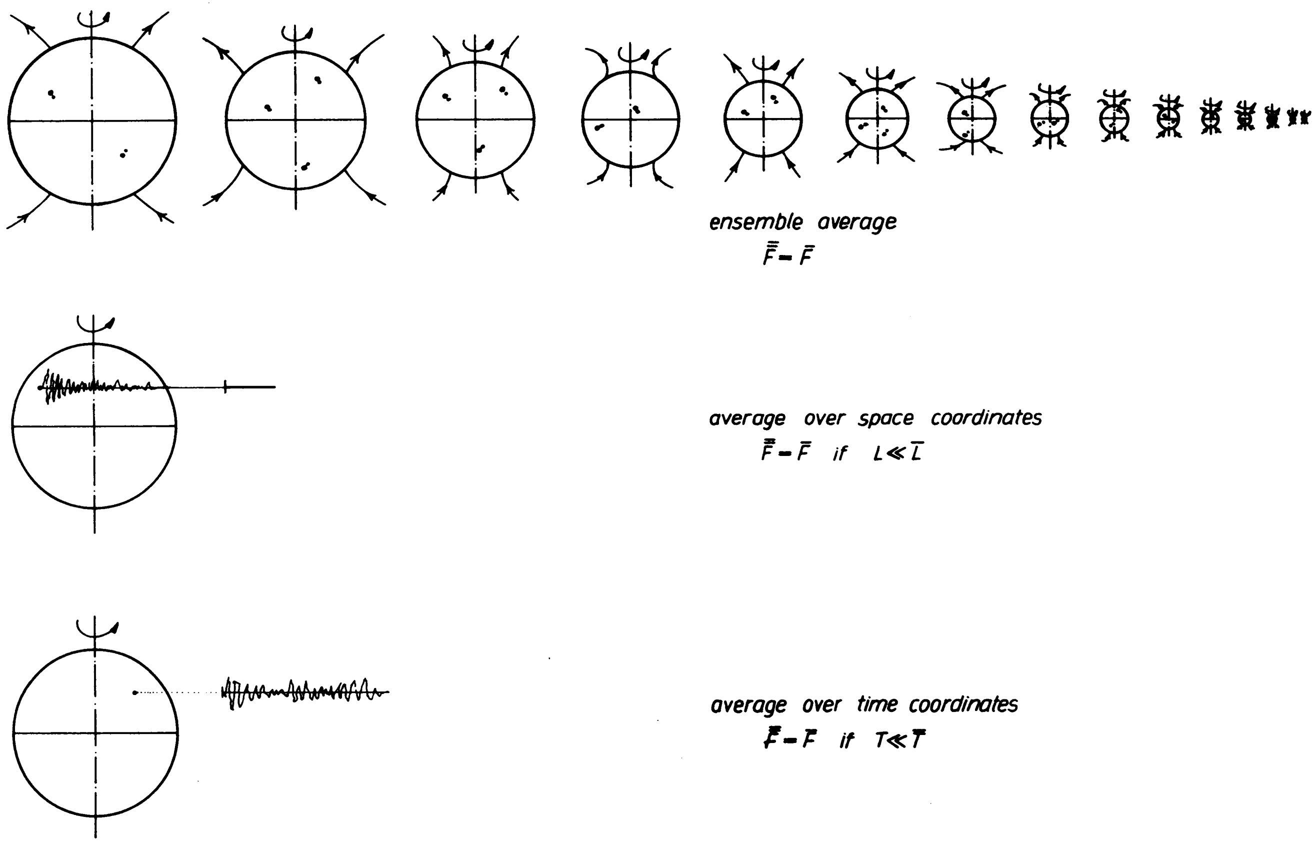

Fig. 1. Schematic representation of different kinds of taking an average. The most convenient one, used in theoretical investigations, is the ensemble average. The idea is that there is a set of a great number of Suns and the average is taken over this set of Suns. By taking each cycle as a representation of a Sun and averaging over all cycles, a possibility for an ensemble average is given. From observational data one can in most cases only take the average with respect to space or time coordinates. Then especially the rule $\bar{\bar{F}} = \bar{F}$ is only approximately valid. (L, T scales of the turbulent field; $\bar{L}$, $\bar{T}$ scales of the mean field.)

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For an average with respect to the space or time coordinates the relations (5) or (6) are only approximately valid, and that if the mean quantities behave nearly constant over the scales used for the average. More precisely, let $\bar{L}$ and $\bar{T}$ be the scales of the mean field, and L and T those of the turbulent field, i.e. L the correlation length and T the correlation time. Then (5) and (6) are valid only in the sense

$$\bar{\bar{F}} = \bar{F} + 0\left(\frac{L}{\bar{L}}, \frac{T}{\bar{T}}\right), \qquad \overline{F'} = 0\left(\frac{L}{\bar{L}}, \frac{T}{\bar{T}}\right). \tag{7}$$

Thus the application of the theory is indicated only if

$$L \ll \bar{L}, \qquad T \ll \bar{T}, \tag{8}$$

and is questionable whenever (8) is violated.

For the convection we have $L \approx 10^3$ km, $T \approx 3 \times 10^2$ s. Compared with the scales of the mean magnetic field, $\bar{L} \approx R_0 = 7 \times 10^5$ km, $\bar{T} = 6 \times 10^8$ s (period of the solar cycle), the relation (8) is, indeed, rather well fulfilled. With respect to the space coordinates it is less satisfactory for the supergranulation.

