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Diametrically Maximal and Constant Width Sets in Banach Spaces

Dedicated to the memory of Simon Fitzpatrick

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Abstract. We characterize diametrically maximal and constant width sets in $C(K)$, where K is any compact Hausdorff space. These results are applied to prove that the sum of two diametrically maximal sets needs not be diametrically maximal, thus solving a question raised in a paper by Groemer. A characterization of diametrically maximal sets in ℓ_1^3 is also given, providing a negative answer to Groemer's problem in finite dimensional spaces. We characterize constant width sets in $c_0(I)$, for every I, and then we establish the connections between the Jung constant of a Banach space and the existence of constant width sets with empty interior. Porosity properties of families of sets of constant width and rotundity properties of diametrically maximal sets are also investigated. Finally, we present some results concerning non-reflexive and Hilbert spaces.

1 Introduction

Most books and surveys on convexity devote special chapters to convex bodies of constant width. The existence of such sets has been known for a long time, certainly since the time of Euler, and the literature concerning finite-dimensional sets of constant width is extensive. For papers before 1983, for instance, the reader can see the more than 260 items cited in the authoritative survey article by Chakerian and Groemer [4]. The survey by Heil and Martini [14] contains a few additions and a complete account up to 1993. The related concept of diametrically maximal sets (also known as *complete* sets) was introduced at the beginning of last century by Meissner. Though both notions are equivalent in finite dimensional Euclidean spaces, Eggleston showed in his fundamental paper [8] that, even in finite dimensional spaces, diametrically maximal sets need not have constant width. He also gave one of the most useful characterizations of diametrically maximal sets, namely the *spherical intersection property*, later developed by Sallee [24, 25] and others. For recent results on this topic, the reader is referred to the survey of Martini and Swanepoel [18]. It was only in the mid-1980s that the first papers dealing with the subject in the more general infinite dimensional setting appeared. Berhends and Harmand [3] proved, among other things, that Banach spaces containing a nontrivial *pseudoball* (a particular case of a constant width set) necessarily contains a copy of c_0. Berhends [2], Payá

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and Rodriguez-Palacios [21] and the latter author [22, 23] pursued the study of constant width sets and their relation to M-ideals. However, in spite of these pioneering works, we are quite far from understanding all the close connections between these notions and the geometry of Banach spaces. Stability, rotundity and porosity properties of constant width and diametrically maximal sets — some of the questions we are concerned with in this paper are natural both for the finite and the infinite dimensional settings.

Our paper is organized as follows. Section 2 is devoted to definitions and basic results that will be used later, sometimes without explicit mention. In Section 3 we deal with the problem of constant width sets with empty interior, discussing the connections with the Jung constant. We characterize diametrically maximal and constant width sets in $c_0(I)$ endowed with the usual norm. Section 4 is devoted to the study of these concepts in $C(K)$, where K is a compact Hausdorff space, endowed with the sup norm. We prove that the only convex sets of constant width in these spaces are points and balls. We characterize diametrically maximal sets and prove that this family is not closed under sums, thus answering a question of Groemer [13]. In Section 5 we characterize the diametrically maximal sets of ℓ_1^3 (the 3-dimensional space endowed with the ℓ_1 norm) to give a negative answer to Groemer's question in the finite dimensional setting. We prove, however, that the sum of a diametrically maximal set in ℓ_1^3 and a constant width set is again diametrically maximal. This result does not hold in general, as we show with an (infinite dimensional) example. In Section 6, we are concerned with the porosity properties of the family of constant width sets in the case that it is different from the family of all diametrically maximal sets. Section 7 is devoted to rotundity properties of diametrically maximal sets. We prove that they are uniformly convex when the space is uniformly convex. The last section contains some final remarks and open questions.

2 Definitions and Basic Results

A finite dimensional bounded closed convex set C is said to be of *constant width* λ if the distance between any two parallel supporting hyperplanes of C equals λ. In an infinite dimensional Banach space X with unit ball B, the definition takes the following form: for every $f \in X^*$, $\|f\| = 1$, we have

$$\sup f(C) - \inf f(C) = \lambda. \tag{2.1}$$

Note that (2.1) is the same as saying that $\sup f(C - C) = \lambda$ for all such f. The latter form would lead one to suspect that $C - C$ must be the ball λB. This is indeed the case in finite dimensions, and it is almost the case in infinite dimensions. Namely, Payá and Rodriguez-Palacios have shown the following [21]: suppose that C is a bounded closed convex set of positive diameter $\lambda = \sup\{\|x - y\| : x, y \in C\}$; then the following are equivalent:

(i) *$C - C$ is dense in the ball λB,*
(ii) *$C - C$ contains the interior of λB,*
(iii) *C is of constant width λ.*

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This elementary result is a bit deeper than it looks; the proof of the implication "(i) implies (ii)" utilizes some of the same machinery used to prove the closed graph theorem. The above proposition *almost* shows that a set C has constant width 1 if and only if the unit ball can be decomposed in the form $B = C - C$. Of course, if C is compact (*e.g.,* if X is finite dimensional), then C is of constant width 1 (if and) only if $C - C = B$, a classical result. More generally, this is the case

(a) if X is reflexive, since $C - C$ is closed in that case;
(b) if X is any dual space.

(Indeed, a constant width set C is weak* compact since, as will be seen below, it is an intersection of balls, therefore $C - C$ is weak* compact hence norm closed.) It is worth remarking that when X is not reflexive, there is always an equivalent norm $\|\cdot\|$ and a constant width set C (under the new norm) such that $C - C$ is not closed. Obviously, if X is a dual space, then $\|\cdot\|$ cannot be a dual norm. We postpone the proof of this fact to the last section.

In Euclidean space, there are a number of notions equivalent to constant width, the most important being the notion of diametrically maximal set. Denoting the diameter of a bounded set C by $\operatorname{diam} C$, we say that C is *diametrically maximal* if

$$\operatorname{diam}\bigl(C \cup \{x\}\bigr) > \operatorname{diam} C$$

for every $x \notin C$. This concept appears in the literature under different names: complete, diametrically complete, entire. It is a simple application of the separation theorem that in every Banach space X, sets of constant width are diametrically maximal. Eggleston [8] proved that the converse is valid when $\dim X = 2$ and that it fails for certain spaces of $\dim X = 3$. Using standard arguments from Banach space theory it can be proved that, actually, $\dim X = 2$ if and only if, for every equivalent norm, diametrically maximal sets have constant width. (We include the proof of this result in the last section.) The difference between sets of constant width and diametrically maximal sets is also sharply drawn in Example 4.6, where it is shown that there exists a diametrically maximal set C such that $C - C$ is contained in a closed hyperplane. If this set were of constant width, $C - C$ would contain a ball.

Eggleston also gave a fundamental characterization: a set C with $\operatorname{diam} C = d$ is diametrically maximal if and only if it has the *spherical intersection property*, that is, if and only if

$$C = \bigcap_{x \in C} (x + dB),$$

a fact that is readily verified in any normed linear space. It is interesting to note that if there is $F \subset C$ such that $\bigcap_{x \in F}(x + dB) = C$ and $\operatorname{diam} C = d$, then C is diametrically maximal. Indeed, if $y \notin \bigcap_{x \in F}(x + dB)$, there is $x \in F \subset C$ such that $\|x - y\| > d$, so $\operatorname{diam}(C \cup \{y\}) > d$.

A reader who has never come across these concepts may wonder why a set of constant width is not a ball. The answer is easy: it is a ball provided it is (centrally) symmetric. More generally, *a diametrically maximal set is a ball if (and only if) it is symmetric*. For the sake of completeness, we will prove the sufficiency. Let C be a diametrically maximal set and assume that it is symmetric with respect to the origin (otherwise we consider a translation carrying the center of symmetry to the origin). Consider $r = \sup\{\|x\| : x \in C\}$. Clearly, $C \subset rB$ and we just need to check that the inclusion is in fact an equality. Since for every $n \in \mathbb{N}$ there is $x_n \in C$ such that $\|x_n\| > r - 1/n$ and $-x_n \in C$, we conclude that $\operatorname{diam} C \geq 2r$. Since $\operatorname{diam}(rB) = 2r = \operatorname{diam} C$, $C \subset rB$ and C is diametrically maximal, this means that $rB = C$.

3 Interior of Diametrically Maximal Sets

In a finite dimensional space X, a diametrically maximal set C which is not a single point has always nonempty interior. Indeed, the cases $\dim X = 1$ and $\dim X = 2$ are easy since, in these spaces, diametrically maximal sets have constant width. For the general case we proceed by induction. Assume that the result is true for the n-dimensional case. If $C \subset \mathbb{R}^{n+1}$ has empty interior then, since C is convex, it lies in an n-dimensional subspace Y. Notice that C is also diametrically maximal in Y and hence, by hypothesis, has nonempty relative interior. Consider $x \in C$ and $\lambda > 0$ such that $(x + \lambda B) \cap Y$ is in the relative interior of C and let $d = \operatorname{diam} C$. It follows that $\sup\{\|x - y\| : y \in C\} \leq d - \lambda$. Hence, if $z \in (x + \lambda B) \setminus Y$, then for all $y \in C$, $\|z-y\| \leq \|z-x\|+\|x-y\| \leq \lambda+(d-\lambda) = d$; since C is diametrically maximal, we get the contradiction that $z \in C \subset Y$. However, C can have empty interior in the infinite dimensional case, even if C has constant width; this will be seen in the next example. It is obvious that, even if a nontrivial constant width set C has empty interior, its linear span must be dense (otherwise, there would exist a nonzero functional which vanished on C). This is in contrast to the case for diametrically maximal sets, as will be seen in Example 4.6 in the next section.

