l-SIMPLE LATTICE-ORDERED GROUPS

by A. M. W. GLASS†

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1. Introduction

Let G be a lattice-ordered group (l-group) and H a subgroup of G. H is said to be an *l-subgroup* of G if it is a sublattice of G. H is said to be *convex* if $h_1, h_2 \in H$ and $h_1 \leqq g \leqq h_2$ imply $g \in H$. The normal convex l-subgroups (l-ideals) of an l-group play the same role in the study of lattice-ordered groups as do normal subgroups in the investigation of groups. For this reason, an l-group is said to be *l-simple* if it has no non-trivial l-ideals. As in group theory, a central task in the examination of lattice-ordered groups is to characterise those l-groups which are l-simple.

Let $\langle S, \leqq \rangle$ be a totally ordered set. Aut $(\langle S, \leqq \rangle)$ is an l-group under the ordering: $f \leqq g$ if and only if $f(s) \leqq g(s)$ for all $s \in S$. Also

$$(g \vee h)(s) = \max\{g(s), h(s)\} \text{ and } (g \wedge h)(s) = \min\{g(s), h(s)\}.$$

Theorem (Holland (3) Theorem 2). *If G is an l-group, there exists a totally ordered set $\langle S, \leqq \rangle$ such that G can be l-embedded (a group embedding which preserves the lattice operation) in* Aut $(\langle S, \leqq \rangle)$.

This theorem is the analogue of the theorem of Cayley for groups and, in this case, we will say that G is represented on $\langle S, \leqq \rangle$.

For the rest of this paper $\langle S, \leqq \rangle$ will denote a totally ordered set and Aut $(\langle S, \leqq \rangle)$ will be the l-group described above.

Let G be an l-subgroup of Aut $(\langle S, \leqq \rangle)$. G is said to be *transitive on S* if, for any $s, t \in S$, there exists $g \in G$ such that $g(s) = t$.

Theorem ((3), Corollary 2). *Every l-simple l-group can be represented transitively on some set $\langle S, \leqq \rangle$.*

An equivalence relation $\mathscr{E}$ on $\langle S, \leqq \rangle$ is called a *convex G-congruence on* $\langle S, \leqq \rangle$ if each equivalence class of $\mathscr{E}$ is convex and, for all $s, t \in S$, $g(s)\mathscr{E}g(t)$ for all $g \in G$ whenever $s\mathscr{E}t$. If there are no non-trivial convex G-congruences on $\langle S, \leqq \rangle$, then G is said to be *o-primitive* on $\langle S, \leqq \rangle$.

In attempting to classify the l-simple l-groups, the first step is to characterise those that have a transitive o-primitive representation.

Let G be an l-subgroup of Aut $(\langle S, \leqq \rangle)$. G is *o-2 transitive* on $\langle S, \leqq \rangle$ if,

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for all $s, t, u, v \in S$ such that $s<t$ and $u<v$, there exists $g \in G$ such that $g(s) = u$ and $g(t) = v$. Let $\langle \bar{S}, \leqq \rangle$ be the Dedekind closure of $\langle S, \leqq \rangle$. If there exists a positive $f_0 \in \text{Aut}(\langle \bar{S}, \leqq \rangle)$ such that f_0 generates the centraliser of G in $\text{Aut}(\langle \bar{S}, \leqq \rangle)$ and $\{f_0^n(s_0): n \in \mathbf{Z}\}$ is unbounded (above and below) in $\langle \bar{S}, \leqq \rangle$ for some $s_0 \in S$, then G is said to be *periodic* and f_0 is called the *period of G*.

Theorem (Holland (**4**) and McCleary (**7**)). *Let G be a transitive o-primitive l-subgroup of* $\text{Aut}(\langle S, \leqq \rangle)$. *Then either:*

(*i*) *G is a subgroup of the real numbers* $(\langle \mathbf{R}, \leqq \rangle)$ *in its regular representation;*

or (*ii*) *G is periodic;*

or (*iii*) *G* is *o*-2 *transitive on* $\langle S, \leqq \rangle$.

If G falls into either of the first two categories, it is *l*-simple (see (**7**)). Let $g \in G$. The *support* of g is the set of points of S moved by g; i.e.

$$\text{supp}(g) = \{s \in S: \ g(s) \neq s\}.$$

If there exist $x, y \in S$ such that $\text{supp}(g) \subseteq [x, y]$, then g is said to have *bounded support*. If G is *o*-2 transitive on $\langle S, \leqq \rangle$ and has no element of bounded support other than the identity, G is called *pathological*.

