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p-VARIATION OF VECTOR MEASURES WITH RESPECT TO BILINEAR MAPS

O. BLASCO and J. M. CALABUIG

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Abstract

We introduce the spaces $\mathrm{V}^p_{\mathcal{B}}(X)$ (respectively $\mathcal{V}^p_{\mathcal{B}}(X)$) of the vector measures $\mathcal{F}:\Sigma\to X$ of bounded $(p,\mathcal{B})$-variation (respectively of bounded $(p,\mathcal{B})$-semivariation) with respect to a bounded bilinear map $\mathcal{B}:X\times Y\to Z$ and show that the spaces $L^p_{\mathcal{B}}(X)$ consisting of functions which are p-integrable with respect to $\mathcal{B}$, defined in by Blasco and Calabuig ['Vector-valued functions integrable with respect to bilinear maps', *Taiwanese Math. J.* to appear], are isometrically embedded in $\mathrm{V}^p_{\mathcal{B}}(X)$. We characterize $\mathcal{V}^p_{\mathcal{B}}(X)$ in terms of bilinear maps from $L^{p'}\times Y$ into Z and $\mathrm{V}^p_{\mathcal{B}}(X)$ as a subspace of operators from $L^{p'}(Z^*)$ into Y^*. Also we define the notion of cone absolutely summing bilinear maps in order to describe the $(p,\mathcal{B})$-variation of a measure in terms of the cone-absolutely summing norm of the corresponding bilinear map from $L^{p'}\times Y$ into Z.

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1. Notation and preliminaries

Throughout the paper X denotes a Banach space, (Ω,Σ,μ) a positive finite measure space, $\mathcal{D}_E$ the set of all partitions of $E\in\Sigma$ into a finite number of pairwise disjoint elements of Σ of positive measure and $\mathcal{S}_\Sigma(X)$ the space of simple functions, $\mathbf{s}=\sum_{k=1}^n x_k\mathbf{1}_{A_k}$, where $x_k\in X$, $(A_k)_k\in\mathcal{D}_\Omega$ and $\mathbf{1}_A$ denotes the characteristic function of the set $A\in\Sigma$. Also Y and Z denote Banach spaces over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\mathcal{B}:X\times Y\to Z$ a bounded bilinear map. We use the notation B_X for the closed unit ball of X, $\mathcal{L}(X,Y)$ for the space of bounded linear operators from X to Y and $X^*=\mathcal{L}(X,\mathbb{K})$.

For a vector measure $\mathcal{F}:\Sigma\to X$ we use the notation $|\mathcal{F}|$ and $\|\mathcal{F}\|$ for the nonnegative set functions $|\mathcal{F}|:\Sigma\to\mathbb{R}^+$ and $\|\mathcal{F}\|:\Sigma\to\mathbb{R}^+$ given by

$$|\mathcal{F}|(E)=\sup\left\{\sum_{A\in\pi}\|\mathcal{F}(A)\|_X:\pi\in\mathcal{D}_E\right\}$$

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and

$$\|\mathcal{F}\|(E) = \sup\{|\langle \mathcal{F}, x^*\rangle|(E) : x^* \in \mathrm{B}_{X^*}\},$$

respectively. In the case of operator-valued measures $\mathcal{F} : \Sigma \to \mathcal{L}(Y, Z)$ we use $|||\mathcal{F}|||$ for the strong-variation defined by

$$|||\mathcal{F}|||(E) = \sup\left\{\sum_{A\in\pi} \|\mathcal{F}(A)y\|_Z : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_E\right\}.$$

Given a norm τ defined on the space $Y \otimes_\tau X$ satisfying $\|y \otimes x\|_\tau \leq C\|y\| \cdot \|x\|$ we write $Y\widehat{\otimes}X$ for its completion. In [1] Bartle introduced the notion of Y-semivariation of a vector measure $\mathcal{F} : \Sigma \to X$ with respect to τ by the formula

$$\beta_Y(\mathcal{F}, \tau)(E) = \sup\left\{\left\|\sum_{A\in\pi} y_A \otimes \mathcal{F}(A)\right\|_\tau : y_A \in \mathrm{B}_Y, \pi \in \mathcal{D}_E\right\}$$

for every $E \in \Sigma$. This is an intermediate notion between the variation and semivariation, since for every $E \in \Sigma$ we clearly have

$$\|\mathcal{F}\|(E) \leq \beta_Y(\mathcal{F}, \tau)(E) \leq |\mathcal{F}|(E).$$

If $Y\widehat{\otimes}_\varepsilon X$ and $Y\widehat{\otimes}_\pi X$ stand for the injective and projective tensor norms, respectively, then we actually have

$$\|\mathcal{F}\|(E) = \beta_Y(\mathcal{F}, \varepsilon)(E) \leq \beta_Y(\mathcal{F}, \tau)(E) \leq \beta_Y(\mathcal{F}, \pi)(E) \leq |\mathcal{F}|(E).$$

We refer the reader to [11] for a theory of integration of Y-valued functions with respect to X-valued measures of bounded Y-semivariation initiated by Jefferies and Okada and to [3] for the study of this notion in the particular cases $X = L^p(\mu)$, $Y = L^q(\nu)$ and τ the norm in the space of vector-valued functions $L^p(\mu, L^q(\nu))$.

We are going to use notions of $\mathcal{B}$-variation (or $\mathcal{B}$-semivariation) which allow us to obtain all of the previous cases for particular instances of bilinear maps.

Recall that, for $1 < p < \infty$, the p-variation and p-semivariation of a vector measure $\mathcal{F}$ are defined by

$$|\mathcal{F}|_p(E) = \sup\left\{\left(\sum_{A\in\pi} \frac{\|\mathcal{F}(A)\|_X^p}{\mu(A)^{p-1}}\right)^{1/p} : \pi \in \mathcal{D}_E\right\} \tag{1.1}$$

and

$$\|\mathcal{F}\|_p(E) = \sup\left\{\left(\sum_{A\in\pi} \frac{|\langle \mathcal{F}(A), x^*\rangle|^p}{\mu(A)^{p-1}}\right)^{1/p} : x^* \in \mathrm{B}_{X^*}, \pi \in \mathcal{D}_E\right\}. \tag{1.2}$$

We use $\mathrm{V}^p(X)$ and $\mathcal{V}^p(X)$ to denote the Banach spaces of vector measures for which $|\mathcal{F}|_p(\Omega) < \infty$ and $\|\mathcal{F}\|_p(\Omega) < \infty$, respectively.

The limiting case $p = 1$ corresponds to $\|\mathcal{F}\|_1(E) = |\mathcal{F}|(E)$ and $\|\mathcal{F}\|_1(E) = \|\mathcal{F}\|(E)$. For $p = \infty$ $\mathrm{V}^\infty(X) = \mathcal{V}^\infty(X)$ is given by vector measures satisfying the

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property that there exists $C > 0$ such that $\|\mathcal{F}(A)\| \leq C\mu(A)$ for any $A \in \Sigma$ and the ∞-variation of a measure is defined by

$$\|\mathcal{F}\|_\infty(E) = \sup\left\{\frac{\|\mathcal{F}(A)\|_X}{\mu(A)} : A \in \Sigma,\ A \subset E,\ \mu(A) > 0\right\}. \tag{1.3}$$

We use $L^0(X)$ and $L^0_{\text{weak}}(X)$ to denote the spaces of strongly and weakly measurable functions with values in X and write $L^p(X)$ and $L^p_{\text{weak}}(X)$ for the space of functions in $L^0(X)$ and $L^0_{\text{weak}}(X)$ such that $\|f\| \in L^p$ and $\langle f, x^*\rangle \in L^p$ for every $x^* \in X^*$, respectively. As usual for $1 \leq p \leq \infty$ the conjugate index is denoted by p', that is $1/p + 1/p' = 1$.

For each $f \in L^p(X)$, $1 \leq p \leq \infty$, one can define a vector measure

$$\mathcal{F}_f(E) = \int_E f\,d\mu, \quad E \in \Sigma$$

which is of bounded p-variation and $|\mathcal{F}_f|_p(\Omega) = \|f\|_{L^p(X)}$. On the other hand, the converse depends on the Radon–Nikodým property (RNP), that is, given $1 < p \leq \infty$, X has the RNP if and only if for any X-valued measure $\mathcal{F}$ of bounded p-variation there exists $f \in L^p(X)$ such that $\mathcal{F} = \mathcal{F}_f$.

For general Banach spaces X, $\mathrm{V}^\infty(X)$ can be identified with the space of operators $\mathcal{L}(L^1, X)$ by means of the map $\mathcal{F} \to T_{\mathcal{F}}$ where

$$T_{\mathcal{F}}(\mathbf{1}_E) = \mathcal{F}(E), \quad E \in \Sigma,$$

and for $1 < p < \infty$ the space $\mathrm{V}^p(X)$ can be identified (isometrically) with the space $\Lambda(L^{p'}, X)$, formed by the cone absolutely summing operators from $L^{p'}$ into X with the π_1^+ norm (see [14, 2]). We refer the reader to [9, 8, 11, 14] for the notions appearing in the paper and the basic concepts about vector measures and their variations.

