CONNECTIONS SATISFYING A GENERALIZED RICCI LEMMA

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1. Introduction. In this paper we shall consider a generalization of a very old problem in differential geometry; namely, given a second-order covariant tensor field $a_{ij}(x)$ on an n-dimensional manifold, when does there exist a connection $\Gamma_j{}^i{}_k\,(x)$ such that the covariant derivative, defined by

$$a_{ij|k} = a_{ij,k} - a_{rj}\Gamma_i{}^r{}_k - a_{ir}\Gamma_j{}^r{}_k, \qquad a_{ij,k} = \frac{\partial}{\partial x^k}a_{ij}, \tag{1.1}$$

vanishes?

The earliest question of this type arose in the case when $a_{ij} = a_{\underline{ij}}$ is symmetric and positive definite. A solution connection of the problem is then given by the Christoffel symbols

$$\left\{{}_j{}^i{}_k\right\} = \tfrac{1}{2}a^{\underline{ir}}(a_{\underline{jr},k} + a_{\underline{kr},j} - a_{\underline{jk},r}), \qquad a^{\underline{ir}}a_{\underline{rj}} = \delta^i{}_j. \tag{1.2}$$

The symbol $\delta^i{}_j$ denotes the Kronecker delta, which has value 1 when $i = j$ and otherwise vanishes. Indicating covariant differentiation using the Christoffel symbols by a semicolon, we may easily prove the identity

$$a_{\underline{ij};k} = 0, \tag{1.3}$$

known as Ricci's lemma. This important result is sometimes referred to as the fundamental theorem of Riemannian geometry, relating as it does the metrical and affine structures of a differentiable manifold. We shall take it as a prototype of our problem; however, we shall only assume that the manifold is once continuously differentiable, i.e. is C^1, and that the determinant

$$|a_{\underline{ij}}| \neq 0, \qquad a_{\underline{ij}} = \tfrac{1}{2}(a_{ij} + a_{ji}). \tag{1.4}$$

We shall refer to connections for which (1.1) holds as "metric."

It may be worth noting that several related problems have already been attacked. For example, Eisenhart **(1)** considered spaces which admit more than one symmetric tensor with vanishing covariant derivative and Hlavatý **(3**, p. 48, Theorem 1.1**)** has obtained a result concerning the above problem when $n = 4$. There are many known results that associate types of connections with geometrical structures on manifolds; however, in spite of the frequent occurrence of non-symmetric tensors, especially in the unified field theories, there does not seem to be a full treatment of the above basic problem.

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In §2 we establish our notation and reformulate the question using the mixed tensor $b^i{}_j$ defined by

$$b^i{}_j = a^{\underline{ir}} a_{\underset{\vee}{rj}}, \qquad a_{\underset{\vee}{ij}} = \tfrac{1}{2}(a_{ij} - a_{ji}). \tag{1.5}$$

In §3 we consider some of the analytical properties of the tensor $b^i{}_j$. Two theorems are proved, one dealing with an associated system of matrix differential equations and one with the differential properties of matrices which transform $b^i{}_j$ to canonical form. Section 4 is occupied with a proof of the principal theorem of this paper. This theorem gives a complete solution of the problem posed above; namely, the necessary and sufficient condition for a *metric* connection to exist is that the Jordan normal form of $b^i{}_j$ be *constant*.

Section 5 is devoted to corollaries of the main theorem and to representations of the metric connection in special cases. Theorems 4 and 5 reformulate the principal result when $b^i{}_j$ is diagonalizable or $a_{\underline{ij}}$ is positive definite. Theorem 6 gives the form of the general metric connection when $n = 3$.

2. Preliminary results. The notation for the symmetric and skew-symmetric parts of a_{ij} and for partial and covariant derivatives given in the Introduction will be used throughout this paper. Since much of our work involves second-order tensors, it is often convenient to use matrix notation. For example,

$$A = (a_{ij}) = (a_{\underline{ij}}) + (a_{\underset{\vee}{ij}}) = \underline{A} + \underset{\vee}{A}; \qquad B = (b^i{}_j) = \underline{A}^{-1}\underset{\vee}{A}. \tag{2.1}$$

In all such definitions the indices i and j refer to row and column respectively. The transpose of a matrix is denoted by a dash so that

$$\underline{A} = \underline{A}', \qquad \underset{\vee}{A} = -\underset{\vee}{A}'. \tag{2.2}$$

The symbol $M_{,k}$ will represent the matrix whose elements are the elements of M partially differentiated with respect to x^k. In §4 we shall use symbols such as $M_{;k}$ which are defined according to the tensor character of the entries of M.