If G is *o*-2 transitive on $\langle S, \leqq \rangle$, then the set of elements of G of bounded support is an *l*-ideal of G. Consequently,

Theorem (Holland (**5**)). *If G is o-2 transitive on* $\langle S, \leqq \rangle$, *then G is l-simple if all its elements are of bounded support. If, in addition, G contains an element—other than the identity—of bounded support, the converse is true.*

We must now examine pathological *o*-2 transitive *l*-groups. Examples of such groups are sparse in the literature; essentially, the only ones known can be found in (**4**) and (**8**) and they are *l*-simple. This led to the conjecture that every pathological *o*-2 transitive *l*-group is *l*-simple (see (**6**) and (**8**)). Were it true, it would yield a complete classification of those *l*-simple *l*-groups which have a transitive *o*-primitive representation. Actually, it is false. To prove this, we will provide new pathological *o*-2 transitive *l*-groups and, in particular, will prove a theorem concerning free *l*-groups (free in the category of *l*-groups) on an infinite set of generators.

For a further discussion of ordered permutation groups, (**6**) is an excellent expository article.

2. Ultraproducts of pathological *o*-2 transitive *l*-groups

Suppose that for each $i \in I$, G_i is a pathological *o*-2 transitive *l*-subgroup of $\text{Aut}(\langle S_i, \leqq_i \rangle)$. Let D be an ultrafilter on I. Then $G = \Pi_D G_i$ is a pathological *o*-2 transitive *l*-subgroup of $\text{Aut}(\langle S, \leqq \rangle)$ where $S = \Pi_D S_i$ and $s_D \leqq t_D$ if and only if $\{i \in I: \ s(i) \leqq_i t(i)\} \in D$ (see (**1**) and (**2**) for further background on ultraproducts). This fact is easily verified.

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Let $I = \omega$, the first infinite ordinal and let D be any non-principal ultrafilter on I. For each $i \in I$, let $G_i = H = \{h \in \text{Aut}(\langle \mathbf{R}, \leqq \rangle)\colon\ hf_0^n = f_0^n h$ for some positive integer $n\}$ where $f_0 \in \text{Aut}(\langle \mathbf{R}, \leqq \rangle)$ is defined by: $f_0(r) = r+1$ for all $r \in \mathbf{R}$. Then H is an l-simple pathological o-transitive l-group (**4**). Observe that $f_0^{n+1} \nleqq h^{-1}f_0^n h$ for any $h \in H$ and any positive integer n (for if n is a positive integer and $h \in H$ are such that $f_0^{n+1} \leqq h^{-1}f_0^n h$, then for all

$r \in \mathbf{R}$, $$f_0^{n+1}(h^{-1}(r)) \leqq h^{-1}f_0^n h(h^{-1}(r)) = h^{-1}(r+n);$$
i.e.
$$h^{-1}(r)+n+1 \leqq h^{-1}(r+n). \qquad (*)$$

There exists a positive integer m such that $f_0^m h = hf_0^m$. Replacing r by 0, n, $2n, \ldots, mn$ in (*) we obtain

$$h^{-1}(0)+m(n+1) \leqq h^{-1}(mn) = h^{-1}(0+mn) = h^{-1}(0)+mn,$$

Aut $(\langle \Pi_D \mathbf{R}, \leqq \rangle)$ which is not l-simple since the l-ideal generated by
a contradiction). Then $G = \Pi_D G_i$ is a pathological o-2 transitive l-subgroup of

$$\{k_D\colon\ k(i) = k(j) \text{ for all } i, j \in I\}$$

is proper (cf. the result for groups, see (**2**)). It can also be shown that G is not "periodic" in any sense even though each H_i is (this is a more general notion of periodic than that given in the introduction). This example indicates that the conjectures of (**8**) are false. Moreover, "l-simple" cannot be expressed in first-order logic (for if a first order sentence is true in $\mathfrak{A}_i$ for each $i \in I$, then it is true in $\Pi_D \mathfrak{A}_i$ for any ultrafilter D on I).

3. Free *l*-groups

Let κ be an infinite cardinal. F_κ will denote the free l-group generated by $\{x_\alpha\colon\ \alpha < \kappa\}$ where $x_\alpha \neq x_\beta$ if $\alpha, \beta < \kappa$ and $\alpha \neq \beta$. Let $\mathbf{Z}$ be the set of integers and $\langle \mathbf{Q}, \leqq \rangle$ the set of rationals under the usual ordering.