Example Let C be the subset of c_0 consisting of all $x = (x_n)$ such that $0 \leq x_n \leq 1$. This is clearly closed, convex and bounded. To see that C has constant width 1, we need only show that $C - C$ is the unit ball B. Clearly, $C - C \subset B$. Further (and equally trivially), if $x \in B$, then $x_n \to 0$ and $-1 \leq x_n \leq 1$ for all n, and we can write $x_n = x_n^+ - x_n^-$, $0 \leq x_n^+, x_n^- \leq 1$, getting $x = x^+ - x^- \in C - C$. Finally, if $x = (x_n) \in C$, then, letting e_n denote the n-th basis vector, define $y^n = x - (x_n + 1/n)e_n$. Then $y^n_n = -1/n < 0$, so $y^n \notin C$ but $\|y^n - x\| = |x_n + 1/n| \to 0$, hence x is not an interior point of C.

The example above appears in [3] as an example of a pseudoball, a closed bounded convex set which can be characterized by the fact that its weak*-closure in X^{**} is a ball (hence must be of constant width). The analogous set in ℓ_∞ is also of constant width but it has nonempty interior, since it is, in fact, a translate by $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$ of $\frac{1}{2}B_{\ell_\infty}$. The following result shows that the example above is a particular case of a more general situation.

Proposition 3.1 *In $c_0(I)$ with the usual* sup *norm, a set C is of constant width if (and only if) it is diametrically maximal. Moreover, C is of constant width if and only if $C = D \cap c_0(I)$ where D is a ball in $\ell_\infty(I)$.*

Proof When dealing with intersections of balls, as in the case of diametrically maximal sets, it is helpful to have an easy-to-use representation, such as the one appearing in [12]: C is an intersection of balls in $(c_0(I), \|\cdot\|_\infty)$ if and only if there exists $k > 0$ such that

$$C = \bigcap_i f_i^{-1}[a_i, b_i],$$

where $\{f_i\}$ are the coordinate functionals and $-k \le a_i \le b_i \le k$ for all $i \in I$. When $D = \bigcap_i f_i^{-1}[c_i, d_i]$ is another (nonempty) intersection of balls, then $C + D = \bigcap_i f_i^{-1}[a_i + c_i, b_i + d_i]$ [12]. (In particular, the sum of two such sets is closed.) Now assume that C has diameter d and is an intersection of balls; then C is a set of constant width if and only if $C - C = dB = \bigcap_i f_i^{-1}[-d, d]$. Since $-C = \bigcap_i f_i^{-1}[-b_i, -a_i]$, this means that C is of constant width if and only if it is nonempty and

$$C - C = \bigcap_i f_i^{-1}[a_i - b_i, b_i - a_i] = \bigcap_i f_i^{-1}[-d, d]. \tag{3.1}$$

That is, $a_i + d = b_i$ for all $i \in I$. Suppose now that $D = \bigcap_i f_i^{-1}[c_i, d_i]$ is a diametrically maximal set with diameter d. Notice that $d = \sup_i (d_i - c_i)$. Assume that there is an index $i_0 \in I$ such that $d_{i_0} - c_{i_0} < d$. Then, the set

$$D' = \bigcap_{i \neq i_0} f_i^{-1}[c_i, d_i] \cap f_{i_0}^{-1}[c_{i_0}, c_{i_0} + d]$$

has diameter d, contains D but $D' \neq D$. For instance,

$$x = \sum_{i \neq i_0} d_i e_i + (c_{i_0} + d) e_{i_0} \in D' \setminus D$$

($\{e_i\}$ being the canonical basis of $c_0(I)$). Therefore, if D is diametrically maximal, then $c_i + d = d_i$ for every i and it is nonempty, so it is of constant width. Finally, observe that we have shown that $D \subset c_0(I)$ is diametrically maximal of diameter d (equivalently, is of constant width d) if and only if

$$D = \Big((x_i)_{i \in I} + \frac{d}{2} B \Big) \cap c_0(I),$$

where $(x_i)_{i \in I}$ is the element of $\ell_\infty(I)$ whose coordinates are $x_i = (c_i + d_i)/2$ for every $i \in I$ and B is the unit ball in $\ell_\infty(I)$. ■

The *radius of C with respect to x* is the number $r(C, x) = \sup\{\|x - y\| : y \in C\}$. The *radius of C* is defined as $r(C) = \inf\{r(C, x) : x \in X\}$. If $x \in C$, then $r(C, x) \le \operatorname{diam} C$, hence $r(C) \le \operatorname{diam} C$. Obviously, if C is a ball, then its diameter is twice its radius. Payá and Palacios [21, Lemma 1.8] have shown that for a set C of constant width, the converse is true: *If* $\operatorname{diam} C = 2r(C)$, *then C is a ball.* The set C is said to be *diametral* if $\operatorname{diam} C = r(C, x)$ for every $x \in C$. Finally, recall the definition of the *Jung constant* [16] of a space X:

$$J(X) = \sup\Big\{ \frac{2r(C)}{\operatorname{diam} C} : C \subset X \text{ is nontrivial and bounded} \Big\},$$

where nontrivial means that the set C is not a singleton, since otherwise $\operatorname{diam} C = 0$. Since $r(C) \leq \operatorname{diam} C$, it is clear that $J(X) \leq 2$ for every Banach space X. Note that for the unit ball B, we have $\frac{2r(B)}{\operatorname{diam} B} = 1$, so that $J(X) \geq 1$ for every X.

Theorem 3.2 *If there is a (nontrivial) diametrically maximal set $C \subset X$ with empty interior, then $r(C) = \operatorname{diam} C$ (so C is diametral) and $J(X) = 2$.*

Proof Note first that for any diametrically maximal set C we have

$$r(C) = \inf\{r(C, x) : x \in C\}.$$

If not, there would be a point $y \notin C$ such that $r(C, y) < \inf\{r(C, x) \colon x \in C\}(\leq \operatorname{diam} C)$. This implies that $\operatorname{diam}(C \cup \{y\}) = \operatorname{diam} C$, contradicting the fact that C is diametrically maximal. Next, if x is a boundary point of C, then $r(C, x) = \operatorname{diam} C$. Indeed, if $r(C, x) < \operatorname{diam} C$, take $y \notin C$ such that $\|x - y\| < \operatorname{diam} C - r(C, x)$. For every $z \in C$, we have

$$\|z - y\| \leq \|z - x\| + \|x - y\| \leq r(C, x) + \operatorname{diam} C - r(C, x)$$

and so $\operatorname{diam}(C \cup \{y\}) \leq \operatorname{diam} C$, which again contradicts the fact that C is diametrically maximal. Finally we obtain that if a diametrically maximal set C has empty interior, then every $x \in C$ is a boundary point, so $r(C, x) = \operatorname{diam} C$ for all $x \in C$, hence C is diametral and $r(C) = \operatorname{diam} C$. ∎

As a consequence of the above result we get that in every Banach space X satisfying $J(X) < 2$, every (nontrivial) set of constant width has nonempty interior. This is the case, for instance, for Hilbert spaces, which satisfy $J(X) = \sqrt{2}$ in the infinite dimensional case (the n-dimensional Euclidean space satisfies $J(X) = \sqrt{2n/(n+1)} < \sqrt{2}$). Also, $J(X) < 2$ in every uniformly convex space. For these and other estimates of the Jung constant, we refer to the paper by Amir [1]. On the other hand, our earlier example of a constant width (hence diametrically maximal) set in c_0 with empty interior yields the known fact that $J(c_0) = 2$. For the same reason, it follows from Example 4.6 that for continuous function spaces $X = C(K)$, we have $J(X) = 2$. It is natural to ask whether the converse of this result holds, namely, does $J(X) = 2$ imply the existence of a (nontrivial) constant width set with empty interior? We answer this question by the negative in the following section, by showing that every set of constant width in $C(K)$ is a ball. With respect to Theorem 3.2, note finally that *a bounded closed convex set C which has nonempty interior cannot be diametral*. Indeed, if $x \in \operatorname{int} C$, there exists $r > 0$ such that $B(x, r) \subset C$, hence $r(C, x) \geq r$. Choose $z \in C$ such that $\|x - z\| > r(C, x) - r/2$. Let $t = 1 + r/\|x - z\|$ and $y = tx + (1 - t)z$. Then

$$\|x - y\| = (t - 1)\|x - z\| = (r/\|x - z\|)\|x - z\| = r,$$

so $y \in B(x, r) \subset C$. Also,

$$r(C, y) \geq \|z - y\| = t\|x - z\| = \|x - z\| + r > r(C, x) + r/2 > r(C, x),$$

so C is not diametral.

4 Constant Width and Diametrically Maximal Sets in $C(K)$

The purpose of this section is to study constant width and diametrically maximal sets in $C(K)$, endowed with the usual supremum norm. Among other things, we will obtain an infinite dimensional counterexample to the question of Groemer [13, §4] concerning the stability under sums of the family of all diametrically maximal sets. (A finite dimensional counterexample will be obtained in Proposition 5.2.)