2.3. Under averaging the Maxwell-equations remain unchanged since they are linear; the same holds for the constitutive equation connecting **B** and **H**. However, Ohm's law includes a non-linear term, $\mathbf{u} \times \mathbf{B}$, which gives rise to an additional term in Ohm's law for the mean fields, the turbulent emf

$$\mathscr{E} = \overline{\mathbf{u}' \times \mathbf{B}'}. \tag{9}$$

Thus we find as the basic equations for mean-field electrodynamics

$$\operatorname{curl} \bar{\mathbf{E}} = -\mathring{\bar{\mathbf{B}}}, \qquad \operatorname{curl} \bar{\mathbf{H}} = \bar{\mathbf{j}}, \qquad \operatorname{div} \bar{\mathbf{B}} = 0, \tag{10}$$

with the constitutive equations

$$\bar{\mathbf{B}} = \mu \bar{\mathbf{H}}, \qquad \bar{\mathbf{j}} = \sigma(\bar{\mathbf{E}} + \bar{\mathbf{u}} \times \bar{\mathbf{B}} + \mathscr{E}). \tag{11}$$

The occurrence of the turbulent emf $\mathscr{E}$ in Ohm's law for the mean electromagnetic fields is the crucial point of the theory. It is essential for providing for an additional equation expressing $\mathscr{E}$ by the mean fields. We start from the well-known induction equation

$$\frac{\partial \mathbf{B}}{\partial t} - \operatorname{curl}(\mathbf{u} \times \mathbf{B}) - \frac{1}{\mu\sigma} \Delta \mathbf{B} = 0, \tag{12}$$

which can be easily derived from the Equations (1) and (2). Taking the average we find

$$\frac{\partial \bar{\mathbf{B}}}{\partial t} - \operatorname{curl}(\bar{\mathbf{u}} \times \bar{\mathbf{B}}) - \frac{1}{\mu\sigma} \Delta \bar{\mathbf{B}} = \operatorname{curl}(\overline{\mathbf{u}' \times \mathbf{B}'}). \tag{13}$$

Subtracting (13) from (12) we have the equation for the fluctuations

$$\frac{\partial \mathbf{B}'}{\partial t} - \operatorname{curl}(\bar{\mathbf{u}} \times \mathbf{B}') - \frac{1}{\mu\sigma} \Delta \mathbf{B}' - \operatorname{curl}(\mathbf{u}' \times \mathbf{B}' - \overline{\mathbf{u}' \times \mathbf{B}'})$$
$$= \operatorname{curl}(\mathbf{u}' \times \bar{\mathbf{B}}). \tag{14}$$

While an exact solution of Equation (14) is hardly possible, we can draw the general conclusion that $\mathbf{B}'$ as well as $\mathscr{E}$ is a linear functional of $\bar{\mathbf{B}}$. Thus we have

$$\mathscr{E} = \mathscr{L}(\bar{\mathbf{B}}), \tag{15}$$

with $\mathscr{L}$ denoting the linear functional dependence. This conclusion can be made without any restrictive assumption. If we now assume the relation (8) to be fulfilled, the field $\bar{\mathbf{B}}$ in Equation (14) can be taken constant or a linear function of the space coordinates. Thus $\mathscr{E}$ depends only on the local field, and (15) reduces to

$$\mathscr{E}_i = a_{ik}\bar{B}_k + b_{ikl}\frac{\partial \bar{B}_k}{\partial x_l}, \tag{16}$$

where the error is of the order $0((L/\bar{L})^2, T/\bar{T})$. a_{ik}, b_{ikl} are certain pseudo-tensors depending on the mean properties of the turbulence. Relation (16) is the needed additional constitutive equation. It must be noted that (16) is valid under the assumption (8), which is a guarantee for the applicability of our theory. (16) is valid independent of what turbulence is regarded. A model is characterized by the pseudo-tensors a_{ik}, b_{ikl}.

3. Ohm's Law for the Mean Electromagnetic Quantities in the Solar Convection Zone

3.1. It is furthermore important to know that more explicit expressions for the tensors a_{ik}, b_{ikl} can be obtained if it is taken into account that these quantities are derivable out of the tensorial mean quantities of the turbulent velocity field by tensorial operations.

The simplest example is given by homogeneous isotropic turbulence, where the only available tensorial quantities are the isotropic tensors, i.e. the Kronecker Tensor δ_{ik} and the Levi-Cività Tensor ε_{ikl}. Hence we have

$$a_{ik} = \alpha\delta_{ik}, \qquad b_{ikl} = \beta\varepsilon_{ikl}, \tag{17}$$

where α is a pseudo-scalar and β a scalar. Introducing (17) into (16) and then in (11) we find Ohm's law for the main quantities as

$$\mathbf{j} = \sigma_T(\mathbf{E} + \alpha\mathbf{B}), \tag{18}$$

where the conductivity with respect to the mean field is given by

$$\sigma_T = \frac{\sigma}{1 + \mu\sigma\beta}. \tag{19}$$

We find that the action of homogeneous isotropic turbulence changes conductivity and produces a new effect – named α-effect. We will come back to this point later.

Obviously, in the convection zone of the Sun there is some structure caused mainly by two vectorial quantities, the rotational motion, represented by the angular velocity $\boldsymbol{\omega}$, and the gradient of density $\mathbf{g}$ in the radial direction. Restricting to the linear expressions in the vectors $\boldsymbol{\omega}$ and $\mathbf{g}$ we have the general ansatz for the tensors a_{ik}, b_{ikl}

$$\begin{aligned} a_{ik} &= \alpha_0(\mathbf{g}\cdot\boldsymbol{\omega})\delta_{ik} + \alpha_1(\omega_i g_k + \omega_k g_i) \\ &\quad + \alpha_2(\omega_i g_k - \omega_k g_i) + \gamma\varepsilon_{ikl} g_l \\ &= \alpha_0(\mathbf{g}\cdot\boldsymbol{\omega})\delta_{ik} + \alpha_1(\omega_i g_k + \omega_k g_i) \\ &\quad + \varepsilon_{ikl}(\gamma g_1 + \alpha_2 \varepsilon_{lpq}\omega_p g_q); \end{aligned} \tag{20}$$

$$b_{ikl} = \beta\varepsilon_{ikl} + \beta_1\omega_i\delta_{kl} + \beta_2\omega_k\delta_{il} + \beta_3\omega_l\,\delta_{ik}\,. \tag{21}$$