Now let us suppose that a connection $\Gamma_j{}^i{}_k$ may be chosen such that

$$a_{ij|k} = a_{\underline{ij}|k} + a_{\underset{\vee}{ij}|k} = 0. \tag{2.3}$$

From symmetry considerations it follows that (2.3) is equivalent to

$$(a) \quad a_{\underline{ij}|k} = 0, \qquad (b) \quad a_{\underset{\vee}{ij}|k} = 0. \tag{2.4}$$

We deduce from (2.4) (a), the product rule of covariant differentiation, and the latter part of (1.2) that

$$a^{\underline{ij}}{}_{|k} = 0 \tag{2.5}$$

and hence, by (1.5), that

$$b^i{}_{j|k} = 0. \tag{2.6}$$

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Our problem is to solve (2.3) or (2.4) for the connection $\Gamma_j{}^i{}_k$. Now it is well known that any pair of connection parameters differs by a third-order tensor. Thus, we may put

$$\Gamma_j{}^i{}_k = \{{}_j{}^i{}_k\} + X_j{}^i{}_k, \tag{2.7}$$

where $\{{}_j{}^i{}_k\}$ is the connection defined in (1.2). The relationship between covariant derivatives formed from the two connections is

$$a_{ij|k} = a_{ij;k} - a_{rj}X_i{}^r{}_k - a_{ir}X_j{}^r{}_k. \tag{2.8}$$

In view of the Ricci lemma (1.3) the conditions (2.4) are equivalent to

$$(a)\quad a_{\underline{rj}}X_i{}^r{}_k + a_{\underline{ir}}X_j{}^r{}_k = 0, \qquad (b)\quad a_{\underset{\vee}{ij};k} - a_{\underset{\vee}{rj}}X_i{}^r{}_k - a_{\underset{\vee}{ir}}X_j{}^r{}_k = 0. \tag{2.9}$$

Writing these equations in terms of the tensor X_{ijk} defined by

$$X_{ijk} = a_{\underline{rj}}X_i{}^r{}_k, \qquad X_i{}^j{}_k = a^{\underline{jr}}X_{irk}, \tag{2.10}$$

and using the symmetry properties of $a_{\underline{ij}}$ and $a_{\underset{\vee}{ij}}$ we obtain

$$(a)\quad X_{ijk} + X_{jik} = 0, \qquad (b)\quad a_{\underset{\vee}{ij};k} - b^r{}_jX_{irk} + b^r{}_iX_{jrk} = 0. \tag{2.11}$$

An equivalent formulation of the basic problem is therefore to discover conditions on a_{ij} such that (2.11) has a solution X_{ijk}. Clearly much will depend on the structure of $b^i{}_j$, which we now discuss.

3. Analytical results concerning the tensor $b^i{}_j$. It is clear from the previous section that we shall have to examine the structure of solutions of the first-order linear system of partial differential equations (2.6). To this end we prove the following theorem.

THEOREM 1. *If there exists a C^1 connection $\Gamma_j{}^i{}_k$ and a tensor $b^i{}_j$ such that equations* (2.6) *hold, then $b^i{}_j$ has a constant Jordan normal form.*

Proof. Let $P_0(x_0)$ be any point of the manifold and let $C\colon x^i = x^i(t)$ be any C^1 curve through P_0. From a standard result of matrix theory we know that there exists a set of linearly independent vectors $\bar{t}^i{}_\alpha$ $(\alpha = 1, 2, \ldots, n)$ of the tangent space at P_0, such that

$$b^i{}_j(x_0)\bar{t}^j{}_\alpha = \bar{t}^i{}_\sigma\,\bar{b}^\sigma{}_\alpha, \tag{3.1}$$

where $\bar{b}^\sigma{}_\alpha$ is the Jordan form of $b^i{}_j$ at P_0. Let the vectors $\bar{t}^i{}_\alpha$ be transported by parallelism along C, i.e. let $t^i{}_\alpha$ satisfy

$$Dt^i{}_\alpha/dt = t^i{}_{\alpha|k}\,dx^k/dt = 0; \qquad t^i{}_\alpha(x_0) = \bar{t}^i{}_\alpha. \tag{3.2}$$

This problem involves a system of first-order ordinary differential equations which are linear and have C^1 coefficients. The solution vectors $t^i{}_\alpha$, therefore, exist, are unique, and remain linearly independent along C. From the latter

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remark we conclude that a tensor field $X^i{}_j$ is well defined along C by the equations

$$X^i{}_s t^s{}_\alpha = t^i{}_\sigma \tilde{b}^\sigma{}_\alpha, \tag{3.3}$$

and, by (3.2), it satisfies

$$DX^i{}_s/dt = X^i{}_{s|k}\, dx^k/dt = 0 \tag{3.4}$$

as well as the initial conditions

$$X^i{}_s(x_0) = b^i{}_s(x_0); \tag{3.4$'$}$$

compare (3.3) with (3.1). Since these conditions (3.4) and (3.4′) uniquely define $X^i{}_j$ along C and since $b^i{}_j$ also satisfies them by assumption, we conclude that $b^i{}_j = X^i{}_j$ along C and hence $b^i{}_j$ has the constant Jordan form $\tilde{b}^i{}_j$ along C. Finally, since P_0, C were arbitrary, it follows that $b^i{}_j$ must have the same constant Jordan form throughout the region of the manifold in which $\Gamma_j{}^i{}_k$ remains C^1.

Now let us suppose that $b^i{}_j(x)$ is any C^1 tensor field. It may well happen that the vectors $t^i{}_\alpha(x)$ which transform $b^i{}_j$ into normal form are not themselves C^1; however, in the present case, we have the following theorem.