We need a useful characterization of the sets which are intersections of closed balls in $C(K)$, where K is Hausdorf compact. For every $t \in K$, let us denote by δ_t the Dirac functional defined as $\delta_t(f) = f(t)$ for every $f \in C(K)$. Given bounded real-valued functions f, g on K such that $f(t) \le g(t)$ for every $t \in K$, we let $[f, g] = \{h \in C(K) : f(t) \le h(t) \le g(t)\}$. This can also be written as $[f, g] = \bigcap_{t\in K} \delta_t^{-1}[f(t), g(t)]$. The function $f\colon K \to \mathbb{R}$ is called:

(i) *lower semicontinuous* if $\liminf_{y\to x} f(y) \ge f(x)$;
(ii) *upper semicontinuous* if $\limsup_{y\to x} f(y) \le f(x)$.

The set of points of continuity of a function $f\colon K \to \mathbb{R}$ will be denoted by D_f. When D_f is dense in K we say that f is *densely continuous*. Semicontinuous functions (upper or lower) on arbitrary topological spaces are always continuous on a residual set [11]. Consequently, when defined on a compact space, they are densely continuous. We will call $f, g\colon K \to \mathbb{R}$ an *admissible pair* (see [19]) when:

(a) they are lower and upper semicontinuous, respectively;
(b) for every $x \in K$, $f(x) \le g(x)$ and $\liminf_{y\to x,\, y\in D_g} g(y) \ge \limsup_{y\to x, y\in D_f} f(y)$.

Proposition 4.1 *If K is a compact Hausdorff space, $C \subset C(K)$ is a nonempty intersection of closed balls if and only if $C = [f, g]$, where f, g form an admissible pair.*

Proof The unit ball B in $C(K)$ endowed with the usual sup norm is

$$B = \bigcap_{t\in K} \delta_t^{-1}[-1, 1].$$

Then the ball with center $h \in C(K)$ and radius $r > 0$ is

$$\begin{aligned} h + rB &= h + \bigcap_{t\in K} \delta_t^{-1}[-r, r] \\ &= \bigcap_{t\in K} \delta_t^{-1}[h(t) - r, h(t) + r]\ . \end{aligned}$$

Therefore, if we consider a family of balls $\{B_i = h_i + r_i B\}$ satisfying $C = \bigcap_i B_i \neq \varnothing$ we have

$$\begin{aligned} C = \bigcap_i B_i &= \bigcap_i \bigcap_{t\in K} \delta_t^{-1}[h_i(t) - r_i, h_i(t) + r_i] \\ &= \bigcap_{t\in K} \delta_t^{-1}\big[\sup_i\{h_i(t) - r_i\}, \inf_i\{h_i(t) + r_i\}\big] \\ &= \bigcap_{t\in K} \delta_t^{-1}[f(t), g(t)], \end{aligned}$$

where $f(t) = \sup_i\{h_i(t) - r_i\}$ and $g(t) = \inf_i\{h_i(t) + r_i\}$ for each $t \in K$. It is clear that f is lower semicontinuous and g is upper semicontinuous. Moreover, $f(t) \leq h(t) \leq g(t)$ for every $t \in K$ and every $h \in C \neq \varnothing$, and both functions f and g are bounded. It remains to show that f, g form an admissible pair. Let $h \in [f, g]$ be a continuous function. Then, $\limsup_{s\to t, s\in D_f} f(s) \leq h(t) \leq \liminf_{s\to t, s\in D_g} g(s)$ for every $t \in K$.

Conversely, suppose that $C = \bigcap_{t\in K} \delta_t^{-1}[f(t), g(t)]$ where f, g are lower and upper semicontinuous, respectively. Consider $h \in C(K)$ such that $h \notin C$. There is $t_0 \in K$ such that $h(t_0) \notin [f(t_0), g(t_0)]$. Assume, for instance, that $h(t_0) < f(t_0)$. Notice that f is the supremum of a family of continuous functions on X, say $\{\psi_i\}$. Indeed, since K is normal, f is the pointwise supremum of the set of all continuous functions which are less than or equal to f (which is nonempty, because f is bounded below). Consequently, there is i_0 such that $\psi_{i_0}(t_0) - h(t_0) = 2m > 0$. Take $M > 0$ satisfying $\psi_{i_0}(t) + M \geq g(t)$ for every $t \in K$ (g is bounded above). Finally, consider the ball D centered in $\psi_{i_0} + (M - m)/2$ and having radius $(M + m)/2$. Then, $C \subset D$ but $h \notin D$. As a direct consequence of a sandwich-like result [19, Lemma 2.], we have $C = [f, g] \neq \varnothing$ provided f, g form an admissible pair. ■

Remark 4.2 If $C \subset C(K)$ is an intersection of balls, then we can also write $C = [\tilde{f}, \tilde{g}]$ where this time $\tilde{f}$ is bounded and *upper* semicontinuous and $\tilde{g}$ is bounded and lower semicontinuous and $\tilde{f}(t) \leq \tilde{g}(t)$ for each $t \in K$.

Detail: Since C is bounded, the functions defined by $f(t) = \inf\{\phi(t) : \phi \in C\}$ and $g(t) = \sup\{\phi(t) : \phi \in C\}$ are bounded and upper and lower semicontinuous, respectively. Clearly, $f(t) \leq g(t)$ for all $t \in K$ and $C \subset [f, g]$. Suppose that a function $\phi \in C(K)$ is not in C. Since the latter is an intersection of balls, there exist $\psi \in C(K)$ and $r > 0$ such that $C \subset \psi + rB$ but $\phi \notin \psi + rB$. This means that $\|\phi - \psi\| > r$, that is, $|\phi(t) - \psi(t)| > r$ for some $t \in K$. Suppose that $\phi(t) > \psi(t) + r$, say. Since $C \subset \psi + rB$, we must have $\phi' \leq \psi + r$ for all $\phi' \in C$ and therefore $g \leq \psi + r$. It follows that $\phi(t) > g(t)$ and therefore $\phi \notin [f, g]$. A similar argument applies if $\phi(t) < \psi(t) - r$, hence $C = [f, g]$.

Notice, finally, that the converse of Remark 4.2 is not true, and so this is not a characterization. Indeed, consider $K = [0, 1]$, $g = 1 - \chi_{\{1/2\}}$ and $f = -g$. Every ball containing $C = [f, g]$ must contain the unit ball, so C is not an intersection of balls.

Theorem 4.3 *The only sets of constant width in $C(K)$ are points and balls.*

Proof Consider a set $C \subset C(K)$ of constant width $\operatorname{diam} C = d > 0$. In a series of steps, we will prove that C is a ball of radius $d/2$. We will use the representation stated in Proposition 4.1, namely $C = [f, g] = \bigcap_{t\in K} \delta_t^{-1}[f(t), g(t)]$ where $f, g \colon K \to \mathbb{R}$ are lower and upper semicontinuous, respectively, hence there exists a dense G_δ-set of points at which both are continuous.

Step 1: If f and g are continuous at $t_0 \in K$, then $d = \sup \delta_{t_0}(C - C) = g(t_0) - f(t_0)$. The equality $d = \sup \delta_{t_0}(C - C)$ needs little explanation since C has constant width

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d and δ_{t_0} is a norm-one functional. To prove the second equality, we will only show that $\sup \delta_{t_0}(C) = g(t_0)$ since the proof of $\inf \delta_{t_0}(C) = f(t_0)$ is analogous. Given $\varepsilon > 0$, we need to find $h \in C$ satisfying $h(t_0) \geq g(t_0) - \varepsilon$. Choose a function $\varphi \in C$. If $\varphi(t_0) = g(t_0)$, there is nothing to prove, so we will assume that $\varphi(t_0) < g(t_0)$. Moreover, by taking ε smaller if necessary, we may also assume that $\varphi(t_0) < g(t_0) - \varepsilon$. By continuity of g and φ at t_0, there is a neighborhood G of t_0 such that $t \in G$ implies $g(t) > g(t_0) - \varepsilon$ and $\varphi(t) < g(t_0) - \varepsilon$. Since K is normal, there is a Urysohn function $\psi\colon K \to \mathbb{R}$ satisfying $\psi(t) = 0$ for all $t \in K \setminus G$, $\psi(t_0) = 1$ and $0 \leq \psi(t) \leq 1$ for all $t \in K$. We now modify φ in order to obtain the desired function h, as follows: $h(t) = (1 - \psi(t))\varphi(t) + \psi(t)(g(t_0) - \varepsilon)$. Clearly, $h(t_0) = g(t_0) - \varepsilon$. To check that $h \in C$, we need only show that $f \leq h \leq g$. This is obvious outside of G, where $h = \varphi$. It is readily checked that in G, the function h is a pointwise convex combination of functions which are not less than f nor greater than g.

Notice that Step 1 has the following consequence: if the functions f and g are actually continuous, then $f(t) = g(t) - d$ for every $t \in K$ and so C is the ball with center $(f + g)/2$ and radius $d/2$. However, f and g need not be continuous and, therefore, if this is the case, we must try to replace them by continuous functions. A couple of technical steps are still required.

Step 2: For every $t \in K$ we have

(i) $\limsup_{s\to t} f(s) \leq \inf \delta_t(C) \equiv a_t$ *and*

(ii) $\liminf_{s\to t} g(s) \geq \sup \delta_t(C) \equiv b_t$.