In the construction we have taken into account that both tensors are skew. The expressions (20) and (21) lead to an Ohm's law for the solar convection zone given by

$$\begin{aligned} \bar{\mathbf{j}} = \sigma_T(&\bar{\mathbf{E}} + \bar{\mathbf{u}}\times\bar{\mathbf{B}} + \alpha_0(\mathbf{g}\cdot\boldsymbol{\omega})\bar{\mathbf{B}} + \alpha_1((\mathbf{g}\cdot\bar{\mathbf{B}})\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\bar{\mathbf{B}})\mathbf{g}) \\ &+ \bar{\mathbf{B}}\times(\gamma\mathbf{g} + \alpha_2\boldsymbol{\omega}\times\mathbf{g}) \\ &+ (\beta_2+\beta_3)\,\mathrm{grad}\,(\boldsymbol{\omega}\cdot\bar{\mathbf{B}}) - \mu\beta_3\boldsymbol{\omega}\times\bar{\mathbf{j}}); \end{aligned} \tag{22}$$

σ_T is given by (19).
The only unknown quantities are the scalars α_0, α_1, α_2, β, β_2, β_3, γ.

3.2. A theoretical determination of the quantities is rather complicated and not fully solved, since one is here confronted with the difficult closure problem of the theory of turbulence. There are a lot of papers which present approximative calculations of those constants. We will not go into details here and refer the interested reader to review papers (Krause and Rädler (1971), Roberts (1971), Roberts and Stix (1971), Vainshtein and Zeldovich (1972), Roberts and Soward (1975), Krause (1975)). The most often used approximation neglects the term curl $(\mathbf{u}'\times\mathbf{B}' - \overline{\mathbf{u}'\times\mathbf{B}'})$ in Equation (14). The resulting equation for $\mathbf{B}'$ is linear and can be solved by a Green's function method. We will speak of the second order correlations approximation (SOCA)*, since all correlations of higher than second order are neglected. This approximation is valid if

$$\min\,(R_m, S) \ll 1 \tag{23}$$

where R_m is the magnetic Reynolds number,

$$R_m = \mu\sigma u' L\,, \tag{24}$$

and S the Stroughal Number

$$S = u'\frac{T}{L}\,; \tag{25}$$

u' denotes the turbulent rms velocity.

For convection data $\mu = 4\pi\times10^{-7}\,\Omega S\ \mathrm{m}^{-1}$, $\sigma = 10^3\,\Omega^{-1}\,\mathrm{m}^{-1}$, $u' \approx 300\ \mathrm{m\,s}^{-1}$ we find

$$R_m \approx 3\times10^5\,, \qquad S \approx 10^{-1}\,. \tag{26}$$

* This approximation is sometimes named 'first-order smoothing' or 'Herrings approximation' in hydrodynamic turbulence theory.

The value for S encourages to apply the results derived on the basis of SOCA to the convection zone. If $R_m \gg 1$ one finds in the homogeneous isotropic case

$$\alpha = -\frac{T}{3}\overline{\mathbf{u}' \cdot \operatorname{curl} \mathbf{u}'}, \tag{27}$$

$$\beta = \frac{\overline{\mathbf{u}'^2} T}{3}. \tag{28}$$

(see, e.g., Steenbeck and Krause, 1969a).

From (27) it becomes obvious that $\alpha \neq 0$ is connected with the fact that one kind of helical motions is more probable than the other. For example, α is positive if left-handed helical motions are more probable than right-handed.

3.3. The value of β depends on quantities that can be taken from observations. With the data belonging to (26) we have

$$\beta_{cz} \approx 10^7 \text{ m}^2 \text{ s}^{-1}. \tag{29}$$

If we derive from Ohm's law (18), with $\alpha = 0$, the induction equation for the mean magnetic field, we find

$$\frac{\partial \bar{\mathbf{B}}}{\partial t} - \left(\frac{1}{\mu\sigma} + \beta\right) \Delta \bar{\mathbf{B}} = 0, \tag{30}$$

and we see that β proves to be the turbulent magnetic diffusivity. For the ratio $\beta/(1/\mu\sigma)$ we obtain from (29)

$$\frac{\beta}{1/\mu\sigma} = \mu\sigma\beta \approx 10^4. \tag{31}$$

This implies especially for the decay time of a large-scale magnetic field that it is shortened by a factor 10^4. One is tempted to apply this result to the decay of sunspots, since the numerical agreement with the observations is very good. Furthermore, more detailed discussions of the decay of sunspot areas by a diffusion model based on Equation (30) by Meyer and Schmidt (1973) and us (Krause and Rüdiger (1975)) show not only qualitatively but also quantitatively a nice agreement with the results derived from observations by Bumba (1963).

However, this discussion ignores that there is a rather strong magnetic field in sunspots, which, perhaps, suppresses the turbulence or at least influences the turbulence, and one has to ask why the magnetic field should be unable to brake its own decay. We will revert to this point later.