Lemma. *F_ω is l-isomorphic to an l-subgroup of* Aut $(\langle \mathbf{Q}, \leqq \rangle)$ *which has no element* (*other than the identity*) *of bounded support.*

Proof. Let $\{I_n\colon\ n \in \mathbf{Z}\}$ be any set of bounded non-empty open intervals in $\langle \mathbf{Q}\ \leqq \rangle$ such that if $m, n \in \mathbf{Z}$ and $m < n$, then $I_m < I_n$ (if $q \in I_m$ and $r \in I_n$, then $q < r$) and if $q \in \mathbf{Q}$, there exists $p \in \mathbf{Z}$ such that $p > 0$ and $I_{-p} < q < I_p$. Since the first order theory of dense total ordering without endpoints is ω-categorical, see (**1**), $\langle I_n, \leqq \rangle$ is isomorphic to $\langle \mathbf{Q}, \leqq \rangle$ for each $n \in \mathbf{Z}$. The proof of Theorem 2 of (**3**) shows that F_ω can be l-embedded in Aut$(\langle \mathbf{Q}, \leqq \rangle)$ and so in

$$\text{Aut}\,(\langle I_n, \leqq \rangle)$$

for each $n \in \mathbf{Z}$. Let $\bar{x}_{m,n}$ be the image of x_m in Aut $(\langle I_n, \leqq \rangle)$ for $m, n \in \mathbf{Z}$ and $m \geqq 0$. Define $f_m \in \text{Aut}(\langle \mathbf{Q}, \leqq \rangle)$ as follows: Let $q \in \mathbf{Q}$; if $q \in I_n$ for some $n \in \mathbf{Z}$, let $f_m(q) = \bar{x}_{m,n}(q)$ and if $q \in \bigcup\{I_n\colon\ n \in \mathbf{Z}\}$, let $f_m(q) = q$. Let W be the subgroup of Aut $(\langle \mathbf{Q}, \leqq \rangle)$ generated by $\{f_m\colon\ m \in \omega\}$. The map $\phi\colon F_\omega \to W$

defined by $\phi(x_m) = f_m$ is an l-isomorphism of F_ω on to W. Indeed, by construction, the only element of W of bounded support is the identity so the lemma is proved.

Theorem. *F_ω is l-isomorphic to a pathological o-2 transitive l-subgroup of* Aut $(\langle \mathbf{Q}, \leqq \rangle)$.

Proof. Let $A = \{(q, r)\colon q, r \in \mathbf{Q} \text{ and } q<r\}$ and $B = A \times A$. The cardinality of B is ω and so there exists a one-to-one function ψ of ω onto B. Denote $\psi(m)$ by (a_m, b_m, c_m, d_m). Define a sequence of bounded open intervals X_m in $\langle \mathbf{Q}, \leqq \rangle$ such that $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_m \subseteq \ldots$ and $[a_m, b_m], [c_m, d_m] \subseteq X_m$ for each $m \in \omega$. For each $m \in \omega$, $\langle X_m, \leqq \rangle$ is isomorphic to $\langle \mathbf{Q}, \leqq \rangle$ and so there exists $k_m \in \text{Aut}(\langle X_m, \leqq \rangle)$ such that $k_m(a_m) = c_m$ and $k_m(b_m) = d_m$. Let $\{I_n\colon n \in \mathbf{Z}\}$ be as in the the proof of the lemma subject to the extra condition that

$$I_{-(m+1)} < X_m < I_{m+1}$$

for each $m \in \omega$. Let $\bar{x}_{m\ n}$ be defined as in the proof of the lemma and define $f_m \in \text{Aut}(\langle \mathbf{Q}, \leqq \rangle)$ for $m \in \omega$ as follows: let $q \in \mathbf{Q}$; if $q \in X_m$, let $f_m(q) = k_m(q)$; if $q \notin I_p$ for some $p \in \mathbf{Z}$ such that $|p| > m$, let $f_m(q) = \bar{x}_{m,\ n}(q)$; if $q \notin X_m \bigcup \{I_p\colon |p| > m\}$, let $f_m(q) = q$. Let W be the l-subgroup of Aut $(\langle \mathbf{Q}, \leqq \rangle)$ generated by $\{f_m\colon m \in \omega\}$. By the coding of B, W is an o-2 transitive l-subgroup of Aut $(\langle \mathbf{Q}, \leqq \rangle)$ which is pathological by the construction. The map $\phi\colon F_\omega \to W$ defined by $\phi(x_m) = f_m$ is an l-isomorphism of F_ω on to W.

Corollary 1. *There exist pathological o-2 transitive l-groups which are neither l-simple nor periodic in any sense.*

Note that the proof of the lemma would apply to any free l-group on a finite number of generators in place of F_ω. If κ is an infinite cardinal such that whenever there exists an order-isomorphism between two subsets of S of cardinality less than κ, the order-isomorphism can be extended to some element of G, then G is said to be *o-κ transitive on* $\langle S, \leqq \rangle$. Any l-group which is o-2 transitive on $\langle S, \leqq \rangle$ is o-ω transitive on $\langle S, \leqq \rangle$. However, the ideas of the proof of the theorem could have been used to show directly that F_ω is l-isomorphic to a pathological o-ω transitive l-subgroup of Aut $(\langle \mathbf{Q}, \leqq \rangle)$.