THEOREM 2. *If $b^i{}_j$ is C^1 and has constant Jordan form $\tilde{b}^\alpha{}_\beta$, then there exist C^1 vectors $t^i{}_\alpha$ such that $b^i{}_j t^j{}_\alpha = t^i{}_\beta \tilde{b}^\beta{}_\alpha$.*

Proof. We use the matrix notation $B = (b^i{}_j)$, $\tilde{B} = (\tilde{b}^\alpha{}_\beta)$, $T = (t^i{}_\alpha)$ and put $\Delta(\lambda) = |\lambda I - B|$, where I is the $n \times n$ unit matrix. Let $\mathfrak{B}(x, \lambda)$ be the matrix of cofactors of $\lambda I - B$ so that $(\lambda I - B)\mathfrak{B}(x, \lambda) = \Delta(\lambda) I$.

Now the roots of the polynomial $\Delta(\lambda)$ are the eigenvalues of B and these are constant, by assumption. Thus, the coefficients of $\Delta(\lambda)$ are constant. Let $P_0(x_0)$ be any fixed point of the region under consideration and $\phi_0(\lambda) = \phi(x_0, \lambda)$ be the greatest common divisor of the elements of $\mathfrak{B}(x_0, \lambda)$ (which are polynomials in λ). Then $\phi_0(\lambda)$ is a divisor of $\Delta(\lambda)$ and so we may write $\Delta(\lambda) = \phi_0(\lambda)\psi(\lambda)$. The polynomial $\psi(\lambda)$ is the *minimal polynomial* of $B(x_0)$ **(2**, Chap. IV, §6**)**. But since the Jordan form of B does not depend on position, neither may the minimal polynomial. We therefore conclude that $\phi(\lambda) = \phi(x, \lambda)$ is also independent of x and so we may put $\mathfrak{B}(x, \lambda) = \phi(\lambda)\, \mathfrak{C}(x, \lambda)$.

It will follow that $(\lambda I - B)\, \mathfrak{C}(x, \lambda) = \psi(\lambda) I$ and, by repeated differentiation with respect to λ,

$$[\lambda I - B(x)]\, \mathfrak{C}_\lambda(x, \lambda) + \mathfrak{C}(x, \lambda) = \psi_\lambda I,$$

$$\vdots$$

$$[\lambda I - B(x)]\, \mathfrak{C}_\lambda{}^{(m-1)}(x, \lambda) + (m-1)\, \mathfrak{C}_\lambda{}^{(m-2)}(x, \lambda) = \psi_\lambda{}^{(m-1)} I.$$

Thus, if $\psi(\lambda) = (\lambda - \lambda_0)^m \chi(\lambda)$, $\chi(\lambda_0) \neq 0$, we have

$$B(x)[\mathfrak{C}(x, \lambda_0)] = \lambda_0[\mathfrak{C}(x, \lambda_0)], \dots, B(x)\left[\frac{\mathfrak{C}_\lambda^{(m-1)}(x, \lambda_0)}{(m-1)!}\right]$$
$$= \lambda_0\left[\frac{\mathfrak{C}_\lambda^{(m-1)}(x, \lambda_0)}{(m-1)!}\right] + \left[\frac{\mathfrak{C}_\lambda^{(m-2)}(x, \lambda_0)}{(m-2)!}\right].$$

Therefore, the non-zero columns of $\mathfrak{C}_\lambda^{(k)}(x, \lambda_0)$ $(k = 0, 1, \dots, m-1)$ yield $m \times m$ Jordan blocks corresponding to the eigenvalue λ_0. It can be shown (**2**, pp. 164 ff.) that there are sufficiently many linearly independent columns of $\mathfrak{C}(x, \lambda)$ for each eigenvalue λ_0 that these columns, considered as basis vectors t^i_α, transform B into its Jordan form. Since $B(x)$ is C^1 in x, so is $\mathfrak{B}(x, \lambda)$, consisting as it does of polynomials in the elements of $B(x)$ and λ. Since $\Delta(\lambda)$, $\phi(\lambda)$ are independent of position, it follows that $\mathfrak{C}(x, \lambda)$ and its derivatives with respect to λ are also C^1. The above choice of basis vectors then yields the theorem.

4. The principal theorem. We are now in a position to prove our main result, which completely answers the question posed in the Introduction.

THEOREM 3. *Given a tensor field $a_{ij}(x)$ which is C^1 and whose symmetric part is non-singular, there exists a connection $\Gamma_j{}^i{}_k$ such that the corresponding covariant derivative of a_{ij} vanishes if and only if the Jordan form of $b^i{}_j$ defined by* (1.5) *is constant.*

Proof. We have already seen in §2 that if $a_{ij|k} = 0$, then $b^i{}_{j|k} = 0$. The result of Theorem 1 then proves the necessity of the condition.

If we assume that $B = (b^i{}_j)$ has a constant Jordan form $\tilde{B}$, then, by Theorem 2, there exists a C^1 non-singular matrix $T = (t^i_\alpha)$ such that

$$BT = T\tilde{B}. \tag{4.1}$$

Now the existence of a metric connection $\Gamma_j{}^i{}_k$ is equivalent to the existence of a solution X_{ijk} of the equations (2.11). In matrix form these read

$$(4.2)\qquad \begin{aligned} &(a) && X_k + X_k' = 0,\\ &(b) && \check{A}_{;k} - X_kB + B'X_k' = 0, \end{aligned}$$

where $X_k = (X_{ijk})$ and

$$\check{A}_{;k} = \check{A}_{,k} - C_k\check{A} - \check{A}C_k', \qquad C_k = (\{{}_i{}^j{}_k\}). \tag{4.3}$$