A further point of interest in this connection is the period of the solar cycle. Observations show the Sun's global magnetic field alternating with a period of about 22 years; its characteristic length scale $\bar{L}$ must be expected to be of the order of the thickness of the convection zone, that means $\bar{L} \approx 10^5$ km and hence

$$\frac{\bar{L}^2}{\beta} \approx 10^9 \text{ s} \approx 30 \text{ yr}.$$

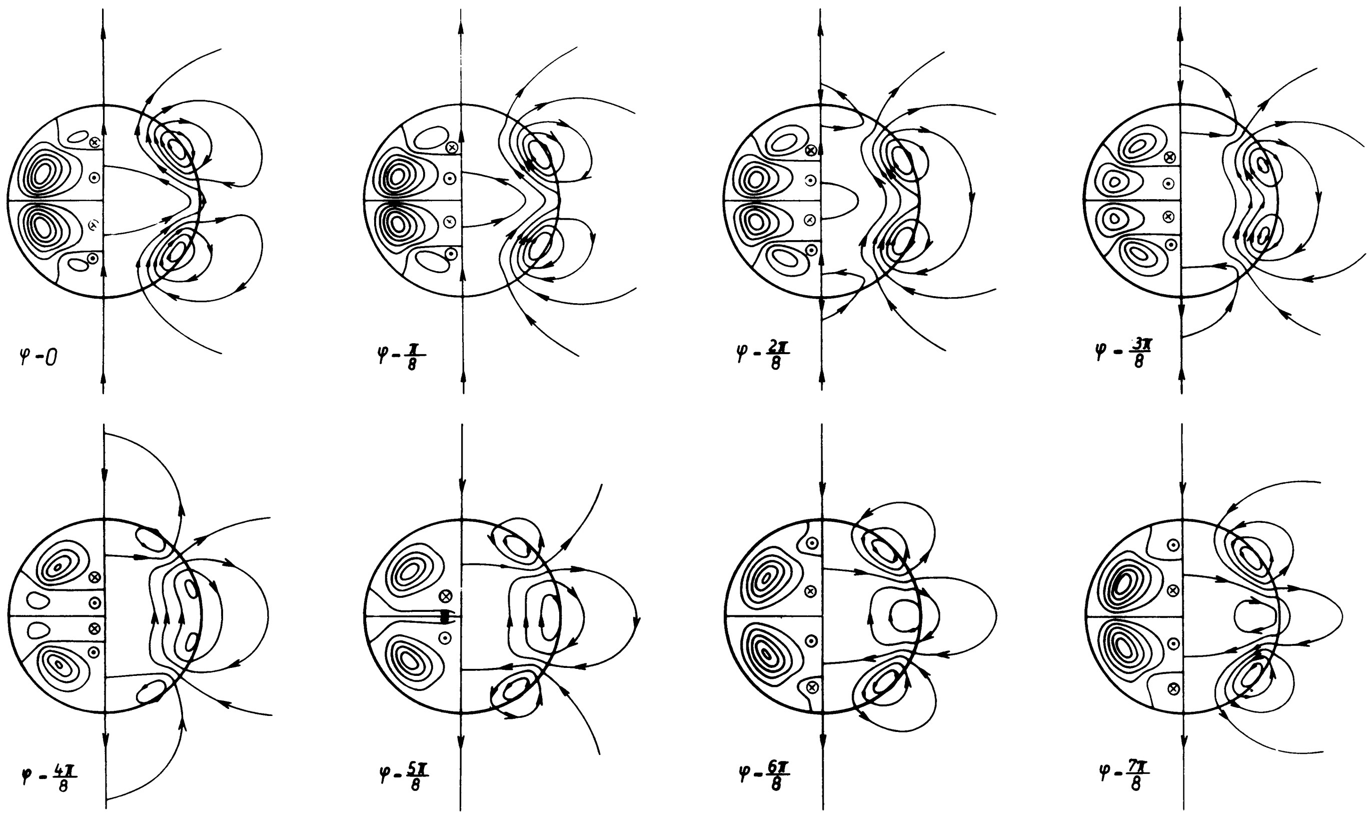

Fig. 2. A model of a self-excited dynamo for the solar cycle. Eight phases of a half cycle of the magnetic field are represented. The phase $\varphi = 0$ corresponds to the phase of maximum activity (maximum toroidal field). At the left half of each phase picture the curves of constant field strength of the toroidal field are drawn, the field lines of the poloidal field at the right. One realizes the migration of the field towards the equator. (After Steenbeck and Krause, 1969a.)

The use of the turbulent magnetic diffusivity leads already to an agreement in the time scales.

3.4. Of greatest interest in dynamo theory is the quantity $\alpha = \alpha_0(\mathbf{g} \cdot \boldsymbol{\omega})$. An estimation from observations is not possible as can be taken from (27), since there is no direct information about the quantity $\overline{\mathbf{u}' \cdot \operatorname{curl} \mathbf{u}'}$. Steenbeck *et al.* (1966) derived an expression in the low-conductivity limit, $R_m \ll 1$. For the high-conductivity limit Krause (1968) derived the relation

$$\alpha = -T^2\overline{\mathbf{u}'^2}\boldsymbol{\omega} \cdot \operatorname{grad}(\ln u'\rho)\,, \tag{32}$$

which for the solar convection zone (Steenbeck and Krause, 1969a) gives

$$\alpha = \frac{T^2\overline{u'^2}\omega}{H} \cos\vartheta \approx 0.26 \cos\vartheta \text{ m s}^{-1} \tag{33}$$

where H is the scale height. Further calculations of α have been given by Moffatt (1970a, b, 1974) and by Stix (unpublished).

