

STONE LATTICES. I: CONSTRUCTION THEOREMS

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1. Introduction. Stone lattices were (named and) first studied in 1957 (5). Since then, a great number of papers have been written on Stone lattices and a very satisfactory theory evolved. Despite the fact that all chains with 0, 1 as well as all Boolean algebras are Stone lattices, it turns out that many of the nice theorems on Boolean algebras have analogues, in fact, generalizations for Stone lattices. To give just two examples: the characterization of Boolean algebras in terms of prime ideals (Nachbin (6)) is generalized in (5) (see also (9)); Stone's representation theory (8) is generalized in (4); see also (2).

Despite a wealth of results, one would be hard-pressed to provide examples of Stone lattices tailored to some specific needs, which would go beyond chains, Boolean algebras and direct products thereof. In this paper we describe a method of construction of Stone lattices. The basic idea is to associate with every Stone lattice two "simpler" structures (and a connecting map), forming a triple. The Basic Theorem states that there is a one-to-one correspondence between Stone lattices and triples.

The Basic Theorem could be viewed as a method of construction of Stone lattices from simpler components. Various theorems of this type are proved in this paper. Furthermore, the Basic Theorem could be interpreted as stating that the structure of Stone lattices can be best studied by investigating the structure of the "simpler" components. This viewpoint will be developed in Part II of this paper (see pp. 895–903 of this issue).

In this paper we construct, after some preliminary considerations (§ 2), the triple associated with a Stone lattice and prove the Uniqueness Theorem (§ 3). Then in § 4 we have the Construction Theorem. A description of homomorphisms and isomorphisms in terms of triples is discussed in § 5. The final section (§ 6) is devoted to "fill in" theorems, which roughly speaking construct the missing entries of triples in certain situations.

2. Preliminaries. We denote the lattice operations by \vee and \wedge . The least and greatest element of a lattice will be denoted by 0 and 1. As usual, a lattice $\langle L; \vee, \wedge \rangle$ will simply be denoted by L .

The *pseudo-complement* x^* of the element x of the lattice L is defined by

$$x \wedge y = 0 \text{ if and only if } y \leq x^*,$$

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that is, x^* is the greatest element “disjoint from” x . If every element of L has a pseudo-complement, then L is a *pseudo-complemented lattice*. Note that a pseudo-complemented lattice always has 0 and 1 ($= 0^*$).

A *Stone lattice* L is a pseudo-complemented, distributive lattice in which

$$(2.1) \quad a^* \vee a^{**} = 1,$$

holds for all $a \in L$.

The following rules of computation (see, e.g., **1**; **3**; **4**) will be used frequently:

$$(2.2) \quad a \leq b \text{ implies } b^* \leq a^*;$$

$$(2.3) \quad a \leq a^{**};$$

$$(2.4) \quad a^{***} = a^*;$$

$$(2.5) \quad a^* \wedge a^{**} = 0;$$

$$(2.6) \quad (a \vee b)^* = a^* \wedge b^*;$$

$$(2.7) \quad (a \wedge b)^{**} = a^{**} \wedge b^{**};$$

$$(2.8) \quad (a \wedge b)^* = a^* \vee b^*;$$

$$(2.9) \quad (a \vee b)^{**} = a^{**} \vee b^{**}.$$

Note that (2.2)–(2.7) hold in any pseudo-complemented lattice, while (2.8) and (2.9) do not hold in general; all rules, (2.1)–(2.9), hold in Stone lattices.

For a distributive lattice L , with 0 and 1, $C(L)$ will denote the set of complemented elements of L , the *centre* of L . Note that $C(L)$ is a sublattice of L .

A *Stone algebra* is a Stone lattice L in which 0, 1, and $*$ are also considered as basic operations. If L_0 and L_1 are Stone algebras, ϕ a *homomorphism* of L_0 into L_1 , then ϕ is a (lattice) homomorphism that preserves 0, 1, and $*$. A *subalgebra* is a sublattice, containing 0, 1, and closed under $*$. If L_0 and L_1 are lattices with 0, 1, an *e-homomorphism* (e for extremal) is a homomorphism, preserving 0 and 1, an *e-sublattice* is a sublattice containing 0 and 1.