Quite recently the authors started studying the spaces of X-valued functions which are p-integrable with respect to a bounded bilinear map $\mathcal{B} : X \times Y \to Z$, that is, functions f satisfying the condition $\mathcal{B}(f, y) \in L^p(Z)$ for all $y \in Y$. Some basic theory was developed and applied to different examples (see [4, 5, 6]). Note that the use of certain bilinear maps, such as

$$\mathcal{B} : X \times \mathbb{K} \to X, \qquad \text{given by } \mathcal{B}(x, \lambda) = \lambda x, \tag{1.4}$$
$$\mathcal{D} : X \times X^* \to \mathbb{K}, \qquad \text{given by } \mathcal{D}(x, x^*) = \langle x, x^*\rangle, \tag{1.5}$$
$$\mathcal{D}_1 : X^* \times X \to \mathbb{K}, \qquad \text{given by } \mathcal{D}_1(x^*, x) = \langle x, x^*\rangle, \tag{1.6}$$
$$\pi_Y : X \times Y \to X\widehat{\otimes}Y, \qquad \text{given by } \pi_Y(x, y) = x \otimes y, \tag{1.7}$$
$$\widetilde{\mathcal{O}}_Y : X \times \mathcal{L}(X, Y) \to Y, \qquad \text{given by } \widetilde{\mathcal{O}}_Y(x, T) = T(x), \tag{1.8}$$
$$\mathcal{O}_{Y,Z} : \mathcal{L}(Y, Z) \times Y \to Z, \qquad \text{given by } \mathcal{O}_{Y,Z}(T, y) = T(y) \tag{1.9}$$

have been around for many years and have been used in different aspects of vector-valued functions, but a systematic study for general bilinear maps was started in [4]

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and used, among other things, to extend the results on boundedness from $L^p(Y)$ to $L^p(Z)$ of operator-valued kernels by Girardi and Weiss [10] to the case where $K : \Omega \times \Omega' \to X$ is measurable and the integral operators are defined by

$$T_K(f)(w) = \int_{\Omega'} \mathcal{B}(K(w, w'), f(w'))\, d\mu'(w').$$

The reader is also referred to [6] for some versions of Hölder's inequality in this setting.

We require some notation and definitions from the previous papers. We write $\Phi_{\mathcal{B}} : X \to \mathcal{L}(Y, Z)$ and $\Psi_{\mathcal{B}} : Y \to \mathcal{L}(X, Z)$ for the bounded linear operators defined by $\Phi_{\mathcal{B}}(x) = \mathcal{B}_x$ and $\Psi_{\mathcal{B}}(y) = \mathcal{B}^y$ where $\mathcal{B}_x$ and $\mathcal{B}^y$ are given by $\mathcal{B}_x(y) = \mathcal{B}^y(x) = \mathcal{B}(x, y)$.

A bounded bilinear map $\mathcal{B} : X \times Y \to Z$ is called admissible (see [4]) if $\Phi_{\mathcal{B}}$ is injective. Throughout the paper we always assume that $\mathcal{B}$ is admissible. However, a stronger condition will also be required for some results: a Banach space X is said to be $(Y, Z, \mathcal{B})$-normed if there exists $k > 0$

$$\|x\|_X \le k\|\mathcal{B}_x\|_{\mathcal{L}(Y,Z)}, \quad x \in X.$$

The bounded bilinear maps (1.4)–(1.9) provide examples of $\mathcal{B}$-normed spaces.

As in [4] we write $\mathcal{L}^p_{\mathcal{B}}(X)$ for the space of functions $f : \Omega \to X$ with $\mathcal{B}(f, y) \in L^0(Z)$ for any $y \in Y$ and such that

$$\|f\|_{\mathcal{L}^p_{\mathcal{B}}(X)} = \sup\{\|\mathcal{B}(f, y)\|_{L^p(Z)} : y \in \mathrm{B}_Y\} < \infty,$$

and we use the notation $L^p_{\mathcal{B}}(X)$ for the space of functions $f \in \mathcal{L}^p_{\mathcal{B}}(X)$ for which there exists a sequence of simple functions $(\mathbf{s}_n)_n \in \mathcal{S}_\Sigma(X)$ such that $\mathbf{s}_n \to f$ almost everywhere and $\|\mathbf{s}_n - f\|_{\mathcal{L}^p_{\mathcal{B}}(X)} \to 0$. In such a case, we write $\|f\|_{L^p_{\mathcal{B}}(X)}$ instead of $\|f\|_{\mathcal{L}^p_{\mathcal{B}}(X)}$ and $\|f\|_{L^p_{\mathcal{B}}(X)} = \lim_{n\to\infty} \|\mathbf{s}_n\|_{L^p_{\mathcal{B}}(X)}$.

In particular, for the examples $\mathcal{B}$ and $\mathcal{D}$ we have that $\mathcal{L}^p_{\mathcal{B}}(X) = L^p(X)$ and $\mathcal{L}^p_{\mathcal{D}}(X) = L^p_{\mathrm{weak}}(X)$. In addition $L^p_{\mathcal{B}}(X) = L^p(X)$ and $L^p_{\mathcal{D}}(X)$ coincides with the space of Pettis p-integrable functions $\mathcal{P}^p(X)$ (see [13, p. 54], for the case $p = 1$).

Observe that, for any $\mathcal{B}$, $L^p(X) \subseteq L^p_{\mathcal{B}}(X)$ and the inclusion can be strict (see [8, p. 53], for the case $\mathcal{B} = \mathcal{D}$). Regarding the connection between $L^p_{\mathcal{B}}(X)$ and $L^p_{\mathrm{weak}}(X)$ it was shown that X is $(Y, Z, \mathcal{B})$-normed if and only if $L^p_{\mathcal{B}}(X) \subseteq L^p_{\mathrm{weak}}(X)$ with continuous inclusion.

Owing to this fact, if $f \in L^1_{\mathcal{B}}(X)$ for some $(Y, Z, \mathcal{B})$-normed space X, then for each $E \in \Sigma$ there exists a unique element of X, denoted by $\int_E^{\mathcal{B}} f\, d\mu$, verifying

$$\int_E \mathcal{B}(f, y)\, d\mu = \mathcal{B}\left(\int_E^{\mathcal{B}} f\, d\mu, y\right), \quad \forall y \in Y.$$

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This allows us to define the vector measure

$$\mathcal{F}_f^{\mathcal{B}}(E) = \int_E^{\mathcal{B}} f \, d\mu, \quad E \in \Sigma.$$

We consider the notion of $(p, \mathcal{B})$-variation which fits with the theory allowing us to show that the $(p, \mathcal{B})$-variation of $\mathcal{F}_f$ coincides with its norm $\|f\|_{L^p_{\mathcal{B}}(X)}$.

The rest of this paper is divided into three sections. In Section 2 we introduce the notion of $\mathcal{B}$-variation, $\mathcal{B}$-semivariation of a vector measure and study their connection with the classical notions. We prove that for $(Y, Z, \mathcal{B})$-normed spaces the $\mathcal{B}$-semivariation is equivalent to the semivariation and that the Y-semivariation considered by Bartle coincides the $\mathcal{B}$-variation for a particular bilinear map $\mathcal{B}$. Of particular interest is the observation that any vector measure with values in $X = L^1(\mu)$ is of bounded $\mathcal{B}$-variation for every $\mathcal{B}$ whenever Z is a Hilbert space. We also show in this section that the measure $\mathcal{F}_f^{\mathcal{B}}$ is μ-continuous and $\|\mathcal{F}_f^{\mathcal{B}}\|_{\mathcal{B}}(\Omega) = \|f\|_{L^1_{\mathcal{B}}(X)}$. In Section 3 the natural notion of $(p, \mathcal{B})$-semivariation is introduced. Starting with the case $p = \infty$ we describe, for $1 < p \leq \infty$, the space of measures with bounded $(p, \mathcal{B})$-semivariation as bounded bilinear maps from $L^{p'} \times Y \to Z$. In Section 4 we deal with the notion of $(p, \mathcal{B})$-variation of a vector measure. Several characterizations are presented and the new notion of 'cone absolutely summing bilinear map' from $L \times Y \to Z$, where L is a Banach lattice, is introduced. This allow us to describe the $(p, \mathcal{B})$-variation of a vector measure as the norm of the corresponding bilinear map from $L^{p'} \times Y$ into Z in this class.

Throughout the paper $\mathcal{F} : \Sigma \to X$ always denotes a vector measure, $\mathcal{B} : X \times Y \to Z$ is admissible and, for each $y \in Y$, $\mathcal{B}(\mathcal{F}, y)$ denotes the Z-valued measure $\mathcal{B}(\mathcal{F}, y)(E) = \mathcal{B}(\mathcal{F}(E), y)$.

2. Variation and semivariation with respect to bilinear maps

DEFINITION 2.1. Let $E \in \Sigma$. We define the $\mathcal{B}$-*variation* of $\mathcal{F}$ on the set E by

$$\begin{aligned} |\mathcal{F}|_{\mathcal{B}}(E) &= \sup\{|\mathcal{B}(\mathcal{F}, y)|(E) : y \in \mathrm{B}_Y\} \\ &= \sup\left\{\sum_{A \in \pi} \|\mathcal{B}(\mathcal{F}(A), y)\|_Z : \pi \in \mathcal{D}_E, \ y \in \mathrm{B}_Y\right\}. \end{aligned}$$

We say that $\mathcal{F}$ has *bounded* $\mathcal{B}$-*variation* if $|\mathcal{F}|_{\mathcal{B}}(\Omega) < \infty$.