Calculations of spherical dynamo models based on the combined action of α-effect and differential rotation have shown that a simulation of details of the solar cycle is possible to a considerable degree (Figure 2 and Figure 3). I am not going to discuss here the coefficients α_0, α_1, α_2 in detail; it will be done in subsequent papers. The term in (22) $(\beta_2 + \beta_3)$ grad $(\boldsymbol{\omega} \cdot \bar{\mathbf{B}})$ is of no further interest since it is compensated by

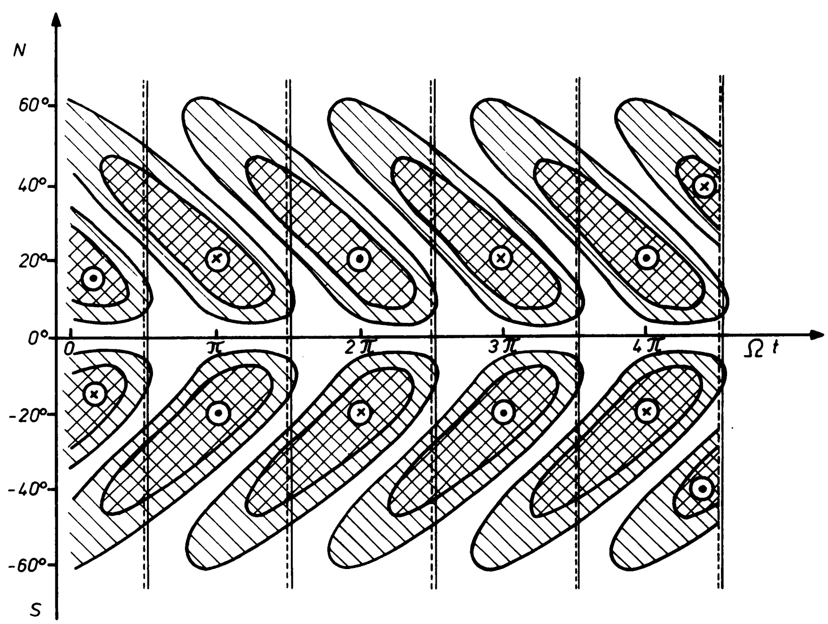

Fig. 3. Butterfly diagram derived from the model described in Figure 2. In the hatched areas the toroidal field strength is larger than $\frac{1}{3}$, in the cross-hatched larger than $\frac{2}{3}$ of its maximum value. Again one realizes the migration towards the equator just as it is observed at the Sun.

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space charges. The last term, $-\mu\beta_3\boldsymbol{\omega}\times\bar{\mathbf{j}}$, can also provide for dynamo action if combined with differential rotation (Rädler, 1969).

A remark on differential rotation may be appropriate here. Ohm's law as given by (22) for the mean electromagnetic fields in the solar convection zone was derived under the assumption of rigid rotation. In the presence of differential rotation further terms will appear describing the turbulent diffusion under the influence of the shear of the differential rotation. In this way Leighton's (1964) heuristic discussion of the random walk of bipolar magnetic regions has found its place in the frame of this deductive theory.

4. The rms of the Magnetic Field Fluctuations

4.1. The decay of a magnetic field in a conductor is governed by a diffusion equation

$$\frac{\partial \mathbf{B}}{\partial t} - \eta\, \Delta \mathbf{B} = 0\,; \tag{34}$$

with the magnetic diffusivity $\eta = 1/\mu\sigma$.

The magnetic energy is transformed into heat. The decay of the mean magnetic field also obeys a diffusion equation, namely

$$\frac{\partial \bar{\mathbf{B}}}{\partial t} - (\eta + \eta_T)\, \Delta \bar{\mathbf{B}} = 0\,, \tag{35}$$

$\eta = 1/\mu\sigma$ and $\eta_T = \beta$, as shown in Section 3.3. But the physical background is quite different: The decay of $\bar{\mathbf{B}}$ is both an Ohmic decay *and* a decay of the mean field in the turbulent field. Especially in a situation as it is met with in the convection zone the decay in the small-scale field is much more efficient than the Ohmic decay.

If we assume $\mathbf{B}'$ to show only Ohmic decay, we have a time scale of

$$\mu\sigma L^2 \approx 10^9\ \mathrm{s} \approx 30\ \mathrm{yr}\,. \tag{36}$$

On the other hand for a large-scale field we have the decay times

$$\bar{T} = \frac{\bar{L}^2}{\eta_T} = \begin{cases} 10^7\ \mathrm{s} \approx 4\ \text{months, if } \bar{L} = 10^4\ \mathrm{km} \\ 10^9\ \mathrm{s} \approx 30\ \text{years, if } \bar{L} = 10^5\ \mathrm{km} \end{cases} \tag{37}$$

Comparing these values we realize that the decay in the small-scale field is more rapid than the Ohmic decay of the small-scale field. Consequently we have to expect in the stationary case that the magnetic energy stored in the fluctuations is larger, or even much larger, than in the mean field.

4.2. According to the general arguments leading to (16) we have the relation

$$\overline{B'^2} = q_{ik} \bar{B}_i \bar{B}_k\,, \tag{38}$$

with a certain tensor q_{ik} depending on the properties of the turbulent motion. The inequalities (8) were a sufficient condition for the validity of (16), and so are they for the validity of (38). An error of the order $0(L/\bar{L},\ T/\bar{T})$ must be expected.

In the isotropic case (38) reduces to

$$\overline{B'^2} = q\bar{B}^2 , \tag{39}$$

where $3q$ is the trace of the tensor q_{ik}. In the framework of SOCA one finds in the high-conductivity limit ($R_m \gg 1$)

$$q = \tfrac{1}{3}\mu\sigma u'^2 L \tag{40}$$

(Bräuer and Krause, 1973), which with the data for the convection zone implies

$$\overline{B'^2} \approx 10^4 \cdot \bar{B}^2 . \tag{41}$$

The relations (40) and (41) had been derived already by Steenbeck and Krause (1969a) using physical arguments.

There have been given other estimations of q by less deductive methods than that leading to (40). Parker, in a paper dating from 1963 expected a proportionality of q to $(R_m^{1/4}/\ln R)^2$, which he retraced in 1969 and in 1973 replaced by the relation

$$q \approx R_m . \tag{42}$$

A numerical investigation of Moss (1970) of a two-dimensional model indicates the behaviour

$$q \approx R_m^{0.7} \tag{43}$$

The result (42) of Parker is in agreement with (40) because it rests on the assumption $S \approx 1$, which is often used in investigations of turbulence.