Let $\mathcal{D}(L)$ denote the set of all dual ideals of the lattice L , ordered under set inclusion. For $a \in L$, the dual ideal generated by a is $[a] = \{x \mid x \geq a\}$. $\mathcal{D}(L)$ is a lattice; if L has a 1, then $\mathcal{D}(L)$ has a zero: $\{1\} = [1]$; $\mathcal{D}(L)$ always has a largest element: L . The map: $x \rightarrow [x]$ is a dual homomorphism of L into $\mathcal{D}(L)$, that is,

$$(2.10) \quad [x \vee y] = [x] \wedge [y],$$

$$(2.11) \quad [x \wedge y] = [x] \vee [y].$$

If L is a distributive lattice with 1, then so is $\mathcal{D}(L)$; then the centre of $\mathcal{D}(L)$ is especially interesting:

$$(2.12) \quad \text{for } a \in L, I \in C(\mathcal{D}(L)), \text{ there exists } b \in L \text{ such that } [b] = I \wedge [a].$$

In other words, an $I \in C(\mathcal{D}(L))$ shares with all principal dual ideals the property that its meet with a principal dual ideal is again a principal dual ideal.

3. The triple associated with a Stone algebra. Let L be a Stone algebra. Then its centre $C(L)$ is a subalgebra of L ; by (2.1), $C(L)$ is a Boolean algebra. It is easy to see that

$$(3.1) \quad a \in C(L) \text{ if and only if } a = b^* \text{ for some } b \in L \text{ if and only if } a = a^{**}.$$

Now set

$$(3.2) \quad x \sim y \text{ if and only if } x^{**} = y^{**}.$$

Then \sim is a congruence relation of the Stone algebra L , and

$$(3.3) \quad L/\sim \cong C(L).$$

Thus, $C(L)$ is a homomorphic image (in fact, a retract) of L under the homomorphism $x \rightarrow x^{**}$. Each congruence class contains exactly one element of $C(L)$, which is the largest element in the congruence class. Hence, \sim partitions L into $\{F_c \mid c \in C(L)\}$, where $F_c = \{x \mid x^{**} = c\}$, $c \in C(L)$.

Note that $x \in F_1$ if and only if $x^* = 0$; therefore an $x \in F_1$ is called a *dense element*; F_1 will be denoted by $D(L)$, and called the *dense set* of L . It is easily seen that $D(L)$ is a dual ideal of L , and $D(L)$ is a distributive lattice with 1, since $1 \in D(L)$.

Let $a, b \in C(L)$, $a \leq b$. Then

$$(3.4) \quad x \rightarrow (x \vee a^*) \wedge b, x \in F_a$$

is an embedding of F_a into F_b . In particular,

$$(3.5) \quad x \rightarrow x \vee a^*$$

embeds F_a into $D(L) = F_1$. It is easy to see that $b \in D(L)$ is of the form $x \vee a^*$, for some $x \in F_a$ if and only if $b \geq a^*$ in L . Thus, F_a is isomorphic with $\{x \mid x \geq a^*, x \in D(L)\}$. Consider the map

$$(3.6) \quad \phi^L: a \rightarrow a\phi^L = \{x \mid x \in D(L), x \geq a^*\}, \quad a \in C(L).$$

Then ϕ^L is an e-homomorphism of $C(L)$ into $\mathcal{D}(D(L))$. We call ϕ^L the *structure map*.

The *triple associated with L* is $\langle C(L), D(L), \phi^L \rangle$. It is clear now that the triple determines all the F_a , $a \in C(L)$. We claim that the triple determines all of L ; in other words, for $x \in F_a, y \in F_b, x \leq y$ in L is determined by the triple. (2.2) implies that if $x \leq y$, then $x^{**} \leq y^{**}$, that is, $a \leq b$. Since $x \leq y$ if and only if

$$a \vee x \leq a \vee y \quad \text{and} \quad x \vee a^* \leq y \vee a^*,$$

and $x \vee a = a, a \leq y \vee a$, we obtain:

$$(3.7) \quad \text{for } x \in F_a, y \in F_b, x \leq y \text{ in } L \text{ if and only if} \\ a \leq b \quad \text{and} \quad x \vee a^* \leq y \vee a^*.$$

Identify an $x \in L, x \in F_a$ with the ordered pair $\langle x \vee a^*, a \rangle$. Then, as noted

above, the pair we obtain is of the form $\langle z, a \rangle$, with $a \in C(L)$, $z \in a\phi^L$, and we have all such pairs.