We will only show (i). Given $\varepsilon > 0$, there is $h \in C$ such that $h(t) < a_t + \varepsilon/2$. Since h is continuous, there is a neighborhood G of t such that $|h(s) - h(t)| < \varepsilon/2$ whenever $s \in G$. Therefore, $h(s) < a_t + \varepsilon$ when $s \in G$. Since $h \in C$, $f(s) \leq h(s)$ for every $s \in K$. Consequently, $f(s) < a_t + \varepsilon$ if $s \in G$ and so $\limsup_{s\to t} f(s) \leq a_t + \varepsilon$, and this holds for every $\varepsilon > 0$.

Step 3: The function $\tilde{f}$ defined as follows:

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if } f \text{ is continuous at } t,\\ a_t & \text{otherwise,} \end{cases}$$

is continuous. Denote by $D_f \subset K$ the set of points of continuity of f, which is dense in K. It is enough to show that $\lim_{s\in D_f,\, s\to t} f(s) = a_t$ for every $t \in K \setminus D_f$. Suppose that this were not so, that is, there exists $t_0 \in K \setminus D_f$ and a net $\{s_i\} \subset D_f$ such that $\lim_i s_i = t_0$ but (by Step 2) $\limsup_i f(s_i) < a_{t_0}$. We now use Step 1 to write

$$\liminf_i g(s_i) \leq \limsup_i g(s_i) = \limsup_i [f(s_i) + d] < a_{t_0} + d = b_{t_0},$$

which contradicts the statement of Step 2, namely that $b_{t_0} \leq \liminf_{s\to t_0} g(s)$. An analogous statement to Step 3 can be made by defining $\tilde{g}$ as follows:

$$\tilde{g}(t) = \begin{cases} g(t) & \text{if } g \text{ is continuous at } t,\\ b_t & \text{otherwise.} \end{cases}$$

As in the preceding case, $\tilde{g}$ is also continuous by a symmetric argument. To finish the proof, notice that

$$C = \bigcap_{t \in K} \delta_t^{-1}[f(t), g(t)] = \bigcap_{t \in K} \delta_t^{-1}[\tilde{f}(t), \tilde{g}(t)],$$

and apply the observation preceding Step 2. The inclusion $\bigcap_{t \in K} \delta_t^{-1}[\tilde{f}(t), \tilde{g}(t)] \subset C$ follows from the fact $[\tilde{f}(t), \tilde{g}(t)] \subset [f(t), g(t)]$ for every $t \in K$. For the reverse inclusion, consider $\varphi \in C$. Then, $\varphi(t) \geq f(t) = \tilde{f}(t)$ for every point $t \in S$. Since $\tilde{f}$ is continuous and D_f is dense, it follows that $\varphi(t) \geq \tilde{f}(t)$ for every $t \in K$. The proof of $\varphi \leq g$ is analogous. ∎

We are indebted to Professor A. Rodriguez-Palacios for calling the following corollary to our attention. Recall that a *pseudoball* is a bounded closed convex set whose weak*-closure in X^{**} is a ball (hence is necessarily of constant width).

Corollary 4.4 *If $X = C_0(J)$ is the sup-normed Banach space of all continuous real-valued functions which vanish at infinity on the locally compact Hausdorff space J, then any nontrivial set C of constant width in X is a pseudoball.*

Proof We first show that the weak* closure $(\widehat{C})^{**}$ of the canonical embedding $\widehat{C}$ of C in X^{**} is of constant width. To this end, we use the characterization of Paya and Rodriguez-Palacios stated in Section 2: C is of constant width $\lambda > 0$ if and only if $\operatorname{int} \lambda B \subset C - C$. Note that the unit ball of X^{**} is the same as $(\widehat{B})^{**}$; moreover, an elementary argument shows that $\operatorname{int} \widehat{B}^{**} \subset (\widehat{\operatorname{int} B})^{**}$. Thus, if C is of constant width $\lambda > 0$, then $\widehat{\operatorname{int} \lambda B} \subset \widehat{C - C} = \widehat{C} - \widehat{C} \subset \widehat{C}^{**} - \widehat{C}^{**}$. The latter set is weak* compact, therefore weak* closed and hence $\operatorname{int} \lambda \widehat{B}^{**} \subset (\widehat{\operatorname{int} \lambda B})^{**} \subset \widehat{C}^{**} - \widehat{C}^{**}$, which implies that C^{**} is of constant width λ in X^{**}.

Next, we use the fact that $X = C_0(J)$ is an abstract (M)-space [6], hence so too is X^{**}. By Kakutani's theorem then X^{**} is linearly isometric to a space $C(K)$ for some compact Hausdorff space K and by Theorem 4.3, $(\widehat{C})^{**}$ is a ball, that is, C is a pseudoball. ∎

Now that we have characterized the sets of constant width in $C(K)$, we focus our attention on diametrically maximal sets. Precisely, we have the following characterization for such sets that will be used later to answer Groemer's question [13].

Theorem 4.5 *The set $C \subset C(K)$ is diametrically maximal of diameter $d > 0$ if and only if $C = [f, g]$ where f, g form an admissible pair and $g(t) - f(t) = d$ whenever f and g are continuous at t.*

Proof First notice that if the set C can be represented as $C = [f, g]$ where $f(t) \leq g(t)$ for every $t \in K$, then the same argument used to prove Step 1 in Theorem 4.3 shows that $g(t) - f(t) = \sup \delta_t(C - C)$ whenever f and g are continuous at t.

If C is diametrically maximal, then C satisfies the spherical intersection property and therefore it is an intersection of closed balls. Thus, we can apply Proposition 4.1 to represent C in the form $[f, g]$ where f and g satisfy the required conditions. Therefore, to prove the necessity, we need simply check that $\sup \delta_{t_0}(C - C) = d$ whenever f and g are continuous at t_0. Since $d = \sup_{t \in K} \sup \delta_t(C - C) \geq \sup \delta_{t_0}(C - C)$, we need only prove that $\sup \delta_{t_0}(C - C) \geq \operatorname{diam} C$. Suppose, on the contrary, that $\sup \delta_{t_0}(C - C) < d$. Choose $\varepsilon > 0$ such that $\varepsilon < (1/2)(d - \sup \delta_{t_0}(C - C))$. There exists a neighborhood G of t_0 such that $|f(t) - f(t_0)| < \varepsilon$ and $|g(t) - g(t_0)| < \varepsilon$ when $t \in G$. Again, there is a Urysohn function $\psi \colon K \to \mathbb{R}$ satisfying $\psi(t) = 0$ for all $t \in K \setminus G$, $\psi(t_0) = 1$ and $0 \leq \psi(t) \leq 1$ for all $t \in K$. Now pick a function $h \in C$. We will modify h to obtain a new function $\tilde{h} \notin C$ such that $\operatorname{diam}(C \cup \{\tilde{h}\}) = d$, contradicting the hypothesis that C is diametrically maximal. The function $\tilde{h}$ is defined as follows: $\tilde{h}(t) = (1 - \psi(t))h(t) + \psi(t)(g(t_0) + \varepsilon)$, so $\tilde{h}(t_0) > g(t_0)$, hence $\tilde{h} \notin C$. Let us check next that $\|\varphi - \tilde{h}\|_\infty \leq d$ for every $\varphi \in C$. Since $h = \tilde{h}$ on $K \setminus G$, the only thing to estimate is $\sup_{t \in G} |\varphi(t) - \tilde{h}(t)|$ when $\varphi \in C$. But,

$$\sup_{t \in G} |\varphi(t) - \tilde{h}(t)| \leq g(t_0) + \varepsilon - (f(t_0) - \varepsilon) \leq d.$$

To prove the sufficiency, assume now that $C = [f, g]$, where f is lower semicontinuous, g is upper semicontinuous and $g(t) - f(t) = \sup \delta_t(C - C) = d > 0$ whenever f and g are continuous at t. We first show that $d = \operatorname{diam} C$. To this end, denote again by D_f and D_g the points of continuity of f and g, respectively, and consider $S = D_f \cap D_g$. It is clear that

$$\operatorname{diam} C = \sup_{s \in K} \sup \delta_s(C - C) \geq \sup_{s \in S} \sup \delta_s(C - C) = d.$$

To prove the reverse inequality, choose $\varepsilon > 0$ and $s_0 \in K$ such that $\operatorname{diam} C \leq \sup \delta_{s_0}(C - C) + \varepsilon/4$; now take $\psi, \varphi \in C$ satisfying $\varphi(s_0) - \psi(s_0) > \sup \delta_{s_0}(C - C) - \varepsilon/4$. Then $\varphi(s_0) - \psi(s_0) + \varepsilon/2 > \operatorname{diam} C$. Let $s_1 \in S$ be close enough to s_0 so that $\varphi(s_1) - \psi(s_1) > \varphi(s_0) - \psi(s_0) - \varepsilon/2$. Then, $\varphi(s_1) - \psi(s_1) > \operatorname{diam} C - \varepsilon$, whence $d = \sup_{s \in S} \sup \delta_s(C - C) \geq \operatorname{diam} C$.