For the convection zone of the Sun we find from (40) for the magnetic field rms

$$B' \approx 100\, \bar{B} , \tag{44}$$

which is nearly the same value as derived from the result of Moss ($R_m^{0.35} \approx 83$); Parker's formula gives about five times this value.

4.3. Because of its simplicity we will present here the derivation of (40) given by Steenbeck and Krause (1969a): We start from the induction equation

$$\frac{\partial \mathbf{B}'}{\partial t} = \operatorname{curl}\,(\mathbf{u}' \times \bar{\mathbf{B}}) , \tag{45}$$

where we have omitted the diffusion term since we integrate over a time interval of the length of the correlation time, which is small compared with $\mu\sigma L^2$. For the correlated part we find from (45)

$$B'_{\text{cor}} \approx \frac{u'T}{L} \bar{B} . \tag{46}$$

This field part is now assumed to exist over the time $\mu\sigma L^2$. We assume the turbulence to produce in every time interval of length T one field part (46), so that the complete field is the statistical sum of such field parts. The number N of the field parts existing at the same point of time is equal to

$$N = \frac{\mu\sigma L^2}{T} ; \tag{47}$$

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hence we get for B' itself from (46) and (47)

$$B' \approx \sqrt{N} B'_{\rm cor} \approx \sqrt{\mu\sigma u'^2 T} \bar{B}, \tag{48}$$

in agreement with (40).

4.4. From these results a very interesting feature becomes visible: According to (44) we have to expect the mean field to be of the order of about 1% of the rms of the fluctuating field. This is, indeed, a theoretical prediction, but if we adopt this result, we must conclude that it is hardly possible to detect the mean field by observation. And this may, perhaps, explain why some observers do not believe in the mean field. However, the mean field must exist, since otherwise there would be no explanation for the existence of the small-scale field. The large-scale field is the exciter field in a separate excited dynamo and the small-scale field is the excited field, which in our case is much larger than the exciter field.

In this light one must see the investigations by Howard (1974a, b) who intended to derive the mean magnetic field on the solar surface by averaging the observational data. I asked Dr Howard for his opinion on how strong the magnetic field rms at the solar surface in a quiet region might be. His estimate lies between 1 G and 10 G.

Hence, according to (44), the mean magnetic field will be between 0.01 G and 0.1 G, which leads to the conclusion, that the averages derived by Howard give no information about the mean field. Answering the same question Dr Stenflo argued that Dr Howard probably underestimates the rms by a factor of about 20. His reasons Dr Stenflo already explained in his talk yesterday when he demonstrated that magnetographs record an average over some area. Hence, according to Stenflo, the rms would be between 20 G and 200 G. Therefore the mean field must be expected in the order of 1 G and in this picture the averages derived by Howard give some correct impression of the mean field. A great accuracy cannot be expected because the mean field is only 1% of the rms.

Against a dynamo theory of the solar cycle there is sometimes the objection raised that the rate of field diffusion and reconnection might be too small. Relation (39) with q from (40) can possibly give an answer, especially if written in the form

$$\overline{B'^2} = \frac{\eta_T}{\eta} \bar{B}^2 . \tag{49}$$

Now it becomes obvious that a small molecular magnetic diffusivity provides for an enrichment of magnetic energy in the small scales.

Our discussion shows that the fluctuating magnetic field can be, and in the case of the Sun in fact is, much larger than the mean field. Since in the frame of SOCA the equation for the fluctuating field had been linearized one might expect that the results (41) and (49) could be out of the range of parameters where the theory can be applied. This is, however, not the case: The derivation of (40) and subsequently of (41) and (49) is restricted to the assumptions (8) and (23) only, wherefrom no restriction of B' follows. This question is discussed in more detail by Krause and Roberts (1976).

5. The Back-Reaction of the Magnetic Field to the Motion

5.1. Up to this point we have discussed situations where the magnetic energy is assumed to be small compared with the kinetic energy,

$$\frac{1}{2\mu}(\bar{B}^2+\overline{B'^2}) \ll \frac{\rho}{2}(\bar{u}^2+\overline{u'^2}) . \tag{50}$$

If we also take into account the back-reaction of the magnetic field on the motion, we have to regard the Lorentz-force in the Navier–Stokes equation. Since we have

$$\overline{\mathbf{j}\times\mathbf{B}} = \bar{\mathbf{j}}\times\bar{\mathbf{B}} + \overline{\mathbf{j}'\times\mathbf{B}'} , \tag{51}$$

an additional force in the Navier–Stokes equation for the mean fields, the Reynolds equation, will appear. As a consequence the dependence of the electromotive force, $\mathscr{E}$, on the mean magnetic field is no longer linear, the tensors a_{ij} and b_{ijk} in Equation (16) will now depend on the mean magnetic field itself:

$$a_{ij} = a_{ij}(\bar{\mathbf{B}}) , \qquad b_{ijk} = b_{ijk}(\bar{\mathbf{B}}) ; \tag{52}$$

and the same is the case for the quantities α, β, etc.

In the incompressible case investigations have been carried out by Rädler (1974) and Rüdiger (1974) assuming a constant mean magnetic field. Roberts and Soward (1975) take into account a gradient of the mean magnetic field. Putting the tensors a_{ij} and b_{ijk} into the induction equation for the mean field (13), one is led to a non-linear equation. First attempts at solving the corresponding non-linear dynamo problem were carried out by Stix (1972) and by Rüdiger (1973).

5.2. I wish to end my talk with some remarks concerning the influence of strong magnetic fields on turbulent motions.

It is often said that strong magnetic fields suppress turbulence. However, experiments with liquid sodium show not a suppression of turbulence but a transformation of its structure, which appears to become two-dimensional: The motion takes place in planes orthogonal to the magnetic field and does not vary along it (Figure 4). Interesting results were published by Kit and Tsinober (1971). Perhaps the rotational motions in prominences as described by Rompolt (1975) indicate a two-dimensional motion occurring in a certain solar structure.