DEFINITION 2.2. Let $E \in \Sigma$. We define the $\mathcal{B}$-*semivariation* of $\mathcal{F}$ on the set E by

$$\begin{aligned} \|\mathcal{F}\|_{\mathcal{B}}(E) &= \sup\{\|\mathcal{B}(\mathcal{F}, y)\|(E) : y \in \mathrm{B}_Y\} \\ &= \sup\{|\langle \mathcal{B}(\mathcal{F}, y), z^*\rangle|(E) : y \in \mathrm{B}_Y, \ z^* \in \mathrm{B}_{Z^*}\} \\ &= \sup\left\{\sum_{A \in \pi} |\langle \mathcal{B}(\mathcal{F}(A), y), z^*\rangle| : \pi \in \mathcal{D}_E, \ y \in \mathrm{B}_Y, \ z^* \in \mathrm{B}_{Z^*}\right\}. \end{aligned}$$

We say that $\mathcal{F}$ has *bounded* $\mathcal{B}$-*semivariation* if $\|\mathcal{F}\|_{\mathcal{B}}(\Omega) < \infty$.

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REMARK 2.3. Let $\mathcal{F}$ be a vector measure and $E \in \Sigma$. Then:

(a) $|\mathcal{F}|_{\mathcal{B}}(E) \le \|\mathcal{B}\| \cdot |\mathcal{F}|(E)$;
(b) $\|\mathcal{F}\|_{\mathcal{B}}(E) \le \|\mathcal{B}\| \cdot \|\mathcal{F}\|(E)$;
(c) $\sup\{\|\mathcal{B}(\mathcal{F}(C), y)\| : y \in \mathrm{B}_Y,\ E \supseteq C \in \Sigma\} \approx \|\mathcal{F}\|_{\mathcal{B}}(E)$.

In particular any measure has bounded $\mathcal{B}$-semivariation for any $\mathcal{B}$.

We can easily describe the $\mathcal{B}$-variation and $\mathcal{B}$-semivariation of vector measures for the bilinear maps given in (1.4)–(1.9). The following results are elementary and left to the reader.

PROPOSITION 2.4. *Let $\mathcal{F}$ be a vector measure and $E \in \Sigma$. Then:*

(a) $|\mathcal{F}|_{\mathcal{B}}(E) = |\mathcal{F}|(E)$ *and* $\|\mathcal{F}\|_{\mathcal{B}}(E) = \|\mathcal{F}\|(E)$;
(b) $|\mathcal{F}|_{\mathcal{D}}(E) = \|\mathcal{F}\|_{\mathcal{D}}(E) = \|\mathcal{F}\|(E)$;
(c) $|\mathcal{F}|_{\mathcal{D}_1}(E) = \|\mathcal{F}\|_{\mathcal{D}_1}(E) = \|\mathcal{F}\|(E)$;
(d) $|\mathcal{F}|_{\pi_Y}(E) = |\mathcal{F}|(E)$ *and* $\|\mathcal{F}\|_{\pi_Y}(E) = \|\mathcal{F}\|(E)$ *(see Proposition 2.7)*;
(e) $|\mathcal{F}|_{\tilde{\mathcal{O}}_Y}(E) = \sup\{|T\mathcal{F}|(E) : T \in \mathrm{B}_{\mathcal{L}(Y,Z)}\}$ *and* $\|\mathcal{F}\|_{\tilde{\mathcal{O}}_Y}(E) = \|\mathcal{F}\|(E)$;
(f) $|\mathcal{F}|_{\mathcal{O}_{Y,Z}}(E) = |\!|\!|\mathcal{F}|\!|\!|(E)$ *and* $\|\mathcal{F}\|_{\mathcal{O}_{Y,Z}}(E) = \|\mathcal{F}\|(E)$.

The notion of $\mathcal{B}$-normed space can be described in terms of vector measures.

PROPOSITION 2.5. *Let $\mathcal{B} : X \times Y \to Z$ be an admissible bounded bilinear map. Then X is $(Y, Z, \mathcal{B})$-normed if and only if for any vector measure $\mathcal{F} : \Sigma \to X$ there exist $C_1, C_2 > 0$ such that*

$$C_1\|\mathcal{F}\|(E) \le \|\mathcal{F}\|_{\mathcal{B}}(E) \le C_2\|\mathcal{F}\|(E)$$

for all $E \in \Sigma$.

PROOF. Obviously $\|\mathcal{F}\|_{\mathcal{B}}(E) \le \|\mathcal{B}\| \cdot \|\mathcal{F}\|(E)$ for any $E \in \Sigma$. Assume that X is $(Y, Z, \mathcal{B})$-normed. Then we have that

$$\begin{aligned}
\|\mathcal{F}\|(E) &= \sup\left\{\left\|\sum_{A\in\pi} \varepsilon_A \mathcal{F}(A)\right\|_X : \pi \in \mathcal{D}_E,\ \varepsilon_A \in \mathrm{B}_{\mathbb{K}}\right\}\\
&\le k \sup\left\{\left\|\mathcal{B}_{\sum_{A\in\pi} \varepsilon_A \mathcal{F}(A)}\right\|_{\mathcal{L}(Y,Z)} : \pi \in \mathcal{D}_E,\ \varepsilon_A \in \mathrm{B}_{\mathbb{K}}\right\}\\
&= k \sup\left\{\left\|\sum_{A\in\pi} \varepsilon_A \mathcal{B}(\mathcal{F}(A), y)\right\|_Z : \pi \in \mathcal{D}_E,\ \varepsilon_A \in \mathrm{B}_{\mathbb{K}},\ y \in \mathrm{B}_Y\right\}\\
&= k\|\mathcal{F}\|_{\mathcal{B}}(E).
\end{aligned}$$

Conversely, for each $x \in X$ select the measure $\mathcal{F}_x(A) = x\mu(A)\mu(\Omega)^{-1}$ and observe that $\|\mathcal{F}_x\|(\Omega) = \|x\|$ and $\|\mathcal{F}_x\|_{\mathcal{B}}(\Omega) = \|\mathcal{B}_x\|$. □

We use $\mathcal{B}^*$ for the 'adjoint' bilinear map from $X \times Z^*$ to Y^*, that is $(\mathcal{B}^*)_x = (\mathcal{B}_x)^*$ or

$$\mathcal{B}^* : X \times Z^* \to Y^*, \text{ given by } \langle y, \mathcal{B}^*(x, z^*)\rangle = \langle \mathcal{B}(x, y), z^*\rangle.$$

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Note that $\mathcal{B}^* = \mathcal{D}$, $\mathcal{D}_1^* = \mathcal{B}$, $(\pi_Y)^* = \widetilde{\mathcal{O}}_{Y^*}$ and $(\mathcal{O}_{Y,Z})^*(T, z^*) = \mathcal{O}_{Z^*,Y^*}(T^*, z^*)$. Let us prove that the $\mathcal{B}$-semivariation and the $\mathcal{B}^*$-semivariation always coincide.

PROPOSITION 2.6. *We have $\|\mathcal{F}\|_{\mathcal{B}}(E) = \|\mathcal{F}\|_{\mathcal{B}^*}(E)$ for all $E \in \Sigma$.*

PROOF. Let us take $E \in \Sigma$. Then

$$\begin{aligned}
\|\mathcal{F}\|_{\mathcal{B}}(E) &= \sup\left\{\sum_{A\in\pi} |\langle \mathcal{B}(\mathcal{F}(A), y), z^*\rangle| : \pi \in \mathcal{D}_E,\ y \in \mathrm{B}_Y,\ z^* \in \mathrm{B}_{Z^*}\right\}\\
&= \sup\left\{\sum_{A\in\pi} |\langle y, \mathcal{B}^*(\mathcal{F}(A), z^*)\rangle| : \pi \in \mathcal{D}_E,\ y \in \mathrm{B}_Y,\ z^* \in \mathrm{B}_{Z^*}\right\}\\
&= \sup\left\{\left|\sum_{A\in\pi} \varepsilon_A \langle y, \mathcal{B}^*(\mathcal{F}(A), z^*)\rangle\right| : \pi \in \mathcal{D}_E,\ y \in \mathrm{B}_Y,\ z^* \in \mathrm{B}_{Z^*},\ \varepsilon_A \in \mathrm{B}_{\mathbb{K}}\right\}\\
&= \sup\left\{\left\|\sum_{A\in\pi} \varepsilon_A \mathcal{B}^*(\mathcal{F}(A), z^*)\right\|_{Y^*} : \pi \in \mathcal{D}_E,\ z^* \in \mathrm{B}_{Z^*},\ \varepsilon_A \in \mathrm{B}_{\mathbb{K}}\right\}\\
&= \sup\left\{\sum_{A\in\pi} |\langle \mathcal{B}^*(\mathcal{F}(A), z^*), y^{**}\rangle| : \pi \in \mathcal{D}_E,\ y^{**} \in \mathrm{B}_{Y^{**}},\ z^* \in \mathrm{B}_{Z^*}\right\}\\
&= \|\mathcal{F}\|_{\mathcal{B}^*}(E).
\end{aligned}$$

□

PROPOSITION 2.7. *Let τ be a norm in $Y \otimes X$ with $\|y \otimes x\|_\tau = \|y\|\|x\|$ for all $y \in Y$ and $x \in X$. Define $\tau_Y : X \times Y \to Y\widehat{\otimes}_\tau X$ given by $(x, y) \to y \otimes x$. Then, for each $E \in \Sigma$,*

$$\beta_Y(\mathcal{F}, \tau)(E) \approx |\mathcal{F}|_{(\tau_Y)^*}(E).$$

PROOF. Taking into account that $Y\widehat{\otimes}_\pi X \subseteq Y\widehat{\otimes}_\tau X$, then $(Y\widehat{\otimes}_\tau X)^*$ can be regarded as a subspace of the space of bounded operators $\mathcal{L}(Y, X^*)$. Moreover $\|T\| \le \|T\|_{(Y\widehat{\otimes}_\tau X)^*}$ for any $T \in (Y\widehat{\otimes}_\tau X)^*$, where the duality is given by

$$\left\langle T, \sum_{j=1}^k y_j \otimes x_j \right\rangle = \sum_{j=1}^k \langle x_j, T(y_j)\rangle.$$

From [3, Theorem 2.1], we have

$$\beta_Y(\mathcal{F}, \tau)(E) \approx \sup\{|T\mathcal{F}|(E) : T \in \mathcal{L}(Y, X^*),\ \|T\|_{(Y\widehat{\otimes}_\tau X)^*} \le 1\}.$$

Hence,

$$\beta_Y(\mathcal{F}, \tau)(E) \approx \sup\{|(\tau_Y)^*(\mathcal{F}, T)|(E) : T \in \mathcal{L}(Y, X^*),\ \|T\|_{(Y\widehat{\otimes}_\tau X)^*} \le 1\}.$$

□

Of course vector measures need not be of bounded $\mathcal{B}$-variation for a general $\mathcal{B}$ (it suffices to take $\mathcal{B}$ such that $|\mathcal{F}|_{\mathcal{B}} = |\mathcal{F}|$), but there are cases where this happens to be true owing to the geometrical properties of the spaces under consideration.