The equations (1.3), which partially define the C_k, may be written

$$\underline{A}_{;k} - C_k\underline{A} - \underline{A}C_k' = O. \tag{4.4}$$

It is easier to deal with (4.2) after transforming B to Jordan form. To this end we introduce the quantities

$$T'X_kT = \tilde{X}_k. \tag{4.5}$$

Since T is non-singular, equations (4.2) are equivalent to

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$$(4.6)\qquad \begin{aligned} &(a) && \tilde{X}_k + \tilde{X}_k' = O,\\ &(b) && T'\underset{\vee}{A}_{;k}T = \tilde{X}_k\tilde{B} - \tilde{B}'\tilde{X}_k'. \end{aligned}$$

In accordance with tensor notation and (4.3) we may put

$$(4.7)\qquad (\underset{\vee}{A}T)_{;k} = (\underset{\vee}{A}T)_{,k} - C_k(\underset{\vee}{A}T) = \underset{\vee}{A}_{;k}T + \underset{\vee}{A}T_{;k},$$

where

$$(4.8)\qquad T_{;k} = T_{,k} + C_k'\,T.$$

Furthermore, we note that since $B = \underline{A}^{-1}\underset{\vee}{A}$, equation (4.1) is equivalent to

$$(4.9)\qquad \underset{\vee}{A}T = \underline{A}T\tilde{B}.$$

Using these relations, their transposes, and the symmetries (2.2), we may evaluate the left-hand side of (4.6) (b) as follows:

$$(4.10)\qquad \begin{aligned} T'\underset{\vee}{A}_{;k}T &= T'[(\underset{\vee}{A}T)_{;k} - \underset{\vee}{A}T_{;k}] = T'(\underline{A}T\tilde{B})_{;k} + \tilde{B}'T'\underline{A}T_{;k}\\ &= (T'\underline{A}T_{;k})\tilde{B} + \tilde{B}'(T'\underline{A}T_{;k}). \end{aligned}$$

In the last equality here, we have used (4.4) and, for the first time, the constancy of $\tilde{B}$.

According to (4.10) the set of equations (4.6) is equivalent to the set consisting of (4.6) (a) and

$$(4.11)\qquad [\tilde{X}_k - (T'\underline{A}T_{;k})]\tilde{B} + \tilde{B}'[\tilde{X}_k - (T'\underline{A}T_{;k})] = O.$$

The general solution of (4.11) is given by

$$(4.12)\qquad \tilde{X}_k = T'\underline{A}T_{;k} - \tfrac{1}{2}Y_k,$$

where the Y_k are any set of matrices satisfying

$$(4.13)\qquad Y_k\tilde{B} + \tilde{B}'Y_k = O.$$

In order that equations (4.6) (a) be satisfied as well as (4.11) we must have

$$T'\underline{A}T_{;k} + T'_{;k}\underline{A}T = \tfrac{1}{2}(Y_k + Y_k').$$

In view of (4.4), (4.8), and its transpose, this condition may be written

$$(4.14)\qquad \tfrac{1}{2}(Y_k + Y_k') = (T'\underline{A}T)_{;k} = (T'\underline{A}T)_{,k}.$$

The existence of a metric connection has thus been reduced to the existence of a solution Y_k of (4.13) and (4.14).

From (4.9) we have $(T'\underline{A}T)\tilde{B} = T'\underset{\vee}{A}T$ and, taking the transpose of this equation and adding, we find that

$$(4.15)\qquad O = (T'\underline{A}T)\tilde{B} + \tilde{B}'(T'\underline{A}T).$$

Since $\tilde{B}$ is constant, differentiation of this relation yields

$$O = (T'\underline{A}T)_{,k}\tilde{B} + \tilde{B}'(T'\underline{A}T)_{,k}.$$

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Therefore a solution Y_k of (4.13), (4.14) exists and is given by $Y_k = Y_k' = (T'\underline{A}T)_{,k} = (T'\underline{A}T)_{;k}$. We conclude that a solution of (4.6) exists, namely,

$$\tilde{X}_k = T'\underline{A}T_{;k} - \tfrac{1}{2}(T'\underline{A}T)_{;k} = \tfrac{1}{2}(T'\underline{A}T_{;k} - T'_{;k}\underline{A}T). \tag{4.16}$$

It follows from (4.5) that X_k is given by

$$X_k = \tfrac{1}{2}[\underline{A}T_{;k}T^{-1} - (T')^{-1}T'_{;k}\underline{A}] \tag{4.17}$$

and the corresponding connection is, by (2.7), (2.10),

$$\Gamma_k = (\Gamma_i{}^j{}_k) = C_k + X_k\underline{A}^{-1} = C_k + \tfrac{1}{2}[\underline{A}T_{;k}T^{-1}\underline{A}^{-1} - (T')^{-1}T'_{;k}]. \tag{4.18}$$

This solution will not be the only possible connection since we can always add to $\tilde{X}_k$ a solution Z_k of the equations

$$(a)\ Z_k\tilde{B} + \tilde{B}'Z_k = O, \qquad (b)\ Z_k + Z_k' = O. \tag{4.19}$$

The general solution connection will thus be given by

$$\Gamma_k = C_k + \tfrac{1}{2}[\underline{A}T_{;k}T^{-1}\underline{A}^{-1} - (T')^{-1}T'_{;k}] + (T')^{-1}Z_kT^{-1}\underline{A}^{-1}, \tag{4.20}$$

where C_k are the Christoffel symbols, T is the matrix whose column vectors transform B to canonical form, and Z_k is any solution of (4.19). This proves the theorem and, as well, exhibits the most general metric connection. It may be worth noting that if Γ_k is a non-real connection, then $\bar{\Gamma}_k$ also is and, hence, so is the real part of Γ_k. It is easy to check that the real part of Γ_k is a metric connection if Γ_k is.