The conception of two-dimensional turbulence occurring in sunspots may open a possibility for explaining the rapid decay of sunspots by turbulent diffusion, although the strong magnetic field influences the turbulent motion, as shown by Krause and Rüdiger (1975). It is of interest that an estimation of the turbulent magnetic diffusivity derived from a comparison of a model proposed by Meyer and Schmidt (1973) with the observational data for long-lived spot groups as published by Bumba (1963), gives just the value (29) derived on the basis of mean-field magnetohydrodynamics. Furthermore, the models show also a relation like Waldmeier's rule of proportionality between the maximum area of a sunspot group and its lifetime. A quantitative comparison yields a turbulent magnetic diffusivity about 8 times the value of (29). This is so obviously because the derivation of Waldmeier includes the short-lived sunspot groups.

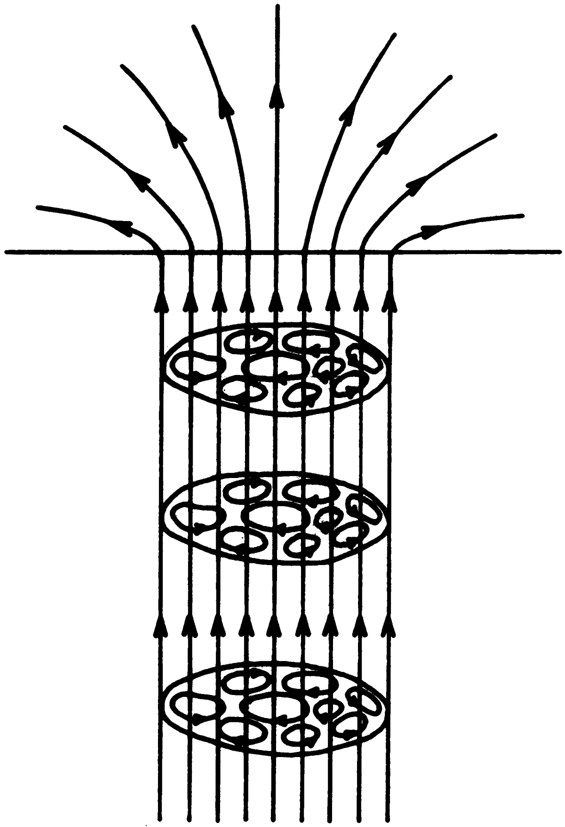

Fig. 4. Schematic drawing of the two-dimensional motion in a sunspot: The magnetic field is assumed to be perpendicular to the Sun's surface. The turbulent motion under the influence of the strong magnetic field assumes a two-dimensional structure. The velocity vectors lie completely in planes orthogonal to the magnetic field and the pattern does not change in the direction of the field.

It is also possible to fit curves describing the development of the areas of sunspot models based on turbulent diffusion to those observed at short-lived sunspot groups. The turbulent magnetic diffusivities derived in these cases are larger than the value of (29) by a factor up to 100. This probably indicates, compared with normal convection, an enhanced turbulence at moments where sunspots emerge until a few days later.

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DISCUSSION

Giovanelli: It worries me that observations show sunspots to decay by the transport of small packets of magnetic flux of field-strength approaching that of the sunspot, whereas theories of this type attempt to explain decay by a continuous spectrum of eddy ridges.

Krause: The turbulent decay describes a decay of the large-scale field in small-scale fields. I think your description of a transport of small packets of magnetic flux fits in this picture.

Zwaan: It should be kept in mind that the lowest decay rates discovered by Bumba apply to a *fraction* of the spots. In fact there is a *range* of decay times, from very high rates down to the lowest decay rate as a lower limit. Therefore I am surprised that your dissipation model predicts at the same time Bumba's *lowest* decay rates, and Waldmeier's *average* relation between sunspot areas and lifetimes.

Krause: This is an interesting point. The turbulent magnetic diffusivity derived from Bumba's long-living sunspot groups is $10^7\ m^2\ s^{-1}$, whereas the same quantity taken from Waldmeier's average relation is $8 \times 10^7\ m^2\ s^{-1}$.

Weiss: Two time scales are involved in the growth and decay of sunspots. A few long-lived spots decay slowly at a steady rate. The majority are formed and decay on a time scale of one to six days. Meyer, Schmidt, Wilson and I interpreted slow decay as a result of turbulent diffusion owing to small-scale convection but the rapid growth and break-up were related to supergranular motion, with the same scales as the sunspot itself. So it seems unlikely that the initial growth and fast decay of sunspot groups can be described by an eddy diffusivity.

Krause: The diffusion model fits also the short-living sunspots groups, but with larger magnetic diffusivity.

Bumba: I have to agree with Dr Zwaan's and Dr Weiss' remarks. There are really two types or two velocities of sunspots area diminution, but they are closely related to the supergranular cells. If the cell is completely filled in by the spot, then the spot is more stable, its area diminishes slower than in the case of small spots which are too small in size to be comparable with supergranules. The number of regular stable spots is of course much smaller than the number of all spots.

Krause: In our paper (Krause and Rüdiger, 1975) we showed that also the more rapidly decaying sunspots groups can be described by a diffusion model, but with a larger magnetic diffusivity.

Deubner: I should like to have classified the type or topology of field you have in mind when you are discussing the observability of the *mean* field. Is it *mainly* poloidal or toroidal, what is its radial component, and finally what is its location with reference to the *observed* field? After all, aren't both quantities derived from identical physical structures, using different observational techniques?