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PROPOSITION 2.8. *Let $X = L^1(\nu)$ for some σ-finite measure ν and let $Z = H$ be a Hilbert space. Then any vector measure $\mathcal{F} : \Sigma \to L^1(\nu)$ is of bounded $\mathcal{B}$-variation for any bounded bilinear map $\mathcal{B} : L^1(\nu) \times Y \to H$ and any Banach space Y.*

PROOF. Recall first that Grothendieck theorem (see [7]) establishes that there exists a constant $\kappa_G > 0$ such that any operator from $L^1(\nu)$ to a Hilbert space H satisfies

$$\sum_{n=1}^{N} \|T(\phi_n)\|_H \leq \kappa_G \|T\| \sup\left\{ \left\| \sum_{n=1}^{N} \varepsilon_n \phi_n \right\|_{L^1(\nu)} : \varepsilon_n \in \mathcal{B}_{\mathbb{K}} \right\}$$

for any finite family of functions $(\phi_n)_n$ in $L^1(\nu)$.

If $\mathcal{F} : \Sigma \to L^1(\nu)$ is a vector measure and π a partition, one has that $\| \sum_{A \in \pi} \varepsilon_A \mathcal{F}(A)\|_{L^1(\nu)} \leq \|\mathcal{F}\|(\Omega)$. Hence, $\mathcal{B}^y \in \mathcal{L}(L1(\eta), H)$ for any $y \in Y$, so one obtains

$$\sum_{A \in \pi} \|\mathcal{B}^y(\mathcal{F}(A))\|_Z \leq \kappa_G \cdot \|\mathcal{B}^y\| \cdot \|\mathcal{F}\|(\Omega).$$

Therefore, one concludes that $|\mathcal{F}|_{\mathcal{B}}(\Omega) \leq \kappa_G \cdot \|\mathcal{F}\|(\Omega)$. □

Recall that a vector measure $\mathcal{F} : \Sigma \to X$ is called μ-continuous if $\lim_{\mu(E) \to 0} \|\mathcal{F}\|(E) = 0$.

THEOREM 2.9. *Let X be $(Y, Z, \mathcal{B})$-normed and $f \in L^1_{\mathcal{B}}(X)$. Then*

$$\mathcal{F}_f^{\mathcal{B}} : \Sigma \to X, \text{ given by } \mathcal{F}_f^{\mathcal{B}}(E) = \int_E^{\mathcal{B}} f \, d\mu \tag{2.1}$$

is a μ-continuous vector measure of bounded $\mathcal{B}$-variation. Moreover, $|\mathcal{F}_f^{\mathcal{B}}|_{\mathcal{B}}(\Omega) = \|f\|_{L^1_{\mathcal{B}}(X)}$.

PROOF. It was shown (see [4, Theorem 1]) that functions in $L^1_{\mathcal{B}}(X)$ are Pettis integrable and $\int_E^{\mathcal{B}} f \, d\mu$ coincides with the Pettis integral. Hence, $\mathcal{F}_f^{\mathcal{B}}$ defines a vector measure.

Using now that, for each $y \in Y$, the vector measure $\mathcal{B}(\mathcal{F}_f^{\mathcal{B}}, y)$ has density $\mathcal{B}(f, y)$ that belongs to $L^1(Z)$, one obtains that, for any $E \in \Sigma$,

$$|\mathcal{B}(\mathcal{F}_f, y)|(E) = \int_E \|\mathcal{B}(f, y)\|_Z \, d\mu.$$

Thus, $|\mathcal{F}_f^{\mathcal{B}}|_{\mathcal{B}}(\Omega) = \|f\|_{L^1_{\mathcal{B}}(X)}$. It remains to show that $\mathcal{F}_f^{\mathcal{B}}$ is μ-continuous. Let us fix $\varepsilon > 0$ and select, using that $f \in L^1_{\mathcal{B}}(X)$, a simple function s such that $\|f - s\|_{L^1_{\mathcal{B}}(X)} \leq \varepsilon$. Thus,

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$$
\begin{aligned}
\|\mathcal{F}_f^{\mathcal{B}}(E)\|_X &\le \left\| \int_E^{\mathcal{B}} (f-s)\,d\mu \right\|_X + \left\| \int_E^{\mathcal{B}} s\,d\mu \right\|_X \\
&= \left\| \int_E^{\mathcal{B}} (f-s)\,d\mu \right\|_X + \left\| \int_E s\,d\mu \right\|_X \\
&\le k \|\mathcal{B}_{\int_E^{\mathcal{B}}(f-s)\,d\mu}\|_{\mathcal{L}(Y,Z)} + \left\| \int_E s\,d\mu \right\|_X \\
&\le k \sup\left\{ \int_E \|\mathcal{B}(f-s,y)\|_Z\,d\mu : y \in \mathrm{B}_Y \right\} + \left\| \int_E s\,d\mu \right\|_X \\
&\le k\varepsilon + \left\| \int_E s\,d\mu \right\|_X .
\end{aligned}
$$

We have the conclusion just taking limits when $\mu(E) \to 0$ and $\varepsilon \to 0^+$. □

COROLLARY 2.10. *Let X is $(Y, Z, \mathcal{B})$-normed and $f \in L^1_{\mathcal{B}}(X)$. If $\int_E^{\mathcal{B}} f\,d\mu = 0$ for all $E \in \Sigma$, then $f = 0$ almost everywhere in Ω.*

3. Measures of bounded $(p, \mathcal{B})$-semivariation.

Extending the notion for $\mathcal{B} = \mathcal{B}$, we say that a vector measure $\mathcal{F} : \Sigma \to X$ is $(\mathcal{B}, \mu)$-continuous if $\lim_{\mu(E)\to 0} \|\mathcal{F}\|_{\mathcal{B}}(E) = 0$. Clearly both concepts coincide for $\mathcal{B}$-normed spaces.

DEFINITION 3.1. We say that $\mathcal{F}$ has *bounded* $(\infty, \mathcal{B})$-*semivariation* if there exists $C > 0$ such that

$$
|\langle \mathcal{B}(\mathcal{F}(A), y), z^*\rangle| \le C \cdot \|y\| \cdot \|z^*\| \cdot \mu(A), \qquad y \in Y,\ z^* \in Z^*,\ A \in \Sigma. \tag{3.1}
$$

The space of such measures is denoted by $\mathcal{V}^\infty_{\mathcal{B}}(X)$ and we set

$$
\begin{aligned}
\|\mathcal{F}\|_{\mathcal{V}^\infty_{\mathcal{B}}(X)} &= \inf\{C : \text{ satisfying } (3.1)\} \\
&= \sup\left\{ \frac{|\langle \mathcal{B}(\mathcal{F}(A), y), z^*\rangle|}{\mu(A)} : y \in \mathrm{B}_Y,\ z^* \in \mathrm{B}_{Z^*},\ A \in \Sigma,\ \mu(A) > 0 \right\}.
\end{aligned}
$$

Observe that every vector measure $\mathcal{F}$ belonging to $\mathcal{V}^\infty_{\mathcal{B}}(X)$ is $(\mathcal{B}, \mu)$-continuous and it has bounded $\mathcal{B}$-variation. Also note that $\mathcal{F}$ has bounded $(\infty, \mathcal{B})$-semivariation if and only if one of the following holds:

$$
\begin{aligned}
\|\mathcal{B}(\mathcal{F}(A), y)\| &\le C\|y\|\mu(A), && y \in Y,\ A \in \Sigma, \\
\|\mathcal{F}\|_{\mathcal{B}}(A) &\le C\mu(A), && A \in \Sigma
\end{aligned}
$$

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or

$$|\mathcal{F}|_{\mathcal{B}}(A) \leq C\mu(A), \qquad A \in \Sigma.$$

It is elementary to see, owing to the admissibility of $\mathcal{B}$, that $\|\mathcal{F}\|_{\mathcal{V}^\infty_{\mathcal{B}}(X)}$ is a norm.