Let α be an ordinal. $\langle S, \leqq \rangle$ is said to be an *α-set* if and only if S has cardinality $\aleph_\alpha$ and whenever $X, Y \subseteq S$ are such that $|X \cup Y| < \aleph_\alpha$ and $X < Y$, there exists $s \in S$ such that $X < \{s\} < Y$. $\langle \mathbf{Q}, \leqq \rangle$ is a 0-set. For any α, there exists at most one α-set (to within isomorphism). Moreover, the existence of α-sets for all α such that $\aleph_\alpha$ is regular is equivalent to the generalised continuum hypothesis (G.C.H.). It was essentially shown in (**9**) that if $\langle S, \leqq \rangle$ is an α-set, it is o-$\aleph_\alpha$ transitive; alternatively, $\langle S, \leqslant \rangle$ is saturated and so homogeneous—that is, o-$\aleph_\alpha$ transitive—(see (**1**)). Therefore the proof of the theorem yields:

Corollary 2 (G.C.H.). *For any ordinal α, F_{ω_α} is l-isomorphic to a pathological o-$\aleph_\alpha$ transitive l-subgroup of* Aut $(\langle S_\alpha, \leqq \rangle)$ *where $\langle S_\alpha, \leqq \rangle$ is an α-set.*

The following result can be found in (**5**):

Let G be a transitive *l*-subgroup of Aut $(\langle S, \leqq \rangle)$ and $e<g\in G$ (e is the identity function). g has bounded support if and only if the sentence

$$\exists h \forall k (k \geqq e \Rightarrow g \wedge k^{-1}h^{-1}ghk = e) \tag{1}$$

holds in G. If G has no positive element (other than the identity) of bounded support, then G has no element (other than the identity) of bounded support. Consequently, the *o*-2 transitive *l*-group G is pathological if and only if the sentence

$$\forall g(g \not> e \vee \forall h \exists k(k \geqq e \,\&\, g \wedge k^{-1}h^{-1}ghk \neq e)) \tag{2}$$

holds in G. This is a Π_2^0 sentence and so is preserved under 1-sandwiches (see (**1**)). It is not preserved under *l*-homomorphic images since it is satisfied in all free *l*-groups (by the Lemma and Corollary 2), every *l*-group is an *l*-homomorphic image of a free *l*-group and there exist non-pathological *o*-2 transitive *l*-groups. Hence (2) is not equivalent to a positive sentence (see (**1**)).

In (**3**, Theorem 3) it was shown that an *l*-group G is *l*-isomorphic to a transitive *l*-subgroup of some Aut $(\langle S, \leqq \rangle)$ if and only if there exists a prime convex *l*-subgroup C of G such that C contains no *l*-ideal of G other than $\{e\}$. (C is *prime* if $f, g \in G$ and $f \wedge g = e$ imply $f \in C$ or $g \in C$.) Such a subgroup C is called a *representing subgroup* of G. G is *l*-isomorphic to an *o*-2 transitive *l*-subgroup of some Aut $(\langle S, \leqq \rangle)$ if and only if there exists a representing subgroup C of G such that if $e \leqq f_i \in G \backslash C$ (i = 1, 2), there exists $g \in C$ such that $Cf_1g = Cf_2$. Such a representing subgroup will be called a *strong representing subgroup*. Thus:

Corollary 3. *Let F be a free l-subgroup and K an l-ideal of F. F/K is l-isomorphic to a pathological o-2 transitive l-subgroup of some* Aut $(\langle S, \leqq \rangle)$ *if and only if there exists a strong representing subgroup C of F such that* $C \supseteq K$, *C contains no l-ideal of F which properly contains K and the sentence* (2) *holds in F/K.*

This yields an algebraic method of determining, inside free *l*-groups, which quotients have pathological *o*-2 transitive *l*-isomorphic images. Unfortunately, this result is very limited since it leaves unanswered many natural questions; e.g. do there exist pathological *o*-2 transitive *l*-groups all of whose elements (other than the identity) have a finite (bounded) set of fixed points? However, a thorough examination of pathological *o*-2 transitive *l*-groups seems necessary so as to make possible a characterisation of the *l*-simple *l*-groups which have an *o*-primitive transitive representation.

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Bowling Green State University
Bowling Green, Ohio 43403
U.S.A.

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