5. Consequences of Theorem 3. A number of results follow from the analysis of §4, the most immediate one being the well-known *Ricci lemma*. If a_{ij} is a non-singular symmetric tensor, there exists a connection for which $a_{ij|k} = 0$. The proof is trivial since $b^i{}_j$ is identically zero in this case. The conditions on T and Z_k in (4.20), namely (4.1) and (4.19), can be satisfied by choosing $T = \underline{T}$, $Z_k = O$. This yields the usual Christoffel-symbol connection.

A less trivial consequence of Theorem 3 is the following theorem.

THEOREM 4. *Necessary conditions in order that a connection exists for which the covariant derivative of a tensor $A = (a_{ij})$ with $|a_{\underline{ij}}| \neq 0$ vanishes are that all the scalars*

$$\text{trace}\,(B^2),\quad \text{trace}\,(B^4),\quad \ldots,\quad \text{trace}\,(B^{2m}),\quad \ldots, \tag{5.1}$$

be constant. These conditions are also sufficient if B is diagonalizable.

Proof. If B is diagonalizable, the constancy of its Jordan form is equivalent to the constancy of its eigenvalues and this in turn is equivalent to the constancy of the elementary symmetric functions σ_α $(\alpha = 1, 2, \ldots, n)$ of the eigenvalues. Now if $\Delta(\lambda) = |\lambda I - B|$, we have

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$$\Delta(-\lambda) = |-\lambda I - B| = (-)^n |\lambda I + B|$$
$$= (-)^n |\underline{A}^{-1}(\lambda I + \underset{\vee}{A}\underline{A}^{-1})\underline{A}| = (-)^n |(\lambda I - B)'| = (-)^n \Delta(\lambda).$$

Thus the characteristic polynomial of B is an even or odd function according as n is even or odd. If we put

$$\Delta(\lambda) = \lambda^n - \sigma_1\lambda^{n-1} + \sigma_2\lambda^{n-2} \ldots + (-)^n \sigma_n, \tag{5.2}$$

this remark yields

$$\sigma_1 = \sigma_3 = \sigma_5 = \ldots = 0. \tag{5.3}$$

The eigenvalues of B therefore occur in positive-negative pairs and there must be a zero eigenvalue if n is odd. More general results can be proved about the structure of B but we shall not need them for the present work.

We quote two well-known results:

(i) If S_α ($\alpha = 1, 2, \ldots, n$) denotes the symmetric function $\lambda_1^\alpha + \lambda_2^\alpha + \ldots + \lambda_n^\alpha$ of the eigenvalues of any matrix B, then (**2**, p. 87, equation 43)

$$S_\alpha = \text{trace } (B^\alpha) \qquad (\alpha = 1, 2, \ldots, n). \tag{5.4}$$

(ii) The sets of σ_α and S_α determine each other uniquely and the relating equations are homogeneous. (**4**, §26; ex. 3).

In the case under consideration it follows from (5.3) and (ii) that

$$S_{2\alpha-1} = O \qquad (\alpha = 1, 2, \ldots). \tag{5.5}$$

A direct proof of this based on (5.4) is:

$$\text{trace } (B^\alpha) = \text{trace } (B')^\alpha = \text{trace } [(-\underset{\vee}{A}\underline{A}^{-1}) \ldots (-\underset{\vee}{A}\underline{A}^{-1})] = (-)^\alpha \text{trace } (B^\alpha),$$

since the trace is independent of transposition or similarity transformations.

It follows that the eigenvalues of B are determined by (5.1) and hence Theorem 3 yields the quoted results.

This theorem provides the most useful criterion in many cases of geometrical interest. We consider the case when $a_{\underline{ij}}$ is positive definite and prove a preliminary lemma.

LEMMA. *If $\underline{A}$ is real, symmetric, and positive definite, then B has a set of n linearly independent eigenvectors t_α which may be so chosen that $t_\alpha' \underline{A}\, t_\beta$ is constant.*