To describe the partial ordering in terms of the triple, we need one more notation. For every $a \in C(L)$, the dual ideal $a\phi^L$ is in the centre of $\mathcal{D}(C(L))$; indeed, the complement of $a\phi^L$ is $a^*\phi^L$. Thus, by (2.12), for every $a \in C(L)$, there is a map ρ_a of $D(L)$ onto $a\phi^L$ given by

$$(3.8) \quad [x\rho_a^L] = a\phi^L \wedge [x], \quad x \in D(L), a \in C(L).$$

Of course, $x\rho_a^L = x \vee a^*$; however, this would not give ρ_a^L intrinsically (i.e., in terms of the triple). Combining (3.7) and (3.8) we obtain:

$$(3.9) \quad \langle x, a \rangle \leq \langle y, b \rangle \text{ if and only if } a \leq b \text{ and } x \leq y\rho_a^L.$$

UNIQUENESS THEOREM. *A Stone algebra L is determined up to isomorphism by the triple $\langle C(L), D(L), \phi^L \rangle$.*

4. The Construction Theorem. How can we abstractly characterize triples associated with Stone algebras? Let us define a *triple* $\langle C, D, \phi \rangle$:

- (i) C is a Boolean algebra;
- (ii) D is a distributive lattice with 1;
- (iii) ϕ is an e-homomorphism from C into $\mathcal{D}(D)$.

CONSTRUCTION THEOREM. *Let $\langle C, D, \phi \rangle$ be a triple. Then we can construct a Stone algebra L such that*

$$\langle C(L), D(L), \phi^L \rangle = \langle C, D, \phi \rangle;$$

that is, $C = C(L)$, $D = D(L)$, and $\phi = \phi^L$.

Remark. The construction in the following proof originated in 1962. A related construction is discussed in (7). The class of algebras considered here and the class considered in (7) do not contain each other, but they have a large intersection. Our structure maps correspond to certain functions from $C \times D$ into D of (7). The algebra constructed in (7) consists of equivalence classes of ordered pairs. Thus, the connections of the two constructions are not clear.

Proof. Set

$$(4.1) \quad L = \{ \langle x, a \rangle \mid a \in C, x \in a\phi \}$$

and

$$(4.2) \quad \langle x, a \rangle \leq \langle y, b \rangle \text{ if and only if } a \leq b, x \leq y\rho_a,$$

where ρ_a is defined by (3.8) in terms of ϕ , that is,

$$(4.3) \quad [x\rho_a] = a\phi \wedge [x], \quad x \in D, a \in C.$$

We will prove that the relation defined in (4.2) makes L a Stone algebra which (with two trivial changes) satisfies the statement of the Construction Theorem.

We start with some formulas concerning ρ_a :

$$(4.4) \quad d\rho_a = d, \text{ for } a \in C, d \in a\phi \quad \text{and} \quad d\rho_a = d \text{ implies } d \in a\phi;$$

$$(4.5) \quad d\rho_a \geq d, \text{ for } a \in C, d \in D;$$

$$(4.6) \quad d\rho_a \wedge d\rho_{a'} = d, \text{ for } a \in C, d \in D \quad (a' \text{ is the complement of } a);$$

$$(4.7) \quad \rho_a\rho_b = \rho_{a\wedge b}, \text{ for } a, b \in C;$$

$$(4.8) \quad d\rho_a \wedge d\rho_b = d\rho_{a\vee b}, \text{ for } a, b \in C, d \in D;$$

$$(4.9) \quad d\rho_{a\wedge b} = d\rho_a \vee d\rho_b, \text{ for } a, b \in C, d \in D.$$

Proof. (4.4)–(4.6) follow immediately from (4.3). Now,

$$[d\rho_a\rho_b] \stackrel{(4.3)}{=} [d\rho_a] \wedge b\phi = [d] \wedge a\phi \wedge b\phi = [d] \wedge (a \wedge b)\phi \stackrel{(4.3)}{=} [d\rho_{a\wedge b}];$$

thus we obtain (4.7). To prove (4.8) we compute:

$$\begin{aligned} [d\rho_{a\vee b}] &= [d] \wedge (a \vee b)\phi = [d] \wedge (a\phi \vee b\phi) \\ &= ([d] \wedge a\phi) \vee ([d] \wedge b\phi) = [d\rho_a] \vee [d\rho_b] \stackrel{(2.11)}{=} [d\rho_a \wedge d\rho_b]. \end{aligned}$$

The proof of (4.9) is similar.