Consider $h \notin C$; we want to show that $\operatorname{diam}(C \cup \{h\}) > d$. There exists $t_0 \in K$ such that $h(t_0) \notin [f(t_0), g(t_0)]$. We may assume, for instance, that $h(t_0) > g(t_0)$. Since g is upper semicontinuous, this implies that

$$\limsup_{t \to t_0} g(t) \leq g(t_0) < h(t_0). \tag{4.1}$$

Since $D_f \cap D_g$ is dense, (4.1) and the continuity of h imply the existence of a point $t_1 \in D_f \cap D_g$, close to t_0, satisfying $g(t_1) < h(t_1)$. Choose $\varepsilon > 0$ such that $\varepsilon < (1/2)(h(t_1) - g(t_1))$. There are $\varphi, \psi \in C$ such that $\varphi(t_1) - \psi(t_1) > \sup \delta_{t_1}(C - C) - \varepsilon = d - \varepsilon$. Hence

$$\begin{aligned} \|h - \psi\| \geq h(t_1) - \psi(t_1) &= h(t_1) - \varphi(t_1) + \varphi(t_1) - \psi(t_1) \\ &> h(t_1) - g(t_1) + d - \varepsilon \\ &\geq d \end{aligned}$$

and the proof is finished. ■

Example 4.6 Consider the two functions $g(t) = \chi_{[\frac{1}{2},1]}(t)$, $f(t) = -\chi_{[0,\frac{1}{2}]}(t)$ and the value $d = 1$; these satisfy the conditions of Theorem 4.5. Therefore, $C = [f, g]$ is diametrically maximal, and so is $-C = [-g, -f]$. However, $\overline{C - C}$ is not diametrically maximal. Moreover, C is contained in a closed hyperplane, hence its linear span is not dense.

Proof First we show that $C - C = [\tilde{f}, \tilde{g}]$, where $\tilde{g} = 1 - \chi_{\{\frac{1}{2}\}}$ and $\tilde{f} = -\tilde{g}$. It is clear that $C - C \subset [\tilde{f}, \tilde{g}]$, since $\varphi(1/2) = 0$ for every function $\varphi \in C$, hence for every function in $-C$ as well. (This shows that C is contained in a closed hyperplane.) To prove the reverse inclusion, consider $h \in [\tilde{f}, \tilde{g}]$ and the usual decomposition $h = h^+ + h^-$, where $h^+ = \max\{h, 0\}$ and $h^- = \min\{h, 0\}$. Noting that $h(\frac{1}{2}) = 0$, it is easy to check that we can also decompose h as follows:

$$h = \left(\chi_{[0,\frac{1}{2}]}h^- + \chi_{[\frac{1}{2},1]}h^+\right) + \left(\chi_{[0,\frac{1}{2}]}h^+ + \chi_{[\frac{1}{2},1]}h^-\right)$$

where $\chi_{[0,\frac{1}{2}]}h^- + \chi_{[\frac{1}{2},1]}h^+ \in C$ and $\chi_{[0,\frac{1}{2}]}h^+ + \chi_{[\frac{1}{2},1]}h^- \in -C$. On the other hand, notice that since $C - C = [\tilde{f}, \tilde{g}]$ and $[\tilde{f}, \tilde{g}] = \bigcap_{t\in[0,1]} \delta_t^{-1}([\tilde{f}(t), \tilde{g}(t)])$ is an intersection of closed sets, the set $C - C$ must be closed. However, we claim that $C - C$ is not an intersection of balls, thus implying that it cannot be diametrically maximal either. To prove the claim, assume (using Proposition 4.1) that $\hat{f}$ and $\hat{g}$ are lower and upper semicontinuous, respectively, and that $C - C = [\tilde{f}, \tilde{g}] = [\hat{f}, \hat{g}]$. When $t \in [0, 1]$ is a point of continuity for f and g we know that $\sup \delta_t(C - C) = \tilde{g}(t)$ and $\inf \delta_t(C - C) = \tilde{f}(t)$. Consequently, in these points $\hat{f}(t) \leq \tilde{f}(t)$ and $\hat{g}(t) \geq \tilde{g}(t)$ and hence $\hat{f}(t) \leq -1$ and $\hat{g}(t) \geq 1$ for every $t \neq 1/2$. Since $\hat{f}$ and $\hat{g}$ are lower and upper semicontinuous, respectively, this implies that $\hat{f}(1/2) \leq -1$ and $\hat{g}(1/2) \geq 1$. Hence $[\hat{f}, \hat{g}]$ contains the unit ball, which contradicts the fact that $[\hat{f}, \hat{g}] = C - C \subset \delta_{\frac{1}{2}}^{-1}(0)$. ∎

Remark 4.7 Fonf and Lindenstrauss [9, Remark 2] have given a simple argument which shows that if C is a bounded closed convex for which $C - C$ has nonempty interior, then for any convex set D containing C, there cannot exist an affine retraction of D onto C. Thus, their result applies to sets of constant width. (Since the nonexistence of such retractions is clear if C itself has nonempty interior, their result is primarily of interest when $\operatorname{int} C = \varnothing$.) That their result does not apply to diametrically maximal sets is shown by the foregoing example, where $C - C$ is contained in a closed hyperplane.

5 Sums of Diametrically Maximal Sets

It is straightforward to verify that given two sets C, D of constant width, the closure of the sum $\overline{C + D}$ necessarily has constant width. Indeed, for every norm-one functional f one has $\sup f(\overline{C + D}) - \inf f(\overline{C + D}) = (\sup f(C) - \inf f(C)) + (\sup f(D) - \inf f(D)) = \operatorname{diam} C + \operatorname{diam} D$. The question of whether the (closure of the) sum of two diametrically maximal sets is again diametrically maximal was answered in the negative in the preceding section. However, it is natural to ask to what extent this

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answer depends on whether the space has infinite dimension. In this section we find another counterexample in a 3-dimensional space, namely ℓ_1^3. To this end, we will obtain a characterization of diametrically maximal sets in this space.

We will make use of this easy observation: the family of all diametrically maximal sets is closed with respect to translations and dilations. To prove, for instance, that λD is diametrically maximal whenever $\lambda \in \mathbb{R}$ and D is diametrically maximal, we just need to use that λD satisfies the spherical intersection property. Then, letting $\operatorname{diam} D = d$, we have

$$\begin{aligned}\lambda D &= \lambda\Big(\bigcap_{x\in D}(x+dB)\Big)\\ &= \bigcap_{x\in D}\lambda(x+dB)\\ &= \bigcap_{y\in\lambda D}(y+\lambda dB)\end{aligned}$$

and, taking into account that $\operatorname{diam} \lambda D = \lambda d$, this implies that λD satisfies also the desired property. In what follows, we will consider the functionals

$$\begin{aligned} f_1(x_1,x_2,x_3) &= x_1+x_2+x_3, & f_2(x_1,x_2,x_3) &= x_1+x_2-x_3,\\ f_3(x_1,x_2,x_3) &= x_1-x_2-x_3, & f_4(x_1,x_2,x_3) &= x_1-x_2+x_3.\end{aligned}$$

Proposition 5.1 *In ℓ_1^3, a set C is diametrically maximal if and only if C is an intersection of balls and* $\sup f_i(C-C) = \operatorname{diam} C$ *for $i = 1,\dots,4$.*

Proof First notice that there is a useful way to calculate the diameter of a set. Recall that the family $\{f_i\}_{i=1}^4$ is norming in ℓ_1^3, that is, $\|x\| = \max\{|f_i(x)|, i = 1,\dots,4\}$ for every $x \in \ell_1^3$. As a consequence, for every set $C \subset \ell_1^3$ we have

$$\operatorname{diam} C = \max\{\sup f_i(C-C), i = 1,\dots,4\}.$$

To prove the sufficiency, since C is an intersection of balls, we will represent C as follows [12]:

$$C = \bigcap_{i=1}^{4} f_i^{-1}([\inf f_i(C), \sup f_i(C)]). \tag{5.1}$$

Now take a point $z \notin C$. There is $i_0 \in \{1,\dots,4\}$ such that

$$f_{i_0}(z) \notin [\inf f_{i_0}(C), \sup f_{i_0}(C)].$$

Then

$$\begin{aligned}\operatorname{diam}\big(C\cup\{z\}\big) &\geq \sup f_{i_0}\big(C\cup\{z\}\big) - \inf f_{i_0}\big(C\cup\{z\}\big)\\ &> \sup(C-C) = \operatorname{diam} C\end{aligned}$$

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thus implying that C is diametrically maximal. Notice that it is not difficult to find examples of sets C satisfying $\sup f_i(C - C) = \operatorname{diam} C$ for $i = 1, \dots, 4$ which are not intersection of balls. These sets cannot be diametrically maximal since obviously they do not satisfy the spherical intersection property.