Proof. Corresponding to such an $\underline{A}$ there exists a real orthogonal matrix P such that

$$\underline{A} = P\{\mu_1, \mu_2, \ldots, \mu_n\}P^{-1}, \qquad PP' = I,$$

where $\{\mu_1, \mu_2, \ldots, \mu_n\}$ is a real diagonal matrix with $\mu_\alpha > 0$ ($\alpha = 1, 2, \ldots, n$). We put $\mu_\alpha^{\frac{1}{2}} = +\sqrt{\mu_\alpha}$ and form $\underline{A}^{\frac{1}{2}} = P\{\mu_1^{\frac{1}{2}}, \ldots, \mu_n^{\frac{1}{2}}\}P^{-1}$. It is easy to check that $\underline{A}^{\frac{1}{2}}\underline{A}^{\frac{1}{2}} = \underline{A}$, $(\underline{A}^{\frac{1}{2}})' = \underline{A}^{\frac{1}{2}}$ and $\underline{A}^{-\frac{1}{2}}\underline{A}^{\frac{1}{2}} = I$, where $\underline{A}^{-\frac{1}{2}} = P\{\mu_1^{-\frac{1}{2}}, \ldots, \mu_n^{-\frac{1}{2}}\}P^{-1}$. Consider then the eigenvalue problem $Bt = \lambda t$ and the associated problem $\underset{\vee}{\tilde{A}}\tau = \lambda\tau$, where $\underset{\vee}{\tilde{A}} = \underline{A}^{-\frac{1}{2}}\underset{\vee}{A}\underline{A}^{-\frac{1}{2}}$, $\tau = \underline{A}^{\frac{1}{2}}t$.

We note first that $\tilde{\underset{\vee}{A}}$ is real and skew-symmetric. Hence, there exists a real orthogonal matrix Q such that

$$\tilde{\underset{\vee}{A}} = Q\left\{\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \lambda_m \\ -\lambda_m & 0 \end{pmatrix}, 0, \ldots, 0\right\}Q^{-1}, \qquad QQ' = I,$$

where λ_r $(r = 1, 2, \ldots, m)$ are real and non-zero.

If we denote the columns of Q by $\xi_1, \eta_1, \ldots, \xi_m, \eta_m, \xi_{2m+1}, \ldots, \xi_n$, we therefore have

$$\tilde{\underset{\vee}{A}}\xi_r = -\lambda_r\eta_r, \qquad \tilde{\underset{\vee}{A}}\eta_r = \lambda_r\xi_r, \qquad \tilde{\underset{\vee}{A}}\xi_s = 0,$$

or

$$\tilde{\underset{\vee}{A}}\tau_r = i\lambda_r\tau_r, \qquad \tilde{\underset{\vee}{A}}\bar{\tau}_r = -i\lambda_r\bar{\tau}_r, \qquad \tilde{\underset{\vee}{A}}\tau_s = 0.$$

Here $r = 1, 2, \ldots, m$, $s = 2m + 1, \ldots, n$, $\tau_r = \xi_r + i\eta_r$, $\tau_s = \xi_s$, i is the complex unit, and bars denote complex conjugates. Let us denote by τ_α the vector τ_r, if $\alpha = r$, $\bar{\tau}_r$, if $\alpha = m + r$, and τ_s, if $\alpha = s$.

Hence, putting $t_\alpha = \underline{A}^{-\frac{1}{2}}\tau_\alpha$, we find that the t_α $(\alpha = 1, \ldots, n)$ provide an eigenbasis for B. For example,

$$Bt_r = \underline{A}^{-1}\underset{\vee}{A}\underline{A}^{-\frac{1}{2}}\tau_r = \underline{A}^{-\frac{1}{2}}\tilde{\underset{\vee}{A}}\tau_r = i\lambda_r\underline{A}^{-\frac{1}{2}}\tau_r = i\lambda_r t_r.$$

The linear independence of the t_α follows from the non-singularity of $\underline{A}^{-\frac{1}{2}}$. Consider finally the scalars $t_\alpha'\,\underline{A}\,t_\beta = \tau_\alpha'\,\underline{A}^{-\frac{1}{2}}\,\underline{A}\,\underline{A}^{-\frac{1}{2}}\,\tau_\beta = \tau_\alpha'\,\tau_\beta$. It is easy to check that these are constant in view of the orthogonality of Q. For example,

$$\tau_{r_1}'\,\bar{\tau}_{r_2} = (\xi_{r_1}' + i\eta_{r_1}')(\xi_{r_2} - i\eta_{r_2}) = 2\delta_{r_1 r_2}.$$

This proves the lemma, which, together with Theorem 4, yields the following theorem.

THEOREM 5. *Necessary and sufficient conditions in order that a connection exists for which the covariant derivative of a tensor* a_{ij}, *with* $a_{\underline{ij}}$ *positive definite, vanishes are that the scalars* (5.1) *be constant.*

In connection with the above lemma we note that whenever it is possible to choose the matrix $T = (t^i{}_\alpha)$ of (4.1) such that $T'\underline{A}T$ is constant, the explicit construction in §4 of a solution connection is simplified. For then, by (4.14), Y_k may be taken to be zero and, by (4.12), $X_k = T'\,\underline{A}\,T_{;k}$. This reduces the expressions for X_k and Γ_k by one term.

It is possible to prove that T may be so chosen in a large class of cases but I have not succeeded in proving this for an arbitrary B.

Concerning the result of Theorem 4 we make the following remarks:

(i) A necessary condition for the existence of a suitable connection in all cases is that the ratio $|\underset{\vee}{A}|/|\underline{A}|$ be constant, since this is the product of the eigenvalues of B **(3)**. When n is odd, this is trivially satisfied since $|\underset{\vee}{A}| = 0$. When $n = 2$ this condition is both necessary and sufficient since the characteristic equation of B reduces to $\lambda^2 + |\underset{\vee}{A}||\underline{A}|^{-1} = 0$ and hence B is diagonalizable if $|\underset{\vee}{A}| \neq 0$. But if $|\underset{\vee}{A}| = 0$, then $\underset{\vee}{A} = O$ and hence $B = O$.