L is a partially ordered set. Indeed, (4.5) implies that \leq is reflexive, (4.4) yields that \leq is anti-symmetric, and (4.7) yields that \leq is transitive. The join and meet in L can be described as follows:

$$(4.10) \quad \langle x, a \rangle \wedge \langle y, b \rangle = \langle x\rho_b \wedge y\rho_a, a \wedge b \rangle;$$

$$(4.11) \quad \langle x, a \rangle \vee \langle y, b \rangle = \langle (x\rho_{b'} \wedge y) \vee (x \wedge y\rho_{a'}), a \vee b \rangle.$$

To prove (4.10), we have to verify first that the right-hand side is in L ; indeed, $x\rho_b \in b\phi$, and $y\rho_a \in a\phi$ by (4.3). Therefore, $x\rho_b \in a\phi \wedge b\phi = (a \wedge b)\phi$ and $y\rho_a \in (a \wedge b)\phi$, thus $x\rho_b \wedge y\rho_a \in (a \wedge b)\phi$. Now $\langle x, a \rangle \geq \langle x\rho_b \wedge y\rho_a, a \wedge b \rangle$ is trivial, since $a \geq a \wedge b$ and $x\rho_{a\wedge b} \stackrel{(4.9)}{=} x\rho_a \vee x\rho_b \geq x\rho_b \wedge y\rho_a$. Thus, the right-hand side of (4.10) is a lower bound for $\langle x, a \rangle$ and $\langle y, b \rangle$. Now let $\langle z, c \rangle$ be any lower bound for $\langle x, a \rangle$ and $\langle y, b \rangle$. Then $a \geq c$, $b \geq c$, and hence $a \wedge b \geq c$. Further, $x\rho_c \geq z$, $y\rho_c \geq z$, hence,

$$(x\rho_b \wedge y\rho_a)\rho_c \stackrel{(4.8) \text{ and } (4.7)}{=} x\rho_{b\wedge c} \wedge y\rho_{a\wedge c} = x\rho_c \wedge y\rho_c \geq z.$$

Thus, $\langle x\rho_b \wedge y\rho_a, a \wedge b \rangle \geq \langle z, c \rangle$, completing the proof of (4.10).

To prove (4.11) we first verify that its right-hand side belongs to L . By (4.4), it is enough to show that it is invariant under $\rho_{a\vee b}$; indeed,

$$\begin{aligned} ((x\rho_{b'} \wedge y) \vee (x \wedge y\rho_{a'}))\rho_{a\vee b} &= (x\rho_{b'}\rho_{a\vee b} \wedge y\rho_{a\vee b}) \vee (x\rho_{a\vee b} \wedge y\rho_{a'}\rho_{a\vee b}) \\ &= (\text{use (4.7) and } x = x\rho_a, y = y\rho_b) (x\rho_{b'\wedge a} \wedge y\rho_{b\rho_a\vee b}) \vee (x\rho_a\rho_{a\vee b} \wedge y\rho_{b\rho_{a'}\wedge b}) \\ &\stackrel{(4.7)}{=} (x\rho_a\rho_{b'} \wedge y\rho_b) \vee (x\rho_a \wedge y\rho_{b\rho_{a'}}) = (x\rho_{b'} \wedge y) \vee (x \wedge y\rho_{a'}). \end{aligned}$$

The inequality

$$\begin{aligned} ((x\rho_{b'} \wedge y) \vee (x \wedge y\rho_{a'}))\rho_a &= (x\rho_{b'}\rho_a \wedge y\rho_a) \vee (x\rho_a \wedge y\rho_{a'}\rho_a) \\ &= (x\rho_{b'}\rho_a \wedge y\rho_a) \vee (x \wedge y\rho_0) = (\text{by (4.3), } y\rho_0 = 1)(x\rho_{b'}\rho_a \wedge y\rho_a) \vee x \geq x, \end{aligned}$$

and the fact that $a \vee b \geq a$ show that

$$\langle (x\rho_{b'} \wedge y) \vee (x \wedge y\rho_{a'}), a \vee b \rangle \geq \langle x, a \rangle,$$

and similarly,

$$\langle (x\rho_{b'} \wedge y) \vee (x \wedge y\rho_{a'}), a \vee b \rangle \geq \langle y, b \rangle.$$

Now let $\langle z, c \rangle \in L$, $\langle z, c \rangle \geq \langle x, a \rangle$ and $\langle y, b \rangle$. Then $c \geq a \vee b$, $z\rho_a \geq x$, and $z\rho_b \geq y$ (by (4.2)). Thus,

$$\begin{aligned} z\rho_{a \vee b} &= z\rho_{(a \wedge b') \vee b} \stackrel{(4.8)}{=} z\rho_{a \wedge b'} \wedge z\rho_b \\ &\stackrel{(4.7)}{=} z\rho_a \rho_{b'} \wedge z\rho_b \geq (\text{since } z\rho_a \geq x, \text{ and } z\rho_b \geq y) x\rho_{b'} \wedge y. \end{aligned}$$

Similarly, $z\rho_{a \vee b} \geq x \wedge y\rho_{a'}$, hence

$$z\rho_{a \vee b} \geq (x\rho_{b'} \wedge y) \vee (x \wedge y\rho_{a'}),$$

establishing

$$\langle z, c \rangle \geq \langle (x\rho_{b'} \wedge y) \vee (x \wedge y\rho_{a'}), a \vee b \rangle.$$

This completes the proof of (4.11).