Of course,

$$\begin{aligned} \|\mathcal{F}\|_{\mathcal{V}^\infty_{\mathcal{B}}(X)} &= \sup\{\|\mathcal{B}(\mathcal{F}, y)\|_{\mathrm{V}^\infty(Z)} : y \in \mathrm{B}_Y\} \\ &= \sup\left\{\frac{\|\mathcal{B}(\mathcal{F}(A), y)\|_Z}{\mu(A)} : y \in \mathrm{B}_Y,\ A \in \Sigma\right\} \\ &= \sup\left\{\frac{\|\mathcal{F}\|_{\mathcal{B}}(A)}{\mu(A)} : A \in \Sigma\right\} \\ &= \sup\left\{\frac{|\mathcal{F}|_{\mathcal{B}}(A)}{\mu(A)} : A \in \Sigma\right\}. \end{aligned}$$

PROPOSITION 3.2. *We have $\mathcal{F} \in \mathcal{V}^\infty_{\mathcal{B}}(X)$ if and only if there exists a bounded bilinear map $\mathcal{B}_{\mathcal{F}}: L^1 \times Y \to Z$ such that*

$$\mathcal{B}_{\mathcal{F}}(\mathbf{1}_A, y) = \mathcal{B}(\mathcal{F}(A), y), \qquad A \in \Sigma,\ y \in Y.$$

Moreover, $\|\mathcal{B}_{\mathcal{F}}\| = \|\mathcal{F}\|_{\mathcal{V}^\infty_{\mathcal{B}}(X)}$.

PROOF. Assume that $\mathcal{F} \in \mathcal{V}^\infty_{\mathcal{B}}(X)$. Define $\mathcal{B}_{\mathcal{F}}$ on simple functions by the formula

$$\mathcal{B}_{\mathcal{F}}\left(\sum_{k=1}^{n} \alpha_k \mathbf{1}_{A_k}, y\right) = \sum_{k=1}^{n} \alpha_k \mathcal{B}(\mathcal{F}(A_k), y).$$

Observe that

$$\left\|\mathcal{B}_{\mathcal{F}}\left(\sum_{k=1}^{n} \alpha_k \mathbf{1}_{A_k}, y\right)\right\|_Z \leq \|\mathcal{F}\|_{\mathcal{V}^\infty_{\mathcal{B}}(X)} \|y\| \sum_{k=1}^{n} |\alpha_k| \mu(A_k).$$

This allows us to extend the bilinear map to $L^1 \times Y \to Z$ with norm $\|\mathcal{B}_{\mathcal{F}}\| \leq \|\mathcal{F}\|_{\mathcal{V}^\infty_{\mathcal{B}}(X)}$. Conversely, one has

$$\|\mathcal{B}(\mathcal{F}(A), y)\|_Z \leq \|\mathcal{B}_{\mathcal{F}}\| \cdot \|y\| \cdot \|\mathbf{1}_A\|_{L^1},$$

which gives $\|\mathcal{F}\|_{\mathcal{V}^\infty_{\mathcal{B}}(X)} \leq \|\mathcal{B}_{\mathcal{F}}\|$. □

We use the notation $\mathrm{Bil}(L^1 \times Y, Z)$ for the space of bounded bilinear maps with its natural norm.

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COROLLARY 3.3. *We have that $\mathcal{V}^\infty_{\mathcal{B}}(X)$ is isometrically embedded in* $\mathrm{Bil}(L^1 \times Y, Z)$. *In the case $\mathcal{B} = \mathcal{O}_{Y,Z}$ we have*

$$\mathcal{V}^\infty_{\mathcal{O}_{Y,Z}}(\mathcal{L}(Y, Z)) = \mathrm{Bil}(L^1 \times Y, Z).$$

Let $L^\infty_{\mathcal{B}}(X)$ stand for the space of measurable functions $f : \Omega \to X$ such that $\mathcal{B}(f, y) \in L^\infty(Z)$ for all $y \in Y$ and write

$$\|f\|_{L^\infty_{\mathcal{B}}(X)} = \sup\{\|\mathcal{B}(f, y)\|_{L^\infty(Z)} : y \in \mathrm{B}_Y\}.$$

Note that $L^\infty_{\mathcal{B}}(X) \subseteq L^1_{\mathcal{B}}(X)$ and $|\mathcal{B}(\mathcal{F}^{\mathcal{B}}_f, y)|(A) = \int^{\mathcal{B}}_A \|\mathcal{B}(f, y)\| d\mu$ for any set $A \in \Sigma$. In particular, if $f \in L^\infty_{\mathcal{B}}(X)$, then the measure $\mathcal{F}^{\mathcal{B}}_f \in \mathcal{V}^\infty_{\mathcal{B}}(X)$ and $\|\mathcal{F}^{\mathcal{B}}_f\|_{\mathcal{V}^\infty_{\mathcal{B}}(X)} = \|f\|_{L^\infty_{\mathcal{B}}(X)}$.

PROPOSITION 3.4. *The following are equivalent:*

(a) *X is $(Y, Z, \mathcal{B})$-normed;*
(b) *$\mathcal{V}^\infty_{\mathcal{B}}(X) = V^\infty(X)$;*
(c) *there exists $k > 0$ such that $\|\mathcal{F}^{\mathcal{B}}_f\|_{\mathrm{V}^\infty(X)} \leq k\|f\|_{L^\infty_{\mathcal{B}}(X)}$ for any $f \in L^\infty_{\mathcal{B}}(X)$.*

PROOF. (a) $\Longrightarrow$ (b) We always have $V^\infty(X) \subseteq \mathcal{V}^\infty_{\mathcal{B}}(X)$. Assume that X is $(Y, Z, \mathcal{B})$-normed and $\mathcal{F} \in \mathcal{V}^\infty_{\mathcal{B}}(X)$. Note that

$$\|\mathcal{F}(A)\| \leq k\|\mathcal{B}_{\mathcal{F}(A)}\| \leq k\|\mathcal{F}\|_{\mathcal{V}^\infty_{\mathcal{B}}(X)}\mu(A).$$

(b) $\Longrightarrow$ (c) Let $f \in L^\infty_{\mathcal{B}}(X)$. Clearly,

$$\|\mathcal{F}^{\mathcal{B}}_f\|_{\mathrm{V}^\infty(X)} \leq k\|\mathcal{F}^{\mathcal{B}}_f\|_{\mathcal{V}^\infty_{\mathcal{B}}(X)} = k\|f\|_{L^\infty_{\mathcal{B}}(X)}.$$

(c) $\Longrightarrow$ (a) Let us take $f_x = x\mathbf{1}_\Omega$ for a given $x \in X$ and observe that $\mathcal{F}^{\mathcal{B}}_{f_x}(A) = x\mu(A)$ for all $A \in \Sigma$. Note that $\|f_x\|_{L^\infty_{\mathcal{B}}(X)} = \|\mathcal{B}_x\|$ and $\|\mathcal{F}^{\mathcal{B}}_{f_x}\|_{\mathrm{V}^\infty(X)} = \|x\|$. □

DEFINITION 3.5. Let $1 \leq p < \infty$. We say that $\mathcal{F}$ has bounded $(p, \mathcal{B})$-semivariation if

$$\|\mathcal{F}\|_{\mathcal{V}^p_{\mathcal{B}}(X)} = \sup\left\{\left(\sum_{A\in\pi} \frac{|\langle\mathcal{B}(\mathcal{F}(A), y), z^*\rangle|^p}{\mu(A)^{p-1}}\right)^{1/p} : y \in \mathrm{B}_Y,\right.$$
$$\left. z^* \in \mathrm{B}_{Z^*}, \pi \in \mathcal{D}_\Omega\right\} < \infty.$$

The space of such measures is denoted by $\mathcal{V}^p_{\mathcal{B}}(X)$.

We have the equivalent formulation

$$\begin{aligned}\|\mathcal{F}\|_{\mathcal{V}^p_{\mathcal{B}}(X)} &= \sup\{\|\mathcal{B}(\mathcal{F}, y)\|_{\mathcal{V}^p(Z)} : y \in \mathrm{B}_Y\}\\ &= \sup\{\|\langle\mathcal{B}(\mathcal{F}, y), z^*\rangle\|_{\mathcal{V}^p} : y \in \mathrm{B}_Y, z^* \in \mathrm{B}_{Z^*}\}.\end{aligned}$$

Let us start with the following description.

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PROPOSITION 3.6. *Let* $1 < p < \infty$. *Then* $\mathcal{F} \in \mathcal{V}^p_{\mathcal{B}}(X)$ *if and only if there exists a bounded bilinear map* $\mathcal{B}_{\mathcal{F}} : L^{p'} \times Y \to Z$ *such that*

$$\mathcal{B}_{\mathcal{F}}(\mathbf{1}_A, y) = \mathcal{B}(\mathcal{F}(A), y), \quad A \in \Sigma, \ y \in Y.$$

Moreover, $\|\mathcal{B}_{\mathcal{F}}\| = \|\mathcal{F}\|_{\mathcal{V}^p_{\mathcal{B}}(X)}$.