(ii) When $n = 2$, $\check{A} \neq O$ and $|\underline{A}|^{-1}|\check{A}|$ is constant, the general solution connection is given by

$$\Gamma_j{}^i{}_k = \{{}_j{}^i{}_k\} + b^i{}_j v_k, \tag{5.6}$$

where v_k is an arbitrary covariant vector field.

Proof. We return to tensor notation and the equations (2.11). When $n = 2$ and $\check{A} \neq O$, the general solution of (2.11) (*a*) may be written $X_{ijk} = a_{\check{i}\check{j}} v_k$. Then (2.11) (*b*) becomes, by (2.1) and symmetry,

$$a_{\check{i}\check{j};k} - a^{\underline{rs}} v_k [a_{\check{r}\check{j}} a_{\underline{i}\check{s}} + a_{\check{i}\check{r}} a_{\check{j}\check{s}}] = a_{\check{i}\check{j};k} = 0.$$

This equation is identically satisfied, if $|a_{\underline{ij}}|^{-1}|a_{\check{i}\check{j}}| = c$, a constant. It is sufficient to check this for $a_{\check{1}\check{2};k}$. Now $|a_{\check{i}\check{j}}| = (a_{\check{1}\check{2}})^2$ and hence $(a_{\check{1}\check{2}})^2 = c|a_{\underline{ij}}|$. Thus

$$\begin{aligned} 2a_{\check{1}\check{2}} a_{\check{1}\check{2};k} &= 2a_{\check{1}\check{2}}[a_{\check{1}\check{2},k} - a_{\check{1}\check{2}}\{{}_1{}^1{}_k\} - a_{\check{1}\check{2}}\{{}_2{}^2{}_k\}] \\ &= c[|a_{\underline{ij}}|_{,k} - 2|a_{\underline{ij}}|\{{}_r{}^r{}_k\}] = 0. \end{aligned}$$

The last equality is a well-known identity which follows from the definition of the Christoffel symbol and the fact that

$$|a_{\underline{ij}}|_{,k} = (\partial|a_{\underline{ij}}|/\partial a_{\underline{rs}}) a_{\underline{rs},k} = (\text{cofactor } a_{\underline{rs}}) a_{\underline{rs},k} = a^{\underline{rs}}|a_{\underline{ij}}| a_{\underline{rs},k}.$$

Substituting $X_{ijk} = a_{\check{i}\check{j}} v_k$ in (2.10) and using (2.7) we obtain the stated result.

(iii) In the case when $n = 3$, a necessary condition for the existence of a metric connection is $b^i{}_j b^j{}_i = b =$ constant. If this constant is non-zero, this is also a sufficient condition since the characteristic equation of $b^i{}_j$ is $\lambda^3 + \sigma_2\lambda = 0$, where $\sigma_2 = -\frac{1}{2}S_2 = -\frac{1}{2}b$, as we see from equation (5.3) and the remarks following it. Hence, if $b \neq 0$, $b^i{}_j$ is diagonalizable and its eigenvalues constant. If $b = 0$, then

$$a^{\check{i}\check{j}} a_{\check{i}\check{j}} = a^{\underline{ir}} a^{\underline{js}} a_{\check{r}\check{s}} a_{\check{i}\check{j}} = 0$$

and, if $a_{\underline{ij}}$ is positive definite, it will follow that $a_{\check{i}\check{j}} = 0$.

We conclude this section with a representation theorem for the case discussed in (iii).

THEOREM 6. *If* $n = 3$,

$$a^{\check{i}\check{j}} a_{\check{i}\check{j}} = b = (\text{constant}) \neq 0,$$

the general connection making $a_{ij|k} = 0$ *may be written*

$$\Gamma_j{}^i{}_k = \{{}_j{}^i{}_k\} \pm (2b)^{-1}(a^i a_{j;k} - a_j a^i{}_{;k}) + b^i{}_j v_k, \tag{5.7}$$

where a^i *is the "dual" of* $a_{\check{i}\check{j}}$ (see below), v_k *is an arbitrary covariant vector field, and the sign of* $(2b)^{-1}$ *is the same as that of* $|a_{\underline{ij}}|$.

Proof. We introduce the permutation tensors ϵ_{ijk}, ϵ^{ijk}. The former is defined, when $|a_{\underline{ij}}| > 0$, by

$$\epsilon_{ijk} = \begin{cases} |a_{\underline{ij}}|^{\frac{1}{2}}, & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3), \\ -|a_{\underline{ij}}|^{\frac{1}{2}}, & \text{if } \quad \text{''} \quad \text{''} \quad \text{''} \quad \text{odd} \quad \text{''} \quad \text{''} \quad \text{''} \\ 0, & \text{otherwise.} \end{cases} \tag{5.8}$$

When $|a_{\underline{ij}}| < 0$, we use $-|a_{\underline{ij}}|$ in place of $|a_{\underline{ij}}|$. The contravariant tensor ϵ^{ijk} is defined similarly using $|a_{\underline{ij}}|^{-1}$ instead of $|a_{\underline{ij}}|$. The following formulae are easily verified:

$$\begin{aligned} &(a) \quad \epsilon^{ijk}\epsilon_{ipq} = \delta^{jk}_{pq} = \begin{vmatrix} \delta^{j}{}_{p} & \delta^{j}{}_{q} \\ \delta^{k}{}_{p} & \delta^{k}{}_{q} \end{vmatrix}, \qquad (b) \quad \epsilon^{ijk}\epsilon_{ijh} = 2\delta^{k}{}_{h}, \\ &(c) \quad \epsilon_{ijk;h} = 0, \qquad (d) \quad \epsilon^{ijk} = \pm\, a^{\underline{ir}} a^{\underline{js}} a^{\underline{kt}} \epsilon_{rst}, \end{aligned} \tag{5.9}$$

where the plus sign is chosen if $|a_{\underline{ij}}| > 0$.