Thus, L is a lattice.

L is a distributive lattice. Let $\langle x, a \rangle$, $\langle y, b \rangle$, $\langle z, c \rangle$ be in L . Now compute:

$$\begin{aligned} A &= (\langle x, a \rangle \wedge \langle y, b \rangle) \vee \langle z, c \rangle \stackrel{(4.10)}{=} \langle x\rho_b \wedge y\rho_a, a \wedge b \rangle \vee \langle z, c \rangle \\ &\stackrel{(4.11)}{=} \langle [(x\rho_b \wedge y\rho_a)\rho_{c'} \wedge z] \vee [x\rho_b \wedge y\rho_a \wedge z\rho_{(a \wedge b)'}], (a \wedge b) \vee c \rangle \end{aligned}$$

and

$$\begin{aligned} B &= (\langle x, a \rangle \vee \langle z, c \rangle) \wedge (\langle y, b \rangle \vee \langle z, c \rangle) \\ &\stackrel{(4.11)}{=} \langle (x\rho_{c'} \wedge z) \vee (x \wedge z\rho_{a'}), a \vee c \rangle \wedge \langle (y\rho_{c'} \wedge z) \vee (y \wedge z\rho_{b'}), b \vee c \rangle \\ &= \langle d, (a \vee c) \wedge (b \vee c) \rangle, \end{aligned}$$

where

$$\begin{aligned} d &= [(x\rho_{c'} \wedge z) \vee (x \wedge z\rho_{a'})]\rho_{b \vee c} \wedge [(y\rho_{c'} \wedge z) \vee (y \wedge z\rho_{b'})]\rho_{a \vee c} \\ &= d_0 \vee d_1 \vee d_2 \vee d_3, \end{aligned}$$

in which

$$d_0 = x\rho_{c'}\rho_{b \vee c} \wedge z\rho_{b \vee c} \wedge y\rho_{c'}\rho_{a \vee c} \wedge z\rho_{a \vee c} \stackrel{(4.7)}{=} x\rho_{b \wedge c'} \wedge y\rho_{a \wedge c'} \wedge z,$$

since $z\rho_{b\vee c} = z\rho_c\rho_{b\vee c} = z\rho_c = z$, and $z\rho_{a\vee c} = z$;

$$\begin{aligned} d_1 &= x\rho_{c'}\rho_{b\vee c} \wedge z\rho_{b\vee c} \wedge y\rho_{a\vee c} \wedge z\rho_{b'}\rho_{a\vee c} \\ &= x\rho_{b\wedge c'} \wedge y\rho_{a\vee c} \wedge z; \\ d_2 &= x\rho_{b\vee c} \wedge y\rho_{a\wedge c'} \wedge z; \\ d_3 &= x\rho_{b\vee c} \wedge y\rho_{a\vee c} \wedge z\rho_{a'\vee b'}. \end{aligned}$$

Observe that

$$y\rho_{a\wedge c'} \stackrel{(4.9)}{=} y\rho_a \vee y\rho_{c'} \geq y\rho_a \geq y\rho_{a\vee c},$$

hence $d_0 \geq d_1$. Similarly, $d_0 \geq d_2$. Thus, $d = d_0 \vee d_3$. Now it is easy to check that $A \geq B$, since $(a \wedge b) \vee c \geq (a \vee c) \wedge (b \vee c)$ and

$$\begin{aligned} &([(x\rho_b \wedge y\rho_a)\rho_{c'} \wedge z] \vee (x\rho_b \wedge y\rho_a \wedge z\rho_{(a\wedge b)'}))\rho_{(a\vee c)\wedge(b\vee c)} \\ &= (x\rho_b\rho_{a\vee c}\rho_{c'} \wedge y\rho_a\rho_{b\vee c}\rho_{c'} \wedge z\rho_{a\vee c}\rho_{b\vee c}) \vee (x\rho_b\rho_{a\vee c} \wedge y\rho_a\rho_{b\vee c} \wedge z\rho_{a'\vee b'}\rho_{a\vee c}\rho_{b\vee c}) \\ &= (x\rho_b\rho_{a\wedge c'} \wedge y\rho_a\rho_{b\wedge c'} \wedge z) \vee (x\rho_b\rho_{a\vee c} \wedge y\rho_a\rho_{b\vee c} \wedge z\rho_{a'\vee b'}) \\ &= (x\rho_{b\wedge c'} \wedge y\rho_{a\wedge c'} \wedge z) \vee (x\rho_b \wedge y\rho_a \wedge z\rho_{a'\vee b'}) \\ &= d_0 \vee (x\rho_b \wedge y\rho_a \wedge z\rho_{a'\vee b'}) \geq d_0 \vee d_3 = d. \end{aligned}$$

Hence, by (4.2), $A \geq B$. Since $A \leq B$ in every lattice, we have proved that $A = B$; that is, L is distributive.