To prove the necessity, if C is diametrically maximal we know that C is an intersection of balls and so we can represent it as in (5.1) [12]:

$$C = \bigcap_{i=1}^{4} f_i^{-1}([\inf f_i(C), \sup f_i(C)]) = \bigcap_{i=1}^{4} f_i^{-1}([a_i, b_i]). \tag{5.2}$$

We want to prove that $\sup f_i(C - C) = \operatorname{diam} C$ for $i = 1, \dots, 4$. Assume, on the contrary, that there is $i \in \{1, \dots, 4\}$ satisfying $\sup f_i(C - C) < \operatorname{diam} C$. We can take, for instance, $i = 1$ since the other cases are analogous. Define $\{b_i'\}$ as $b_i' = a_i + \operatorname{diam} C$ for $i = 2, 3, 4$, $b_1' = b_1$ and consider the set $C' = \bigcap_{i=1}^4 f_i^{-1}([a_i, b_i'])$. Then since $C \subset C'$ and $\operatorname{diam} C = \operatorname{diam} C'$, necessarily $C = C'$. After a suitable translation we may assume $a_i = -b_i'$ and, finally, a dilation $\varphi(x) = (\operatorname{diam} C)^{-1} x$ will make $a_i = -1$, $b_i' = 1$ for $i = 2, 3, 4$. Now consider the set $D = \bigcap_{i=2}^4 f_i^{-1}([a_i, b_i']) = \bigcap_{i=2}^4 f_i^{-1}([-1, 1])$. The diameter of D is exactly $\sup f_1(D - D) = f_1(1, 1, 1) - f_1(-1, -1, -1) = 6$. This means that $-3 \leq a_1$ and $b_1 \leq 3$. Actually, since $\operatorname{diam} C = 2$, either $-3 < a_1$ or $b_1 < 3$. Assume, for instance, that $-3 < a_1$. Then $a_1 < 3$ also, since otherwise either C will reduce to a point or will be empty, which in both cases is impossible. Thus $-3 < a_1 < 3$ and therefore the hyperplane $f_1^{-1}(a_1)$ intersects the interior of D. Take $z \in f_1^{-1}(a_1) \cap \operatorname{int}(D)$ and $\varepsilon > 0$ satisfying $z + \varepsilon B \subset D$ and $3\varepsilon < 2 - (b_1 - a_1)$. Finally, consider $w = z + \varepsilon(-1, -1, -1)$. Then $f_1(z) < a_1$ and so $z \notin C$. We just need to prove that $\operatorname{diam}(\{w\} \cup C) = \operatorname{diam} C$ to obtain a contradiction. Since $w \in D$, $|f_i(w) - f_i(y)| \leq 2$ for arbitrary $y \in C$ and $i = 2, 3, 4$. Finally, $|f_1(y) - f_1(w)| = f_1(y) - f_1(w) \leq b_1 - a_1 + 3\varepsilon < 2$ also for every $y \in C$. ∎

As a consequence of the above result, we get that every diametrically maximal set C with $\operatorname{diam}(C) = 2$ in ℓ_1^3 (up to translation) has the form $C = \bigcap_{i=2}^4 f_i^{-1}([-1, 1]) \cap f_1^{-1}([a_1, b_1])$ where $-3 \leq a_1 \leq 2$ and $b_1 = a_1 + 1$. For instance, if C is the convex hull of the four points $(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)$, then C is diametrically maximal. Indeed, $C = \bigcap_{i=2}^4 f_i^{-1}([-1, 1]) \cap f_1^{-1}([1, 3])$. Next we show that this is not the case for $C - C$.

We will denote the family of all diametrically maximal sets by $\mathcal{D}$ and the family of all sets of constant width by $\mathcal{W}$.

Proposition 5.2 *For the set C defined above, the sum $C + (-C)$ is not an intersection of balls, hence $C - C \notin \mathcal{D}$, implying that $\mathcal{D}$ is not closed with respect to sums in ℓ_1^3. However, in this space, if $C \in \mathcal{W}$ and $D \in \mathcal{D}$ then $C + D \in \mathcal{D}$.*

Proof The set $C - C$ contains the face which is the convex hull of the four points $\{(-1, 0, 1), (0, -1, 1), (0, 1, 1), (1, 0, 1)\}$. Therefore, every ℓ_1-ball containing $C - C$

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necessarily contains the ball with center $\frac{1}{4}(-1,0,1) + \frac{1}{4}(0,-1,1) + \frac{1}{4}(0,1,1) + \frac{1}{4}(1,0,1) = (0,0,1)$ and radius 1 [12], which is not contained in $C - C$.

To prove the second assertion, first notice that $\sup f_i(C+D) - \inf f_i(C+D) = \operatorname{diam} C + \operatorname{diam} D$ for every $i = 1,\dots,4$, so the second condition stated in Proposition 5.1, to check whether $C + D$ is diametrically maximal, is satisfied. It remains to prove that $C + D$ is an intersection of balls. We use a result of [26], where it was proved that the only sets of constant width in ℓ_1^3 (as well as in a wide family of finite dimensional polyhedral spaces) are points and balls. The case of C being a point is trivial, so we may assume that C is a ball. Now we can apply a result from [12], where it was shown that in ℓ_1^3 the sum of a ball with a set which is an intersection of balls is itself an intersection of balls. Consequently, using Proposition 5.1, we get that $C + D$ is diametrically maximal. ∎

The following example shows that, in general, the sum of a diametrically maximal set with a ball does not yield a diametrically maximal set.

Example 5.3 Let $X = \{f \in C[0,1] : 2f(1/2) = f(1)\}$ be a subspace of $C[0,1]$, let C be the same set used in Example 4.6 and define $C' = C \cap X$ $(= \{f \in C : f(\frac{1}{2}) = f(1) = 0\})$. Then $\operatorname{diam} C' = 1$, and C' is diametrically maximal in X (the latter endowed with the sup norm). But, if we denote by B the unit ball of X, then $\overline{C' + B}$ is not diametrically maximal.

Proof It is readily checked that $\operatorname{diam} C' = 1$. To see that C' is diametrically maximal, note that if $f \in X \setminus C'$, then there is either $t \in [0,1/2)$ satisfying $f(t) \notin [-1,0]$ or $t \in (1/2,1)$ satisfying $f(t) \notin [0,1]$. In either case, it is easy to find $f' \in C'$ for which $1 < |f - f'|(t) \le \|f - f'\|$. It is also easy to show that $\operatorname{diam}(C' + B) = \operatorname{diam} C' + 2 = 3$, hence $\operatorname{diam} \overline{C' + B} = 3$. Now, for every $f = f_1 + f_2 \in C' + B$ where $f_1 \in C'$ and $f_2 \in B$, we have $-1 \le f(1) \le 1$, since $f_1(1) = 2f_1(1/2) = 0$. These same inequalities must also hold for functions in $\overline{C' + B}$. Consider, finally, the function $h(t) = 2t$; this is in X but not in $\overline{C' + B}$. It is straightforward to verify that the diameter of $(C' + B) \cup \{h\}$ (therefore of $\overline{C' + B} \cup \{h\}$) is 3, which shows that $\overline{C' + B}$ is not diametrically maximal. ∎

While the above example shows that C being diametrically maximal does not imply the same for $\overline{C + B}$ (recall that B is the unit ball), we next observe that if C is *not* diametrically maximal, then the same is true for for $\overline{C + B}$. Indeed, assume that C is not diametrically maximal. There must exist $x' \in \bigcap_{x \in C}(x + dB) \setminus C$, where $d = \operatorname{diam} C$; say that $\operatorname{dist}(x', C) = \alpha > 0$. Then, we can choose $z \in B$ such that $\operatorname{dist}(x' + z, C + B) > \alpha/2$, say. Since $\operatorname{diam}(\overline{C + B}) = d + 2$, if we can prove that $x' + z \in \bigcap_{y \in \overline{C+B}}(y + (d+2)B)$, then we have finished. To this end, notice that

$$
\begin{aligned}
x' + z \in \Big(\bigcap_{x \in C}(x + dB)\Big) + B &\subset \bigcap_{x \in C}(x + (d+1)B) \\
&\subset \bigcap_{y \in C+B}(y + (d+2)B) = \bigcap_{y \in \overline{C+B}}(y + (d+2)B).
\end{aligned}
$$

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It can also be easily checked that if C is not of constant width, then the same is true of $\overline{C+B}$.

6 Sets of Constant Width and Porosity

We know that $\mathcal{W} = \mathcal{D}$ in two dimensional spaces [8] and that this is also the case for $(c_0(I), \|\cdot\|_\infty)$, as was proved in Proposition 3.1. In spaces with the Binary Intersection Property (P_1-spaces), both classes coincide with the family of closed balls [10]. However, it is a natural question to ask about the size of $\mathcal{W}$ inside of $\mathcal{D}$ in spaces where they are different. We will answer this question when $\mathcal{D}$ is stable under (the closure of) the addition of constant width sets, namely if $\overline{C+D} \in \mathcal{D}$ provided $C \in \mathcal{W}$ and $D \in \mathcal{D}$. In this case, we say that $\mathcal{D}$ is w-stable. For instance, $\mathcal{D}$ is w-stable in ℓ_1^3 (see the preceding section) and in $(C(K), \|\cdot\|_\infty)$ for every Hausdorff compact space K. The latter result is a consequence of Theorems 4.3 and 4.5 in Section 4, together with a result of [19]. On the other hand, Example 5.3 shows that there are spaces in which $\mathcal{D}$ is not w-stable.

We recall the notion of a *uniformly very porous* set [15]. Let M be a metric space, P a subset of M, $B(x, R)$ the closed ball centered at x with radius R and $\gamma(x, R, P)$ the supremum of all r for which there exists $y \in M$ such that $B(y, r) \subset B(x, R) \setminus P$. The number

$$\rho(x, P) = 2 \liminf_{R \to 0} \frac{\gamma(x, R, P)}{R}$$

is called the *extreme porosity* of P at x [27]. We say that P is *uniformly very porous* if there exists an $\varepsilon > 0$ satisfying $\rho(x, P) > \varepsilon$ for every $x \in M$. In our setup, we take $M = \mathcal{D}$, $P = \mathcal{W}$ and x will be an element of $\mathcal{W}$.