Using these results we shall solve equations (2.11) for X_{ijk}. First note that the solution of (2.11) (a) may be written in the form $X_{ijk} = \epsilon_{ijh}X^{h}{}_{k}$, where $X^{h}{}_{k}$ is a mixed tensor field. Equation (2.11) (b) then becomes

$$a_{\underset{\vee}{ij};k} = [a_{s\underset{\vee}{j}}\epsilon_{irt} - a_{s\underset{\vee}{i}}\epsilon_{jrt}]a^{\underline{rs}}X^{t}{}_{k}. \tag{5.10}$$

We put $a^{k} = \epsilon^{kij}\, a_{\underset{\vee}{ij}}$ and note that $a_{\underset{\vee}{ij}} = \frac{1}{2}\epsilon_{ijk}a^{k}$. Substituting in (5.10) and using (5.9) (c) and the skew-symmetry of ϵ_{sih}, we obtain

$$\tfrac{1}{2}\epsilon_{ijh}a^{h}{}_{;k} = \tfrac{1}{2}(\epsilon_{sjh}\epsilon_{irt} + \epsilon_{ish}\epsilon_{jrt})a^{h}a^{\underline{rs}}X^{t}{}_{k},$$

or, on multiplication with ϵ^{ijl} and use of (5.9) (b), (a),

$$a^{l}{}_{;k} = \tfrac{1}{2}(\epsilon_{sjh}\delta^{jl}_{rt} + \delta^{jl}_{sh}\epsilon_{jrt})a^{h}a^{\underline{rs}}X^{t}{}_{k}.$$

If we expand $\delta^{j}{}_{r}{}^{l}{}_{t}$ by (5.9) (a) and use the relation $\epsilon_{rst}a^{\underline{rs}} = 0$, this reduces to

$$a^{l}{}_{;k} = \tfrac{1}{2}(-\epsilon_{sth}\delta^{l}{}_{r} - \epsilon_{hrt}\delta^{l}{}_{s})a^{h}a^{\underline{rs}}X^{t}{}_{k}.$$

Finally, multiplying by $a_{\underline{li}}$ and using (1.3), we find that

$$a_{i;k} = \tfrac{1}{2}(-\epsilon_{ith} - \epsilon_{hit})a^{h}X^{t}{}_{k} = \epsilon_{iht}a^{h}X^{t}{}_{k}. \tag{5.11}$$

The general solution of (2.11) is therefore given by the general solution of (5.11) for $X^{t}{}_{k}$.

Put $\tilde{X}^{h}{}_{k} = \epsilon^{hrs}a_{r;k}a_{s}$. Then, by (5.9) (a),

$$\epsilon_{ijh}a^{j}\tilde{X}^{h}{}_{k} = \epsilon_{ijh}a^{j}\epsilon^{hrs}a_{r;k}a_{s} = a^{j}(a_{i;k}a_{j} - a_{j;k}a_{i}). \tag{5.12}$$

Now, by (5.9) (d),

$$\begin{aligned} a^{i}a_{i} &= a_{\underline{ij}}\epsilon^{irs}a_{\underset{\vee}{rs}}\epsilon^{jpq}a_{\underset{\vee}{pq}} = \pm\, a_{\underline{ij}}a^{\underline{im}}a^{\underline{rh}}a^{\underline{sk}}\epsilon_{mhk}a_{\underset{\vee}{rs}}\epsilon^{jpq}a_{\underset{\vee}{pq}} \\ &= \pm\, \delta^{pq}_{hk}a^{\underline{rh}}a^{\underline{sk}}a_{\underset{\vee}{rs}}a_{\underset{\vee}{pq}} = \pm\, 2a^{\overset{\vee}{hk}}a_{\underset{\vee}{hk}} = \pm\, 2b = \text{constant}, \end{aligned} \tag{5.13}$$

by assumption. Thus covariant differentiation of (5.13) yields $a^{i}a_{i;k} = 0$ and

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hence, by (5.12), $\epsilon_{ijh}a^j\tilde{X}^h{}_k = \pm 2ba_{i;k}$. It follows that $(\pm 2b)^{-1}\tilde{X}^t{}_k$ is a particular solution of (5.11). Since the general solution of $\epsilon_{iht}a^hX^t{}_k = 0$ is $X^t{}_k = a^tv_k$, where v_k is arbitrary, the general solution of (5.11) is

$$X^t{}_k = \pm(2b)^{-1}\epsilon^{trs}a_{r;k}a_s + a^tv_k.$$

From this we obtain $X_{ijk} = \epsilon_{ijt}X^t{}_k$ and, by (2.7), (2.10), the connection (5.7).

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