PROOF. Assume that $\mathcal{F} \in \mathcal{V}^p_{\mathcal{B}}(X)$. As above define $\mathcal{B}_{\mathcal{F}}$ on simple functions by the formula

$$\mathcal{B}_{\mathcal{F}}\Big(\sum_{k=1}^{n} \alpha_k \mathbf{1}_{A_k}, y\Big) = \sum_{k=1}^{n} \alpha_k \mathcal{B}(\mathcal{F}(A_k), y).$$

We use that

$$\begin{aligned}
\|\mathcal{F}\|_{\mathcal{V}^p_{\mathcal{B}}(X)} &= \sup\bigg\{\bigg|\sum_{A\in\pi} \frac{\langle \mathcal{B}(\mathcal{F}(A), y), z^*\rangle \gamma_A}{\mu(A)^{1/p'}}\bigg| : y \in \mathrm{B}_Y, z^* \in \mathrm{B}_{Z^*}, \\
&\qquad \pi \in \mathcal{D}_\Omega, \ (\gamma_A)_A \in \mathrm{B}_{\ell^{p'}}\bigg\} \\
&= \sup\bigg\{\bigg\|\sum_{A\in\pi} \mathcal{B}(\mathcal{F}(A), y)\beta_A\bigg\|_Z : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, \sum_{A\in\pi} \beta_A \mathbf{1}_A \in \mathrm{B}_{L^{p'}}\bigg\} \\
&= \sup\bigg\{\bigg\|\mathcal{B}_{\mathcal{F}}\Big(\sum_{A\in\pi} \beta_A \mathbf{1}_A, y\Big)\bigg\|_Z : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, \sum_{A\in\pi} \beta_A \mathbf{1}_A \in \mathrm{B}_{L^{p'}}\bigg\}.
\end{aligned}$$

Hence, using the density of simple functions we extend to $L^{p'}$ and $\|\mathcal{B}_{\mathcal{F}}\| \le \|\mathcal{F}\|_{\mathcal{V}^p_{\mathcal{B}}(X)}$. The converse also follows from the previous formula. □

It is known that $\mathcal{V}^p(X) = \mathcal{L}(L^{p'}, X)$ (see [12]). The next result is the analogue in the bilinear setting.

COROLLARY 3.7. *Let* $1 < p < \infty$. *Then* $\mathcal{V}^p_{\mathcal{B}}(X)$ *is isometrically embedded in* $\mathrm{Bil}(L^{p'} \times Y, Z)$. *In the case* $\mathcal{B} = \mathcal{O}_{Y,Z}$ *we have*

$$\mathcal{V}^p_{\mathcal{O}_{Y,Z}}(\mathcal{L}(Y, Z)) = \mathrm{Bil}(L^{p'} \times Y, Z).$$

PROPOSITION 3.8. *Let* $\mathcal{B} : X \times Y \to Z$ *be an admissible bounded bilinear map and* $1 < p < \infty$. *Then* X *is* $(Y, Z, \mathcal{B})$*-normed if and only if the space* $\mathcal{V}^p_{\mathcal{B}}(X)$ *is continuously contained into* $\mathcal{V}^p(X)$.

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PROOF. Assume that X is $(Y, Z, \mathcal{B})$-normed. Then

$$\begin{aligned}\|\mathcal{F}\|_{\mathcal{V}^p(X)} &= \sup\bigg\{\bigg|\sum_{A\in\pi}\frac{\langle\mathcal{F}(A), x^*\rangle\gamma_A}{\mu(A)^{1/p'}}\bigg| : x^*\in \mathrm{B}_{X^*}, \pi\in\mathcal{D}_\Omega,\ (\gamma_A)_A\in \mathrm{B}_{\ell^{p'}}\bigg\}\\ &= \sup\bigg\{\bigg\|\sum_{A\in\pi}\frac{\mathcal{F}(A)\gamma_A}{\mu(A)^{1/p'}}\bigg\|_X : \pi\in\mathcal{D}_\Omega, (\gamma_A)_A\in \mathrm{B}_{\ell^{p'}}\bigg\}\\ &\le k\sup\bigg\{\Big\|\mathcal{B}_{\sum\frac{\mathcal{F}(A)\gamma_A}{\mu(A)^{1/p'}}}\Big\|_{\mathcal{L}(Y,Z)} : \pi\in\mathcal{D}_\Omega,\ (\gamma_A)_A\in \mathrm{B}_{\ell^{p'}}\bigg\}\\ &= k\sup\bigg\{\bigg\|\sum_{A\in\pi}\mathcal{B}\bigg(\frac{\mathcal{F}(A)\gamma_A}{\mu(A)^{1/p'}}, y\bigg)\bigg\|_Z : y\in \mathrm{B}_Y, \pi\in\mathcal{D}_\Omega, (\gamma_A)_A\in \mathrm{B}_{\ell^{p'}}\bigg\}\\ &= k\sup\bigg\{\bigg(\sum_{A\in\pi}\frac{\|\mathcal{B}(\mathcal{F}(A), y)\|^p}{\mu(A)^{p-1}}\bigg)^{1/p} : y\in \mathrm{B}_Y, \pi\in\mathcal{D}_\Omega,\bigg\}\\ &= k\|\mathcal{F}\|_{\mathcal{V}^p_{\mathcal{B}}(X)}.\end{aligned}$$

For the converse consider the vector measure $\mathcal{F}_x : \Sigma \to X$ given by $\mathcal{F}_x(A) = x\mu(A)\mu(\Omega)^{-1}$ for each $x \in X$. Note that $\|\mathcal{F}_x\|_{\mathcal{V}^p(X)} = \|x\|$ and $\|\mathcal{F}_x\|_{\mathcal{V}^p_{\mathcal{B}}(X)} = \|\mathcal{B}_x\|$. □

4. Measures of bounded $(p, \mathcal{B})$-variation

DEFINITION 4.1. We say that $\mathcal{F}$ has *bounded* $(p, \mathcal{B})$*-variation* if

$$\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} = \sup\bigg\{\bigg(\sum_{A\in\pi}\frac{\|\mathcal{B}(\mathcal{F}(A), y)\|_Z^p}{\mu(A)^{p-1}}\bigg)^{1/p} : y\in \mathrm{B}_Y, \pi\in\mathcal{D}_\Omega\bigg\} < \infty.$$

The space of such measures is denoted by $\mathrm{V}^p_{\mathcal{B}}(X)$.

It is clear that the norm in the vector space $\mathrm{V}^p_{\mathcal{B}}(X)$ is also given by the expressions

$$\begin{aligned}\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} &= \sup\{\|\mathcal{B}(\mathcal{F}, y)\|_{\mathrm{V}^p(Z)} : y\in \mathrm{B}_Y\}\\ &= \sup\bigg\{\bigg\|\sum_{A\in\pi}\frac{\mathcal{B}(\mathcal{F}(A), y)}{\mu(A)}\mathbf{1}_A\bigg\|_{L^p(Z)} : y\in \mathrm{B}_Y, \pi\in\mathcal{D}_\Omega\bigg\}\\ &= \sup\bigg\{\bigg\|\sum_{A\in\pi}\frac{\mathcal{F}(A)}{\mu(A)}\mathbf{1}_A\bigg\|_{L^p_{\mathcal{B}}(X)} : \pi\in\mathcal{D}_\Omega\bigg\}.\end{aligned}$$

REMARK 4.2. For $p = 1$ and $p = \infty$ this corresponds to $|\mathcal{F}|_{\mathcal{B}}(\Omega)$ and $\|\mathcal{F}\|_{\mathcal{V}^\infty_{\mathcal{B}}(X)}$. Hence, we define $\mathrm{V}^\infty(X) = \mathcal{V}^\infty(X)$.

It is clear that $\mathrm{V}^p_{\mathcal{B}}(X) \subseteq \mathcal{V}^p_{\mathcal{B}}(X)$ and $\|\mathcal{F}\|_{\mathcal{V}^p_{\mathcal{B}}(X)} \le \|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)}$.

On the other hand, since

$$|\mathcal{F}|_{\mathcal{B}}(E) \le \|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)}\|\mathbf{1}_E\|_{L^{p'}}, \quad E\in\Sigma,$$

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one sees that if $\mathcal{F} \in \mathrm{V}^p_{\mathcal{B}}(X)$ then $\mathcal{F}$ has bounded $\mathcal{B}$-variation and it is $(\mathcal{B}, \mu)$-continuous.

REMARK 4.3. Using the inclusions $L^q(X) \subseteq L^p(X)$ for $1 \le p \le q \le \infty$ one also has

$$\mathrm{V}^\infty_{\mathcal{B}}(X) \subseteq \mathrm{V}^q_{\mathcal{B}}(X) \subseteq \mathrm{V}^p_{\mathcal{B}}(X)$$

and

$$\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} \le \mu(\Omega)^{1/q-1/p} \|\mathcal{F}\|_{\mathrm{V}^q_{\mathcal{B}}(X)} \le \mu(\Omega)^{1/q} \|\mathcal{F}\|_{\mathrm{V}^\infty_{\mathcal{B}}(X)}.$$

Let us find different equivalent formulations for the norm in $\mathrm{V}^p_{\mathcal{B}}(X)$.