Proposition 6.1 *The family $\mathcal{W}$ is (topologically) closed. Moreover, when $\mathcal{D}$ is w-stable, $\mathcal{W}$ is uniformly very porous (in $\mathcal{D}$) if and only if $\mathcal{W} \neq \mathcal{D}$.*

Proof Let C be closed, convex and bounded set which is not of constant width. Then, there is $\lambda > 0$ satisfying

$$\lambda = \sup_{\|f\|=1} \{\sup f(C-C)\} - \inf_{\|f\|=1} \{\sup f(C-C)\}.$$

As a consequence, if $d(C, D) < \lambda/4$ (here $d(C, D)$ denotes the distance with respect to the usual Hausdorff metric) then $D \notin \mathcal{W}$. Indeed, consider, for every $n \in \mathbb{N}$, norm one functionals f_n, g_n satisfying $\sup f_n(C-C) - \sup g_n(C-C) > \lambda - 1/n$. Since

$$\sup f_n(D-D) \geq \sup f_n(C-C) - 2d(C, D),$$
$$\sup g_n(D-D) \leq \sup g_n(C-C) + 2d(C, D),$$

we get

$$\sup f_n(D-D) - \sup g_n(D-D) > (\lambda - 1/n) - 4d(C, D),$$

which is positive for sufficiently large n. This shows that $\mathcal{W}$ is closed. To prove that it is uniformly very porous, we will use a similar argument, as follows. Consider now $C \in \mathcal{D}$ and functionals f_n, g_n defined as before, assuming that $0 \in C$ and $\operatorname{diam} C \leq 1$ (if not, a suitable translation and a dilation do the job). Take $D \in \mathcal{W}$, $R > 0$ and define $D_R = \overline{D + RC}$. Since $\mathcal{D}$ is w-stable, $D_R \in \mathcal{D}$. We claim that

$$B_d(D_R, R\lambda/4) \cap \mathcal{W} = \varnothing$$

where $B_d(D_R, R\lambda/4)$ denotes the ball (in the Hausdorff metric) with center D_R and radius $R\lambda/4$. To prove the claim, first notice that for every functional f,

$$\begin{aligned}\sup f(D_R - D_R) &= \sup f(D_R) - \inf f(D_R)\\ &= \sup f(D) + R \sup f(C) - (\inf f(D) + R \inf f(C))\\ &= R \sup f(C - C) + \operatorname{diam} D\end{aligned}$$

and this implies that for every $n \in \mathbb{N}$,

$$\sup f_n(D_R - D_R) - \sup g_n(D_R - D_R) > R(\lambda - 1/n)$$

and, finally, we get

$$\sup_{\|f\|=1} \{\sup f(D_R - D_R)\} - \inf_{\|f\|=1} \{\sup f(D_R - D_R)\} = R\lambda.$$

Now the same argument used to prove that $\mathcal{W}$ is closed applies to prove the claim. To compute the porosity of $\mathcal{W}$ at D, note that $d(D, D_R) \leq R \operatorname{diam} C = R$. This means that $\gamma(D, R + R\lambda/4, \mathcal{W}) \geq R\lambda/4$ thus implying that

$$2 \liminf_{R\to 0} \frac{\gamma(D, R + R\lambda/4, \mathcal{W})}{R + R\lambda/4} \geq \liminf_{R\to 0} \frac{R\lambda/2}{R + R\lambda/4} = \frac{\lambda/2}{1 + \lambda/4}.$$

The above estimate though, proves that $\mathcal{W}$ is uniformly very porous, is not sharp in the sense that we do not know if there is $C' \in \mathcal{D} \setminus \mathcal{W}$ producing a bigger λ. Then we can define

$$\lambda_0 = \sup_{\substack{C \in \mathcal{D}\setminus\mathcal{W}\\ \operatorname{diam} C = 1}} \Big\{ \sup_{\|f\|=1} \{\sup f(C - C)\} - \inf_{\|f\|=1} \{\sup f(C - C)\}\Big\}$$

and a similar proof, together with a standard approximation argument yields, for every $D \in \mathcal{W}$,

$$\rho(D, \mathcal{W}) \geq \frac{\lambda_0/2}{1 + \lambda_0/4}. \qquad \blacksquare$$

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7 Rotundity Properties of Diametrically Maximal Sets

This section is devoted to the question of which rotundity properties of the unit ball are inherited by every diametrically maximal set. Our main result is that in a uniformly convex space, every diametrically maximal set is uniformly convex. We provide an estimate of its modulus of convexity. Recall that *Clarkson's modulus of convexity* of X is the function $\rho_X\colon [0,2] \to [0,1]$ defined by the formula $\rho_X(\varepsilon) = \inf\{1 - \|\frac{1}{2}(x+y)\| : x, y \in B, \|x-y\| \geq \varepsilon\}$. Given a closed, convex and bounded set C with $0 \in \operatorname{int}(C)$, we can analogously define γ_C, the modulus of convexity of C, by replacing in the above definition the norm by the Minkowski functional $\mu_C(x)$ of C: $\mu_C(x) = \inf\{\alpha > 0 : x \in \alpha C\}$. Thus,

$$\gamma_C(\varepsilon) = \inf\Big\{ 1 - \mu_C\Big(\frac{1}{2}(x+y)\Big) : x, y \in C, \mu_C(x-y) \geq \varepsilon \Big\}. \tag{7.1}$$

Note that this modulus of convexity is defined in the interval $[0, \delta_M(C)]$, where $\delta_M(C) = \sup\{\mu_C(x-y) : x, y \in C\}$. The number $\delta_M(C)$ is called the Minkowski diameter of C (see [28] for an interesting discussion on this notion).

Proposition 7.1 *Let C be a diametrically maximal set with diameter $\operatorname{diam} C = d$ and let $0 < r \leq R$ be such that $rB \subset C \subset RB$. Then $\gamma_C(\varepsilon) \geq \frac{d}{R}\rho_X(r\varepsilon)$ for every $\varepsilon \in [0,2]$.*

Proof First notice that $R^{-1}\|z\| \leq \mu_C(z) \leq r^{-1}\|z\|$ for every $z \in X$. Now consider $\varepsilon \in [0,2]$ and let x, y be two points of C such that $\mu_C(x-y) \geq \varepsilon$, hence $\|x-y\| \geq r\mu_C(x-y) \geq r\varepsilon$. Being diametrically maximal, C satisfies the spherical intersection property and so $C = \bigcap\{z + (\operatorname{diam} C)B : z \in C\}$. Since the segment $[x,y]$ is in C, it follows that $[x,y] \subset z + (\operatorname{diam} C)B$ for every $z \in C$. Now, by using the definition of the modulus of convexity in each ball $z + dB$, we get

$$\frac{1}{2}(x+y) + \rho_X(r\varepsilon)dB \subset z + dB \tag{7.2}$$

for every $z \in C$ and, as a consequence, $\frac{1}{2}(x+y)+\rho_X(r\varepsilon)dB \subset C$. Then, since $\frac{1}{R}C \subset B$, we have

$$\frac{1}{2}(x+y) + \frac{1}{R}\rho_X(r\varepsilon)dC \subset C$$

and this implies, letting $w = \frac{1}{2}(x+y)\mu_C(\frac{1}{2}(x+y))^{-1}$, that

$$\begin{aligned} 1 - \mu_C\Big(\frac{1}{2}(x+y)\Big) &= \mu_C(w) - \mu_C\Big(\frac{1}{2}(x+y)\Big) \\ &= \mu_C(w - \frac{1}{2}(x+y)) \\ &\geq \frac{1}{R}\rho_X(r\varepsilon)\operatorname{diam} C \end{aligned}$$

which proves the proposition. ∎

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In [28], the authors extend to convex bodies Gurarii's version of the modulus of convexity, instead of Clarkson's version that we have considered. In other words, they define $\gamma_C(\varepsilon) = \inf\{\max\{1-\mu_C(tx+(1-t)y) : 0 \le t \le 1\} : x, y \in C, \mu_C(x-y) \ge \varepsilon\}$ which is not a suitable extension for our purposes. Indeed, when trying to apply the definition of Gurarii's modulus in (7.2) , we find that for different z and different balls $z + (\operatorname{diam} C)B$, the value $\max\{1 - \mu_C(tx + (1 - t)y) : 0 \le t \le 1\}$ is attained at different t.

Two points $x, y \in C$ are called *diametral points* if $\|x - y\| = \operatorname{diam}(C)$. Obviously, diametral points are always boundary points. Given a closed, convex and bounded set C, we say that $x \in C$ is a *strongly exposed point* of C if there is $f \in X^*$ such that $f(x) = \sup f(C)$ and $x_n \to x$ when $\{x_n\} \subset C$ and $f(x_n) \to f(x)$. Say that a norm is *strongly convex* if every point of the unit sphere S is a strongly exposed point of the unit ball. Recall that a norm is said to be *strictly convex* if every point of the unit sphere is an extreme point of the unit ball.