Krause: If there is outside an insulator (vacuum) the toroidal field on the surface is zero. In this sense any observable magnetic field is a poloidal field. Observed with the magnetographs is in the most cases the component in the direction of line of sight only.

Parker: I do not see that the rapid decay of some sunspots presents a problem. As I pointed out on Monday, the magnetic field configuration of a sunspot is subject to the hydromagnetic exchange instability, unless that instability is blocked by some other force. Dr Krause has made the point in his lecture that a sunspot (once it ceases to accumulate more flux) must decay at least as fast as required by turbulent diffusion. He has also shown how well this idea fits the decay of the more long-lived spots. That some spots decay more rapidly, merely reminds us that there are disruptive forces, in addition to turbulent mixing. Presumably the hydromagnetic exchange instability assists in the decay of many spots.

Newkirk: I'd like to ask a question and also make a comment. The question is what observable parameters, other than the relation between mean and rms field and the decay rate of sunspots which you have mentioned, can be used to test the validity of the dynamo theory. The comment concerns the value of the observed topology of the large scale photospheric fields. Dynamo theories predict a very regular poloidal and toroidal field. Yet the observed field is much more complete with such patterns as a 'dipole' with a large angle to the rotation axis being quite common.

Krause: Both things you mentioned in your question have not really something to do with dynamo theory, however, they play some role in this theory. The most striking result supporting dynamo theory is that it was possible to construct models simulating the solar cycle by a strict integration of the basic equations. As concerns your comment: Dynamo theory predicts a regular *mean* poloidal and regular *mean* toroidal field and mean-field electrodynamics predicts at the surface a very *irregular* field, since the rms field B' is larger by a factor of 100 than the mean field.

Stenflo: In reply to Dr Newkirk let me note that the rms field strength is not 1–2 kG, the value of the field in the flux tubes that carry most of the net magnetic flux through the solar surface. It is much smaller, since the strong-field flux tubes occupy only a small area in the network. When determining the rms field we have to average over the large non-network regions of weaker fields as well.

Weiss: Perhaps I can allay some of Professor Parker's worries about the stability of sunspots. F. Meyer, H. U. Schmidt and I have been studying this problem. The interchange instability occurs when the curved lines of force are concave towards the surrounding plasma, but the configuration can be stabilized by the vertical pressure gradient outside the sunspot. Energy is released by straightening out a curved flux tube but the tube will be restored by buoyancy forces to its original position. Thus the hydrostatic pressure gradient which causes the curvature of field lines can stabilize the magnetic field. For a simple model we find that a spot is stable if the horizontal component of the field decreases upwards at its boundary. Preliminary results suggest that pores and sunspots will indeed be stable.

Howard: I believe that the Kitt Peak observers have placed an upper limit in the neighbourhood of 50 G on the turbulent magnetic field of the quiet Sun.

Stenflo: My estimate of the ratio between the rms and the mean field strengths that you mentioned do not really refer to a turbulent field. It is based on the observation that most of the magnetic flux recorded by solar magnetographs is due to kG fields which occupy only a very small fraction of the solar surface. The rms of such a field is much higher, by approximately a factor of 20, than if the same magnetic flux were uniformly spread out. The turbulent field you are talking about is a very tangled field, with opposite polarities being mixed on a small scale. Such a field, is very difficult to detect by polarization measurements unless the spatial resolution is very high, since opposite polarity fluxes cancel out over a small area. A turbulent field causes however some extra broadening of spectral lines, depending on the Laudé factor. Attempts have been made to detect this minute effect, starting with Unno in 1959, but so far it has only been possible to set an upper limit of a few hundred Gauss.

Giovanelli: We can scarcely expect any fine-scale 'unresolved' field of random polarity distribution to contribute appreciably to the rms field strength, since the total flux contained in the observed strong-field magnetic points is approximately equal to that of the sunspot from which the active-region-field originated.

Krause: My argumentation mainly counts with the average poloidal field in quiet regions (e.g. polar caps) and not in active regions, where the magnetic field mainly has its origin in the toroidal field. Thus I do not expect a relation between the two fluxes.

Vainshtein: (a) From our study of a nonlinear dynamo-theory it is clear that the field growth cannot be without limit. The conclusion of the nonlinear theory is as follows: The electromagnetic forces influence on the first the α-effect and than the diffusivity.

(b) I am studying now the mean field electrodynamics in the presence of plasma oscillations: ionic sound and plasma oscillations Langmuire oscillations). The theory can be applied to solar flare: to explain the fast magnetic damping one must use the turbulent conductivity caused by ionic sound. And if the ionic sound is excited there is generation and reconstruction of magnetic field.

Kuklin: I should like to recall the results of Severny, Grigoriev and others. Practically any background magnetic field observed with large aperture is composed of totality of fine structure magnetic field elements with different areas, field intensities and polarities. All changes of background fields are really variations of number, of size, of field intensity, of polarity of those elements. The principle fact is that background fields consist of changing magnetic field elements of different polarities.

Gilman: Is the mean field theory, particularly estimates of α, β affected by difference in shape between kinetic and magnetic energy spectra with space, in other words differences in spatial scale of v' and B'.?

Krause: As far as we discuss the mean-field electromagnetics the kinetic energy is assumed to be large compared with the magnetic field. α, β, etc. are integral values involving the (uninfluenced) spectrum of the turbulent velocity by field. If the back-reaction of the magnetic field is taken into account such affections may be indicated by a vanishing or, taking into account diffusion, nearly vanishing denominator involved in the integral expressions.

Stenflo: I would just like to add that Livingston and Harvey have recently recorded with high spatial resolution some kind of tangled field with mixed polarities inside supergranulation cells. Since the visibility of this field is quite seeing dependent, it is not clear how large fluxes and field strengths are involved, but still this observation shows the existence of some kind of tangled or 'turbulent' magnetic field away from the network.