Using (4.10) and (4.11), the following formulas are easy to check:

$$(4.12) \quad \langle 1, 0 \rangle \leq \langle x, a \rangle \leq \langle 1, 1 \rangle;$$

$$(4.13) \quad \langle x, a \rangle^* = \langle 1, a' \rangle;$$

$$(4.14) \quad C(L) = \{ \langle 1, a \rangle \mid a \in C \};$$

$$(4.15) \quad D(L) = \{ \langle d, 1 \rangle \mid d \in D \};$$

$$(4.16) \quad \langle 1, a \rangle \phi^L = \{ \langle d, 1 \rangle \mid d \in a\phi \};$$

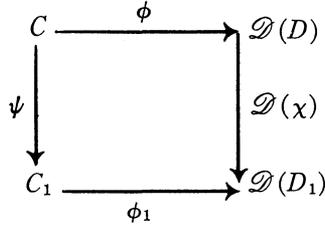
$$(4.17) \quad \langle x, a \rangle^* \vee \langle x, a \rangle^{**} = \langle 1, 1 \rangle.$$

Thus, L is a distributive lattice with 0 and 1 by (4.12), L is pseudo-complemented by (4.13), L is a Stone lattice by (4.17), and the triple associated with L is $\langle C(L), D(L), \phi^L \rangle$ as given by (4.14)–(4.16). Thus, if we identify $C(L)$ with C , and $D(L)$ with D , then by (4.16), $\phi^L = \phi$; hence, this Stone algebra satisfies the requirements of the Construction Theorem.

5. Isomorphisms and homomorphisms. The Uniqueness Theorem and the Construction Theorem can be combined to obtain the Basic Theorem which states that there is a one-to-one correspondence between Stone algebras and triples. However, this is true only up to isomorphism. Hence, to state this precisely we have to define the isomorphism of triples:

An *isomorphism* of the triples $\langle C, D, \phi \rangle$ and $\langle C_1, D_1, \phi_1 \rangle$ is a pair $\langle \psi, \chi \rangle$,

where ψ is an isomorphism of C and C_1 , χ is an isomorphism of D and D_1 , such that the diagram



is commutative.

In the diagram, $\mathcal{D}(\chi)$ stands for the isomorphism of $\mathcal{D}(D)$ and $\mathcal{D}(D_1)$ induced by χ .

BASIC THEOREM. *Two Stone algebras are isomorphic if and only if the associated triples are isomorphic. Every triple is isomorphic to a triple associated with a Stone algebra.*

This theorem follows from the Uniqueness Theorem and the Construction Theorem.

A homomorphism from the triple $\langle C, D, \phi \rangle$ into $\langle C_1, D_1, \phi_1 \rangle$ is a pair $\langle \psi, \chi \rangle$, where ψ is a homomorphism of C into C_1 , χ is a homomorphism of D into D_1 such that $1_\chi = 1$, and

$$(5.1) \quad \rho_a \chi = \chi \rho_{a\psi} \quad \text{for } a \in C.$$

THEOREM 1. *Let L and L_1 be Stone algebras, $\langle C, D, \phi \rangle$ and $\langle C_1, D_1, \phi_1 \rangle$ the associated triples, respectively. Let α be a homomorphism of L into L_1 , and α_C, α_D the restriction of α to C and D , respectively. Then $\langle \alpha_C, \alpha_D \rangle$ is a homomorphism of the triples. Conversely, let $\langle \psi, \chi \rangle$ be a homomorphism of the triples. For $x \in L$, define*

$$(5.2) \quad x\alpha = x^{**}\psi \wedge (x \vee x^*)\chi.$$

Then α is a homomorphism of L into L_1 , and $\alpha_C = \psi, \alpha_D = \chi$.

In other words, homomorphisms of Stone algebras are the same as homomorphisms of triples.