Proposition 7.2 *Let C be a closed, convex and bounded set and let $x, y \in C$ be diametral points of C. Then:*

(i) *if $(y - x)\|y - x\|^{-1}$ is an extreme point, (resp., strongly exposed point) of B, then y is an extreme point (strongly exposed point) of C;*
(ii) *if $\|\cdot\|$ is strictly convex, then x, y are exposed points of C;*
(iii) *if $\|\cdot\|$ is strongly convex, then x, y are strongly exposed points of C.*

Proof First, note that letting $d = \operatorname{diam} C = \|y - x\|$, then $(y - x)/d$ is an extreme point of B, and so y is an extreme point of $x + dB$. Since $C \subset x + dB$ and y lies in the boundary of C, necessarily y is an extreme point of C. In the case that $(y - x)/d$ is strongly exposed in B by $0 \ne f \in X^*$, then y is strongly exposed by f in $x + dB$ and thus in C.

To prove the second part, let $0 \ne g \in X^*$ be a support functional of B at $(y-x)/d$, that is, $\|g\| = \sup g(B) = g(y-x)/d$. Since $\|\cdot\|$ is strictly convex, $S \cap \{z \in X : g(z) = \|g\|\} = \{(y - x)/d\}$. As a consequence, g also supports the ball $x + dB$ only at y. Again, $C \subset x + dB$ and so g also supports C at y. A symmetric argument shows that x is also an exposed point, in this case by the functional $-g$, for $-g$ supports the unit ball at the point $(x - y)/d$. The proof of (iii) is entirely similar to (ii). ■

When a constant width set C is a weakly compact and has nonempty interior, then every boundary point x of C has a diametral companion. Indeed, if $f \in X^*$ satisfies $f(x) = \sup f(C)$, there is $y \in C$ such that $f(y) = \inf f(C)$, hence $\|x - y\| \ge f(x - y) = \sup f(C) - \inf f(C) = \operatorname{diam}(C)$. As a consequence, if X is reflexive and has a strictly convex norm, then every boundary point of C is exposed, a result by Dalla and Tamvakis [5]; if it has a strongly convex norm, then every boundary point of C is strongly exposed. The same argument can be applied when C is a weak* compact set in a dual space, to deduce similar statements replacing reflexive spaces by dual spaces. Finally, when the space has finite dimension, then we can replace *constant width* by *diametrically maximal* since, in these spaces, every boundary point of a diametrically maximal set has a diametral companion.

8 Final Remarks

We include in this section, among other things, detailed proofs of two results that were stated without proof in Section 1, the first of them yielding the characterization of two dimensional spaces as those with the property that, under every equivalent norm, diametrically maximal sets always have constant width.

Proposition 8.1 *If X is a Banach space of dimension at least 3, then (after being given an equivalent norm) it contains a set C which is diametrically maximal but not of constant width.*

Proof By hypothesis, we can write $X = Y \times Z$, where Z has dimension 3 and Y is a closed subspace. We can assume that Z contains a bounded closed convex subset D and an equivalent norm (with unit ball B_1, say) with respect to which D has diameter 2 and is diametrically maximal but not of constant width. For instance, we can simply consider $Z = \ell_1^3$. Let B_Y be the unit ball of Y with respect to the norm induced by X, and renorm $X = Y \times \ell_1^3$ by $\|(y,z)\| = \max\{\|y\|, \|z\|_1\}$, with the corresponding unit ball $B_X = B_Y \times B_1$. Using standard Banach space notation, this can be written as $X = Y \oplus_\infty \ell_1^3$. We claim that the set $B_Y \times D \subset X$ is diametrically maximal but not of constant width. On the one hand, we have $\operatorname{diam} C = \max\{\operatorname{diam} B_Y, \operatorname{diam} D\} = 2$. To prove that C is diametrically maximal, consider $(y,z) \notin C$. Then, either $y \notin B_Y$ or $z \notin Z$. In any case,

$$\operatorname{diam}\bigl(C \cup \{(y,z)\}\bigr) = \max\Bigl\{\operatorname{diam}\bigl(B_Y \cup \{y\}\bigr), \operatorname{diam}\bigl(D \cup \{z\}\bigr)\Bigr\} > 2.$$

On the other hand, $C - C \neq 2B_X$, implying that C does not have constant width. Indeed, choose $z \in 2B_1 \setminus (D - D)$; then $(0,z) \in 2B_X \setminus (C - C)$. ■

The second result concerns sets of constant width in non-reflexive Banach spaces. Note that it characterizes reflexive spaces as those Banach spaces with the property that, for every equivalent norm and every set C of constant width, the set $C - C$ is closed.

Proposition 8.2 *If the Banach space X is not reflexive, then there exists an equivalent norm and a constant width set C (under this norm) such that $C - C$ is not closed.*

Proof Indeed, let f be a norm-one functional on X which does not attain its norm. Let $D = \ker f \cap B$ where B denotes the unit ball of the original norm. Given $0 < \lambda < 1/2$, define

$$C = \overline{co}\bigl[D \cup \{x : f(x) \geq 0, \|x\| \leq \lambda\}\bigr]$$

and renorm X with the new norm whose unit ball is $\overline{C - C}$. Then C becomes a set of constant width and we just need to check that $C - C$ is not closed. To this end, consider $x \in (1/2)B$ such that $f(x) = \lambda$. We claim that x is in the closure of $C - C$. To see this, first notice that for every $n \in \mathbb{N}$ such that $1/n < 1/2 - \lambda$, we have

$x + (\lambda + 1/n)B \subset x + (1/2)B \subset B$. Since $\inf f(x + (\lambda + 1/n)B) = -1/n$ (and $\sup f(x + (\lambda + 1/n)B) > 0$), there exists

$$x_n \in x + (\lambda + 1/n)B \cap \ker f \subset D \subset C.$$

Thus, the balls $x_n + \lambda B$ and $x + (1/n)B$ must intersect; let y_n be a point in their intersection. This implies that $y_n \to x$; also, since $f(y_n) \to f(x) > 0$, for large enough n we have $y_n \in D + \lambda B \cap \{x : f(x) \geq 0\}$. Now the latter set is contained in $C - C$; indeed, given a point $d + \lambda b$ with $d \in D$, $b \in B$ and $f(b) \geq 0$, we can write it as $\lambda b - (-d)$ and note that $\lambda b \in C$ and $-d \in D \subset C$. Thus, $x \in \overline{C - C}$. On the other hand, since $f < \lambda$ on $D \cup \{x : f(x) \geq 0, \|x\| \leq \lambda\}$, we also have $f < \lambda$ on its closed convex hull: in fact, if $z \in C$, then $z = \lim t_n d_n + (1 - t_n)\lambda b_n$, where $d_n \in D$, $b_n \in B$ and $t_n \in [0, 1]$. We may assume, by taking a subsequence, that $t_n \to t \in [0, 1]$ and so $z = \lim t d_n + (1 - t)\lambda b_n$. If $t = 0$, then $z \in \lambda B$ and hence $f(z) < \lambda$. If $t > 0$, then $f(z) \leq \lim(1 - t)\lambda f(b_n) \leq \lambda(1 - t) < \lambda$. Therefore, $f < \lambda$ on $C - C$, and this proves that $C - C$ is not closed, since $x \notin C - C$. ■

There are examples of nonreflexive Banach spaces in which $C - C$ is closed for every constant width set C. Indeed, this is the case of $c_0(I)$ with the usual sup norm. The reason is that in this space the sum of two intersections of balls is again an intersection of balls [12]. We do not know how big the topological size might be of the family of norms having constant width sets C whose difference $C - C$ is not closed. We do not even know whether this family is dense. Our last result concerns sets of constant width in Hilbert spaces.

Theorem 8.3 *A bounded closed convex subset C of a Hilbert space H is of constant width λ if and only if $P(C)$ has constant width λ, for every orthogonal projection P of H onto a closed hyperplane.*

Proof Since $C - C$ is closed, C has constant width λ (if and) only if $C - C = \lambda B$, hence, with P as above, $P(C) - P(C) = P(C - C) = P(\lambda B) = \lambda P(B) = \lambda B_{P(H)}$, so $P(C)$ has constant width λ. To prove the converse, suppose that each such $P(C)$ has constant width λ but that C does not, so that $C - C$ is a proper (and closed) subset of λB. It follows that there exists a continuous linear functional f of norm one such that $\sup f(C - C) < \lambda$. That is, there exists a point $u \in H$, $\|u\| = 1$, such that $f(x) = \langle u, x\rangle$ for all $x \in H$ with the property that $\sup\langle u, C - C\rangle < \lambda$. Choose any point $v \in H$, $\|v\| = 1$ such that $\langle u, v\rangle = 0$ and let P be the projection defined for each $x \in H$ by $P(x) = x - \langle x, v\rangle v$, which maps H onto the hyperplane $\{x \in H : \langle v, x\rangle = 0\}$. Note that for any $x \in H$ we have $\langle u, Px\rangle = \langle u, x\rangle$, and consequently, $\sup f|_{P(H)}(P(C - C)) = \sup_{x \in C - C}\langle u, x\rangle < \lambda$. Since $u \in P(H)$, the restriction of f to $P(H)$ still has norm 1, so we have a contradiction. ■

Let us finish now by noting some open questions that arise from the results of this paper. First, it would be interesting to know whether the condition of w-stability is essential in Proposition 6.1. Second, we do not know whether the main result of Section 7 remains true if we replace *uniformly convex* by *locally uniformly convex*, namely: is every diametrically maximal set locally uniformly convex provided the norm is locally uniformly convex? Finally, it is not clear whether diametral points exist in diametrically maximal sets.

Note added in proof While this manuscript was being processed, we became aware that Naszodi and Visy [20] have found a counterexample in $\mathbb{R}^3$, answering Groemer's question in the negative.

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