PROPOSITION 4.4. *We have*

$$\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} = \sup\Big\{ \sum_{A\in\pi} \|\mathcal{B}(\mathcal{F}(A), \beta_A y)\|_Z : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, \sum_{A\in\pi} \beta_A \mathbf{1}_A \in \mathrm{B}_{L^{p'}} \Big\}. \tag{4.1}$$

$$\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} = \sup\Big\{ \Big\| \sum_{A\in\pi} \mathcal{B}^*(\mathcal{F}(A), z^*_A) \Big\|_{Y^*} : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, \sum_{A\in\pi} z^*_A \mathbf{1}_A \in \mathrm{B}_{L^{p'}(Z^*)} \Big\}. \tag{4.2}$$

PROOF. Given a partition $\pi \in \mathcal{D}_\Omega$, $\alpha_A \in \mathbb{R}$ and $\beta_A = (\alpha_A/\mu(A)^{1/p'})$ one has that the simple function $g = \sum_{A\in\pi} \beta_A \mathbf{1}_A$ satisfies $\|g\|_{L^{p'}} = \|(\alpha_A)_{A\in\pi}\|_{\ell^{p'}}$. Therefore,

$$\begin{aligned}
\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} &= \sup\Big\{ \Big\| \Big(\Big\| \mathcal{B}\Big(\frac{\mathcal{F}(A)}{\mu(A)^{1/p'}}, y \Big) \Big\|_Z \Big)_{A\in\pi} \Big\|_{\ell^p} : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega \Big\} \\
&= \sup\Big\{ \sum_{A\in\pi} \Big\| \mathcal{B}\Big(\frac{\mathcal{F}(A)}{\mu(A)^{1/p'}}, y \Big) \Big\|_Z |\alpha_A| : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, (\alpha_A)_{A\in\pi} \in \mathrm{B}_{\ell^{p'}} \Big\} \\
&= \sup\Big\{ \sum_{A\in\pi} \Big\| \mathcal{B}\Big(\mathcal{F}(A) \frac{\alpha_A}{\mu(A)^{1/p'}}, y \Big) \Big\|_Z : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, (\alpha_A)_{A\in\pi} \in \mathrm{B}_{\ell^{p'}} \Big\} \\
&= \sup\Big\{ \sum_{A\in\pi} \Big\| \mathcal{B}(\mathcal{F}(A), \beta_A y) \Big\|_Z : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, \sum_{A\in\pi} \beta_A \mathbf{1}_A \in \mathrm{B}_{L^{p'}} \Big\}.
\end{aligned}$$

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We obtain (4.2) from the duality $(\ell^1(Z))^* = \ell^\infty(Z^*)$ and (4.1). Indeed,

$$\begin{aligned}
\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} &= \sup\bigg\{\sum_{A\in\pi} \|\mathcal{B}(\mathcal{F}(A), \beta_A y)\|_Z : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, \\
&\qquad \sum_{A\in\pi} \beta_A \mathbf{1}_A \in \mathrm{B}_{L^{p'}}\bigg\} \\
&= \sup\bigg\{\bigg|\sum_{A\in\pi} \langle \mathcal{B}(\mathcal{F}(A), \beta_A y), z_A^*\rangle\bigg| : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, z_A^* \in \mathrm{B}_{Z^*}, \\
&\qquad \sum_{A\in\pi} \beta_A \mathbf{1}_A \in \mathrm{B}_{L^{p'}}\bigg\} \\
&= \sup\bigg\{\bigg|\sum_{A\in\pi} \langle y, \mathcal{B}^*(\mathcal{F}(A), \beta_A z_A^*)\rangle\bigg| : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, z_A^* \in \mathrm{B}_{Z^*}, \\
&\qquad \sum_{A\in\pi} \beta_A \mathbf{1}_A \in \mathrm{B}_{L^{p'}}\bigg\} \\
&= \sup\bigg\{\bigg\|\sum_{A\in\pi} \mathcal{B}^*(\mathcal{F}(A), z_A^*)\bigg\|_{Y^*} : \pi \in \mathcal{D}_\Omega, \sum_{A\in\pi} z_A^* \mathbf{1}_A \in \mathrm{B}_{L^{p'}(Z^*)}\bigg\}. \quad \square
\end{aligned}$$

Let us give a characterization of the vector measures in the space $\mathrm{V}^p_{\mathcal{B}}(X)$ using only scalar-valued functions $\{\varphi_y \mid y \in \mathrm{B}_Y\} \subseteq L^p$.

THEOREM 4.5. *We have $\mathcal{F} \in \mathrm{V}^p_{\mathcal{B}}(X)$ if and only if there exist $0 \le \varphi_y \in L^p$ for each $y \in Y$ such that:*

(a) $\sup\{\|\varphi_y\|_{L^p} : y \in \mathrm{B}_Y\} < \infty$; *and*
(b) $\|\mathcal{B}(\mathcal{F}(E), y)\| \le \int_E \varphi_y \, d\mu$ *for every $y \in Y$ and $E \in \Sigma$.*

Moreover, $\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} = \sup\{\|\varphi_y\|_{L^p} : y \in \mathrm{B}_Y\}$.

PROOF. Let $\mathcal{F} \in \mathrm{V}^p_{\mathcal{B}}(X)$. Then we have that $\mathcal{B}(\mathcal{F}, y) \in \mathrm{V}^p(Z)$ for all $y \in \mathrm{B}_Y$ and $|\mathcal{B}(\mathcal{F}, y)|$ is a nonnegative μ-continuous measure that has bounded variation. Using the Radon–Nikodým theorem there exists a nonnegative integrable function φ_y such that for all $E \in \Sigma$

$$|\mathcal{B}(\mathcal{F}, y)|(E) = \int_E \varphi_y \, d\mu. \tag{4.3}$$

In fact φ_y can be chosen belonging to L^p and verifying that

$$\|\varphi_y\|_{L^p} = \|\mathcal{B}(\mathcal{F}, y)\|_{\mathrm{V}^p(Z)}.$$

Then, for every $E \in \Sigma$ and $y \in \mathrm{B}_Y$,

$$\begin{aligned}
\|\mathcal{B}(\mathcal{F}(E), y)\| &\le |\mathcal{F}|_{\mathcal{B}}(E) = \sup\{|\mathcal{B}(\mathcal{F}, y)|(E) : y \in \mathrm{B}_Y\} \\
&= \sup\bigg\{\int_E \varphi_y \, d\mu : y \in \mathrm{B}_Y\bigg\}
\end{aligned}$$

and we obtain the result.

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Conversely observe that using Hölder's inequality we have that

$$\|\mathcal{B}(\mathcal{F}(E), y)\| \leq \int_E \varphi_y \, d\mu \leq \left(\int_E \varphi_y^p \, d\mu\right)^{1/p} \mu(E)^{1/p'}$$

for all $E \in \Sigma$ and $y \in \mathrm{B}_Y$. Hence, for every $\pi \in \mathcal{D}_\Omega$

$$\sum_{A \in \pi} \frac{\|\mathcal{B}(\mathcal{F}(A), y)\|^p}{\mu(A)^{p-1}} \leq \int_\Omega \varphi_y^p \, d\mu.$$

This shows that $\mathcal{F} \in \mathrm{V}^p_{\mathcal{B}}(X)$ and $\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} \leq \sup\{\|\varphi_y\|_{L^p} : y \in \mathrm{B}_Y\}$. $\square$

Let us now see the analogue to Theorem 2.9 in the cases $1 < p < \infty$.

THEOREM 4.6. *Assume X is $(Y, Z, \mathcal{B})$-normed and $1 < p < \infty$. If $f \in L^p_{\mathcal{B}}(X)$ then $\mathcal{F}^{\mathcal{B}}_f \in \mathrm{V}^p_{\mathcal{B}}(X)$ and $\|\mathcal{F}^{\mathcal{B}}_f\|_{\mathrm{V}^p_{\mathcal{B}}(X)} = \|f\|_{L^p_{\mathcal{B}}(X)}$.*

PROOF. Let us take $f \in L^p_{\mathcal{B}}(X)$. From Theorem 2.9 one know that $\mathcal{F}^{\mathcal{B}}_f : \Sigma \to X$ is a vector measure of bounded variation. Now, for each $y \in Y$, $\mathcal{B}^y\mathcal{F}^{\mathcal{B}}_f : \Sigma \to Z$ is a vector measure verifying that

$$\mathcal{B}^y\mathcal{F}^{\mathcal{B}}_f(E) = \mathcal{B}(\mathcal{F}^{\mathcal{B}}_f(E), y) = \mathcal{B}\left(\int_E^{\mathcal{B}} f \, d\mu, y\right) = \int_E \mathcal{B}(f, y) \, d\mu, \quad E \in \Sigma.$$

Therefore

$$\begin{aligned}\|f\|_{L^p_{\mathcal{B}}(X)} &= \sup\{\|\mathcal{B}(f, y)\|_{L^p(Z)} : y \in \mathrm{B}_Y\} = \sup\{\|\mathcal{B}(\mathcal{F}^{\mathcal{B}}_f, y)\|_{\mathrm{V}^p(Z)} : y \in \mathrm{B}_Y\} \\ &= \|\mathcal{F}^{\mathcal{B}}_f\|_{\mathrm{V}^p_{\mathcal{B}}(X)}.\end{aligned}$$ $\square$

COROLLARY 4.7. *If X is $(Y, Z, \mathcal{B})$-normed, then $L^p_{\mathcal{B}}(X)$ is isometrically contained in $\mathrm{V}^p_{\mathcal{B}}(X)$.*

From the definition one clearly has the following interpretations of $\mathrm{V}^p_{\mathcal{B}}(X)$ as operators:

> $\mathrm{V}^p_{\mathcal{B}}(X)$ *is isometrically embedded in* $\mathcal{L}(Y, \mathrm{V}^p(Z))$ *by composition, that is* $\mathcal{F} \to \Phi_{\mathcal{F}}(y) = \mathcal{B}^y\mathcal{F}$.