Proof. To prove the first statement we have to verify (5.1) with $\psi = \alpha_C, \chi = \alpha_D$. Then (5.1) reads:

$$(5.3) \quad x\rho_a\alpha = (x\alpha)\rho_{a\alpha}.$$

Recall that $x\rho_a = x \vee a^*$, $(x\alpha)\rho_{a\psi} = (x\alpha) \vee (a\psi)^*$, hence

$$x\rho_a\alpha = (x \vee a^*)\alpha = x\alpha \vee a^*\alpha = x\alpha \vee (a\alpha)^* = (x\alpha)\rho_{a\alpha}.$$

Conversely, let (5.1) hold. We represent the elements of L and L_1 as in (4.1). Then the definition of α reads:

$$\langle x, a \rangle \alpha = \langle x\chi, a\psi \rangle.$$

Obviously, $\alpha_C = \psi, \alpha_D = \chi$. To verify that α is a homomorphism we have to verify the following three formulas:

$$(5.4) \quad \langle x, a \rangle \wedge \langle y, b \rangle \alpha = \langle x, a \rangle \alpha \wedge \langle y, b \rangle \alpha,$$

$$(5.5) \quad \langle x, a \rangle \vee \langle y, b \rangle \alpha = \langle x, a \rangle \alpha \vee \langle y, b \rangle \alpha,$$

$$(5.6) \quad \langle x, a \rangle^* \alpha = (\langle x, a \rangle \alpha)^*.$$

Using (4.10), (4.11), and (4.13), these are equivalent to

$$(5.7) \quad (x\rho_b \wedge y\rho_a)\chi = (x\chi)\rho_{b\psi} \wedge (y\chi)\rho_{a\psi},$$

$$(5.8) \quad ((x\rho_{b'} \wedge y) \vee (x \wedge y\rho_{a'}))\chi = ((x\chi)\rho_{(b\psi)'} \wedge y\chi) \vee (x\chi \wedge (y\chi)\rho_{(a\psi)'}),$$

together with the fact that ψ and χ are homomorphisms and $1\chi = 1$. (5.7) and (5.8) are trivial by (5.1). This completes the proof of Theorem 1.

COROLLARY. *A homomorphism $\alpha: L \rightarrow L_1$ is onto (one-to-one) if and only if the restriction of α to $C(L)$ and $D(L)$ are onto (one-to-one).*

In conclusion, we mention that (5.1) is equivalent to

$$(5.9) \quad a\phi \subseteq a\psi\phi_1 \quad \text{for } a \in C.$$

In some applications, (5.9) is more convenient to use.

6. Fill in theorems. For a given Boolean algebra C , distributive lattice D with 1, when does there exist a ϕ such that $\langle C, D, \phi \rangle$ is a triple?

THEOREM 3. *$\langle C, D, ? \rangle$ can always be filled in to make it a triple if C is a Boolean algebra and D a distributive lattice with 1, provided $|C| > 1$. If $|C| = 1$, then $|D| = 1$.*

In other words, the centre and the dense set are *independent*.

Proof. Trivial: we take an arbitrary prime ideal P of C and set $x\phi = D$ for $x \notin P$ and $x\phi = \{1\}$ for $x \in P$.

Triples, associated with subalgebras, can easily be determined.

LEMMA 1. *Let L_1 be a subalgebra of the Stone algebra L . Then $C_1 = L_1 \cap C(L)$ is a subalgebra of $C(L)$ and $D_1 = L_1 \cap D(L)$ is a sublattice of $D(L)$ containing 1. The triple associated with L_1 is $\langle C_1, D_1, \phi_1 \rangle$, where ϕ_1 is given by $a\phi_1 = a\phi \cap D$, for $a \in C_1$.*

By an easy computation, one obtains the proof.

The converse of Lemma 1 is again a fill in theorem.

THEOREM 4. *Let L be a Stone lattice, C_1 a subalgebra of $C(L)$, and D_1 a sublattice of $D(L)$ containing 1. We can fill in $\langle C_1, D_1, ? \rangle$ such that it will become the triple associated with a subalgebra of L if and only if*

$$(6.1) \quad d\rho_a^L \in D_1 \quad \text{for } d \in D_1, a \in C_1.$$

Proof. If L_1 is a subalgebra, and $\langle C_1, D_1, \phi_1 \rangle$ the associated triple, then for $d \in D_1, a \in C_1$, we have (see the comment following (3.8)):

$$d\rho_a^L = d \vee a^* \in L_1,$$

and

$$(d \vee a^*)^* \stackrel{(2.6)}{=} d^* \wedge a^{**} = 0 \wedge a^{**} = 0,$$

thus $d\rho_a^L \in L_1 \cap D(L) = D_1$, verifying (6.1).