Let us see other processes that generate operators from vector measures: given a vector measure $\mathcal{F} : \Sigma \to X$ and a bounded bilinear map $\mathcal{B} : X \times Y \to Z$ we can consider the operators $T^{\mathcal{B}}_{\mathcal{F}}$ or $S^{\mathcal{B}}_{\mathcal{F}}$ defined on Y-valued simple functions $s = \sum_{k=1}^n y_k \mathbf{1}_{A_k}$ or Z^*-valued simple functions $t = \sum_{k=1}^n z_k^* \mathbf{1}_{A_k}$, respectively, by

$$T^{\mathcal{B}}_{\mathcal{F}}(s) = \sum_{k=1}^n \mathcal{B}(\mathcal{F}(A_k), y_k)$$

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and

$$S_{\mathcal{F}}^{\mathcal{B}}(t) = \sum_{k=1}^{n} \mathcal{B}^*(\mathcal{F}(A_k), z_k^*).$$

Observe that actually $S_{\mathcal{F}}^{\mathcal{B}} = T_{\mathcal{F}}^{\mathcal{B}^*}$.

THEOREM 4.8. *Let* $1 < p < \infty$. *Then* $\mathrm{V}_{\mathcal{B}}^{p}(X)$ *is continuously contained into* $\mathcal{L}(L^{p'} \widehat{\otimes} Y, Z)$.

PROOF. Let $\mathcal{F} \in \mathrm{V}_{\mathcal{B}}^{p}(X)$. Consider the linear operator $T_{\mathcal{F}}^{\mathcal{B}}$ defined on Y-valued simple functions and with values in Z. Note that for any partition π, $\phi = \sum_{A \in \pi} \alpha_A \mathbf{1}_A$ and $y \in Y$

$$\|T_{\mathcal{F}}^{\mathcal{B}}(\phi \otimes y)\|_Z \leq \sum_{A \in \pi} \|\mathcal{B}(\mathcal{F}(A), \alpha_A y)\|_Z.$$

Using (4.1) and the definition of projective tensor product one obtains $\|T_{\mathcal{F}}^{\mathcal{B}}\| \leq \|\mathcal{F}\|_{\mathrm{V}_{\mathcal{B}}^{p}(X)}$. □

THEOREM 4.9. *Let* $1 < p < \infty$. *Then* $\mathrm{V}_{\mathcal{B}}^{p}(X)$ *is isometrically embedded into* $\mathcal{L}(L^{p'}(Z^*), Y^*)$.

PROOF. Let $\mathcal{F} \in \mathrm{V}_{\mathcal{B}}^{p}(X)$. Consider the linear operator $S_{\mathcal{F}}^{\mathcal{B}}$ from the space of Z^*-valued simple functions into Y^*. Note that for any partition π

$$\left\| S_{\mathcal{F}}^{\mathcal{B}}\left(\sum_{A \in \pi} z_A^* \mathbf{1}_A \right) \right\|_{Y^*} = \left\| \sum_{A \in \pi} \mathcal{B}^*(\mathcal{F}(A), z_A^*) \right\|_{Y^*}.$$

Using (4.2) and the density of simple functions in $L^{p'}(Z^*)$ one obtains $\|S_{\mathcal{F}}^{\mathcal{B}}\| = \|\mathcal{F}\|_{\mathrm{V}_{\mathcal{B}}^{p}(X)}$. □

Note that $\mathrm{V}_{\mathcal{B}}^{p}(X) \subseteq \mathcal{V}_{\mathcal{B}}^{p}(X)$ and, from Corollary 3.7, $\mathcal{V}_{\mathcal{B}}^{p}(X)$ is embedded in $\mathrm{Bil}(L^{p'} \times Y, Z)$. Hence, $\mathrm{V}_{\mathcal{B}}^{p}(X)$ is continuously contained in $\mathrm{Bil}(L^{p'} \otimes Y, Z)$ by means of the mapping $\mathcal{F} \to \mathcal{B}_{\mathcal{F}} : L^{p'} \times Y \to Z$ given by

$$\mathcal{B}_{\mathcal{F}}(s, y) = \sum_{k=1}^{n} \mathcal{B}(\mathcal{F}(A_k), \alpha_k y)$$

where $s = \sum_{k=1}^{n} \alpha_k \mathbf{1}_{A_k}$. Let us find which special class of bilinear maps represents elements in $\mathrm{V}_{\mathcal{B}}^{p}(X)$.

In the case of $Y = \mathbb{K}$ the corresponding operators would correspond to the class of cone absolutely summing operators.

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DEFINITION 4.10. Let L be a Banach lattice, Y and Z be Banach spaces and $\mathcal{U} : L \times Y \to Z$ be a bounded bilinear map. We say that $\mathcal{U}$ is *cone absolutely summing* if there exists $C > 0$ such that

$$\sup\left\{\sum_{n=1}^{N} \|\mathcal{U}(\varphi_n, y)\|_Z : y \in \mathrm{B}_Y\right\} \leq C \sup\left\{\sum_{n=1}^{N} |\langle \varphi_n, \psi\rangle| : \psi \in \mathrm{B}_{L^*}\right\}$$

for any finite family $(\varphi_n)_n$ of positive elements in L.

We denote by $\Lambda(L \times Y, Z)$ the space of such bilinear maps and we endow the space with the norm $\pi^+(\mathcal{U})$ given by the infimum of the constants satisfying the above inequality.

THEOREM 4.11. *If* $\mathcal{F} \in \mathrm{V}^p_{\mathcal{B}}(X)$, *then* $\mathcal{B}_{\mathcal{F}} \in \Lambda(L^{p'} \times Y, Z)$ *and* $\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} = \pi^+(\mathcal{B}_{\mathcal{F}})$.

PROOF. Given $\mathcal{F} \in \mathrm{V}^p_{\mathcal{B}}(X)$, then $\mathcal{B}_{\mathcal{F}} : L^{p'} \times Y \to Z$ is bounded. Let us show that $\mathcal{B}_{\mathcal{F}} \in \Lambda(L^{p'} \times Y, Z)$ and $\pi^+(\mathcal{B}_{\mathcal{F}}) = \|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)}$.

From Theorem 4.5 there exists $0 \leq \varphi_y \in L^p$ such that

$$\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} = \sup\{\|\varphi_y\|_{L^p} : y \in \mathrm{B}_Y\}$$

and

$$\|\mathcal{B}_{\mathcal{F}}(\mathbf{1}_A, y)\| \leq \int_\Omega \mathbf{1}_A \varphi_y \, d\mu, \quad A \in \Sigma.$$

Using linearity and density of simple functions one also extends to

$$\|\mathcal{B}_{\mathcal{F}}(\psi, y)\| \leq \int_\Omega \psi \varphi_y \, d\mu,$$

for any $0 \leq \psi \in L^{p'}$ and $y \in Y$.

Now, given a finite family $0 \leq \psi_n \in L^{p'}$ and $y \in Y$, we can write

$$\begin{aligned}
\sum_{n=1}^{N} \|\mathcal{B}_{\mathcal{F}}(\psi_n, y)\| &\leq \sum_{n=1}^{N} \int_\Omega \psi_n \varphi_y \, d\mu \\
&= \sum_{n=1}^{N} \|\varphi_y\|_{L^p} \left\langle \psi_n, \frac{\varphi_y}{\|\varphi_y\|_{L^p}} \right\rangle d\mu \\
&\leq \|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} \sup\left\{\sum_{n=1}^{N} |\langle \psi_n, \varphi\rangle| : \varphi \in \mathrm{B}_{L^p}\right\}.
\end{aligned}$$

This shows that $\pi^+(\mathcal{B}_{\mathcal{F}}) \leq \|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)}$.

On the other hand, given a partition π, a sequence $(\alpha_A)_A \in \ell^{p'}$ and denoting $\psi_A = (|\alpha_A|/\mu(A)^{1/p'})\mathbf{1}_A$ one can apply the condition of cone absolutely summing

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bilinear map to obtain

$$\sum_{A\in\pi}\left\|\mathcal{B}\left(\frac{\mathcal{F}(A)}{\mu(A)^{1/p'}},\alpha_A y\right)\right\|_Z=\sum_{A\in\pi}\|\mathcal{B}_{\mathcal{F}}(\psi_A,y)\|_Z$$
$$\le\pi^+(\mathcal{B}_{\mathcal{F}})\|y\|\ \sup\left\{\sum_{A\in\pi}\int_\Omega\psi_A|\varphi|\,d\mu:\varphi\in\mathrm{B}_{L^p}\right\}$$
$$=\pi^+(\mathcal{B}_{\mathcal{F}})\|y\|\ \sup\left\{\sum_{A\in\pi}\frac{|\alpha_A|}{\mu(A)^{1/p'}}\int_A|\varphi|\,d\mu:\varphi\in\mathrm{B}_{L^p}\right\}$$
$$\le\pi^+(\mathcal{B}_{\mathcal{F}})\|y\|\ \sup\left\{\sum_{A\in\pi}|\alpha_A|\left(\int_A|\varphi|^p\right)^{1/p}d\mu:\varphi\in\mathrm{B}_{L^p}\right\}$$
$$\le\pi^+(\mathcal{B}_{\mathcal{F}})\cdot\|y\|\cdot\|(\alpha_A)_A\|_{\ell^{p'}}.$$

Now (4.1) allows us to conclude that $\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)}\le\pi^+(\mathcal{B}_{\mathcal{F}})$. □

COROLLARY 4.12. *We have that* $\mathrm{V}^p_{\mathcal{B}}(X)$ *is isometrically embedded in* $\Lambda(L^{p'}\times Y,Z)$.

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O. BLASCO, Department of Mathematics, Universitat de València,
Burjassot 46100 (València), Spain
e-mail: oscar.blasco@uv.es

J. M. CALABUIG, Department of Applied Mathematics,
Universitat Politècnica de València, València 46022, Spain
e-mail: jmcalabu@mat.upv.es

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