Now, let us assume (6.1), and define $\phi_1: C_1 \rightarrow \mathcal{D}(D_1)$ by $a\phi_1 = a\phi^L \cap D_1$, for $a \in C_1$. Obviously, ϕ_1 preserves 0 and 1. Furthermore, for $a, b \in C_1$,

$$(a \wedge b)\phi_1 = (a \wedge b)\phi^L \cap D_1 = (a\phi^L \cap b\phi^L) \cap D_1 = a\phi_1 \cap b\phi_1.$$

Finally, for $a, b \in C_1$,

$$\begin{aligned} (a \vee b)\phi_1 &= (a \vee b)\phi^L \cap D_1 = (a\phi^L \vee b\phi^L) \cap D_1 \\ &\supseteq (a\phi^L \cap D_1) \vee (b\phi^L \cap D_1) = a\phi_1 \vee b\phi_1; \end{aligned}$$

now if $x \in (a \vee b)\phi_1$, then

$$x \in (a \vee b)\phi^L = (a \wedge (b \wedge a'))\phi^L = a\phi^L \vee (b \wedge a')\phi^L,$$

and therefore $x = x_a \wedge x_b$, with $x_a \in a\phi^L, x_b \in (b \wedge a')\phi^L = b\phi^L \cap a'\phi^L$, where $x_a = x\rho_a^L$ and $x_b \in x\rho_{a'}^L$. Thus, by (6.1), $x_a \in D_1, x_b \in D_1$; hence, we obtain $x = x_a \wedge x_b \in (a\phi^L \cap D_1) \vee (b\phi^L \cap D_1) = a\phi_1 \vee b\phi_1$, completing the proof of Theorem 4.

The connection of homomorphism and triples was completely cleared up in § 5. A corresponding “fill in” problem is given by the following situation:

$$\begin{array}{ccc} \langle C, D, \phi \rangle & & \\ f \downarrow & & \downarrow g \\ \langle C_1, D_1, ? \rangle & & \end{array}$$

where f and g are onto homomorphisms.

THEOREM 5. *We are given the triple $\langle C, D, \phi \rangle$, the “defective” triple $\langle C_1, D_1, ? \rangle$, and a pair of onto homomorphisms $f: C \rightarrow C_1, g: D \rightarrow D_1$ (preserving 1). There exists a ϕ_1 , making $\langle C_1, D_1, \phi_1 \rangle$ a triple, and $\langle f, g \rangle$ a homomorphism of the associated Stone algebras, if and only if*

$$(6.2) \quad (a\phi)g = \{1\} \quad \text{for all } a \in 0f^{-1}.$$

Proof. The necessity of (6.2) follows either by direct computation or by using the commutative diagram of § 5 and the fact that $0\phi_1 = \{1\}$.

To prove the sufficiency, we first verify that (6.2) implies the following condition:

$$(6.3) \quad af = bf \text{ implies } (a\phi)g = (b\phi)g \quad \text{for } a, b \in C.$$

Indeed, $af = b\phi$ implies $(a \wedge b')f = af \wedge b'f = 0$; thus by (6.2), and $a \wedge b' \in 0f^{-1}$, we obtain $(a \wedge b')\phi g = \{1\}$, that is, $a\phi g \wedge b'\phi g = \{1\}$. Since $b\phi g$ is the complement of $b'\phi g$ in $\mathcal{D}(D_1)$, we conclude that $a\phi g \subseteq b\phi g$. By symmetry, $a\phi g \supseteq b\phi g$, and we have proved (6.3).

For $b \in C_1$, set $b\phi_1 = a\phi g$, where $b \in a\phi$. By (6.3), ϕ_1 is well-defined. The remainder of the proof is routine.

Finally, we state, without proof, a theorem, describing the connection of triples and direct products.

THEOREM 6. *Let $\langle C_i, D_i, \phi_i \rangle$ be the triple associated with the Stone algebra L_i , $i = 0, 1, 2$. Then $L_0 = L_1 \times L_2$ implies that*

$$(6.4) \quad C_0 = C_1 \times C_2 \quad \text{and} \quad D_0 = D_1 \times D_2,$$

and

$$(6.5) \quad \langle a_1, a_2 \rangle \phi_0 = a_1 \phi_1 \times a_2 \phi_2.$$

Conversely, if for $\langle C_0, D_0, \phi_0 \rangle$, and for the “defective” triples $\langle C_i, D_i, ? \rangle$, $i = 1, 2$, (6.4) holds, then (and only then) the “defective” triples can be filled in by ϕ_1, ϕ_2 , such that $L_0 = L_1 \times L_2$ holds for the associated Stone algebras.

Remark. The statements of the theorem are true “up to isomorphism”.

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