DERIVABLE NETS[1)]

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Up to a duality, the known finite projective planes which are not translation planes all are equivalent to affine planes which contain the type of structure defined below to be a "derivable net". (Insofar as the known finite planes are concerned, this means that the intimate connection between projective geometry and linear algebra still holds for non-Desarguesian planes.)

An affine plane which contains a derivable net is called a derivable plane. (The precise sense in which we are using the word "contain" will be explained below.) There has been no complete determination of all derivable planes. A step in the derivation of such a determination would be the determination of all transversals of a derivable net. One of the consequences of our results is that finding all of the transversals of a derivable net is related to finding the dual translation planes which contain the net and are of dimension two in the sense of Andre' [1].

A collineation of a derivable plane which carries the included derivable net into itself will induce a collineation of the net. A large class of derivable planes has the property that its members can admit no collineations which do not induce collineations of the corresponding derivable nets. For such planes, the collineation group of the plane is a subgroup of the collineation group of the derivable net. We have determined the complete collineation group of a derivable net.

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Thus much of the theory of derivable planes depends upon the theory of derivable nets and their transversals. Besides the results already indicated (and related to them) we obtain representations of derivable nets and their transversals in terms of Andre''s congruence [1] and Bruck's spreads [2, 3].

In general, a net of order n and degree k is a set of n^2 points and kn lines with n points on each line. The lines fall into k parallel classes, so that the two lines in the same parallel class have no points in common, while lines in different parallel classes have exactly one point in common. Each point lies on exactly one line of each parallel class; two points lie on at most one line. A net of order n and degree $n + 1$ is an affine plane.

Definition. A net of order q^2 and degree $q + 1$ is said to be derivable if and only if it is isomorphic to the net N described below.

Let F be a field of order q. Let V be a two-dimensional vector space over F. The points of N are ordered pairs (x, y) where x and y belong to V. For each α in F and each b in V, the set of points (x, y) such that $y = x\alpha + b$ is a line of N. (The indicated operations are vector addition and multiplication by a scalar.) For each c in V, the set of points such that $x = c$ is a line of N. There are no other lines of N.

It will be convenient to let $t, 1$ represent basis elements of V, so that each element of V may be written in the form $t\alpha + \beta$, where α, β belong to F and each point (x, y) may be written $(tx_1 + x_2, ty_1 + y_2)$ where x_1, x_2, y_1, y_2 belong to F. In general, we shall denote elements of F either by small Greek letters or by x_1, x_2, y_1, y_2.

We shall say that N is contained (or included) in an affine plane π if (1) The points of N and π are identical and (2) Every line of N is a line of π.

A transversal of N is a set of q^2 distinct points, no two of which lie on a line of N. Thus if π includes N, every line of π is either a line of N or a transversal of N.

Definition. Let $x \to f(x)$ be a 1 - 1 correspondence between the elements of V. Then we shall say that f(x) is a transversal function if, for every distinct c, d in V the vectors f(c) - f(d) and c - d are independent.

THEOREM 1. If T is any transversal of N, then there is a transversal function f(x) such that T consists of the set of points (x, y) for which $y = f(x)$. Conversely, every transversal function defines a transversal.

Proof. There is a natural 1 - 1 correspondence between the lines x = constant and y = constant such that two lines correspond if their point of intersection is on T. Thus we have a 1 - 1 mapping $x \to f(x)$ such that (x, y) is on T if and only if $y = f(x)$. Now two points (c, d) and (r, s) with $c \neq r$ are on a line $y = x\alpha + b$ if and only if $d - s = (c - r)\alpha$. Thus f(c) - f(r) and c - r must be independent if $y = f(x)$ is to be a transversal. Conversely, if f(c) - f(r) and c - r are independent for all choices of c, r $c \neq r$ and f is 1 - 1, then $y = f(x)$ must be a transversal.

LEMMA 1. If f(x) is a transversal function, then $f(x)\alpha + x\beta + b$ is a transversal function for each non-zero α in F, each β in F, and each b in V.

Proof. The proof is almost immediate from the definition of a transversal function and the properties of a vector space.

THEOREM 2. If T is a transversal of N, then there is an affine plane π which contains N and has T as one of its lines.

Proof. The points of π will, of course, be the points of N. The lines of π will be the lines of N together with the transversals given by the transversal functions $f(x)\alpha + x\beta + b$.

Let us examine the intersections of two such transversals:

$$y = f(x)\alpha + x\beta + b \quad \text{and} \quad y = f(x)\gamma + x\delta + c .$$

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The x coordinate of any point of intersection must satisfy

$$f(x)(\alpha - \gamma) + x(\beta - \delta) + (b - c) = 0 .$$

If $\alpha = \gamma$ and $\beta = \delta$ there is no solution unless $b = c$ - i.e. the lines are parallel. If $\alpha \neq \gamma$ the left hand side of the above equation is a transversal function and there must be exactly one value of x for which this transversal function takes on the value zero. If $\alpha = \gamma$ but $\beta \neq \delta$, we obviously get a unique solution for x.

Thus we get one parallel class for each choice of $\alpha \neq 0$ and β, which gives us $q^2 - q$ parallel classes. The $q + 1$ parallel classes in N give us a total of $q^2 + 1$ parallel classes. Lines in different parallel classes intersect in exactly one point. The q^2 distinct lines of any one given parallel class contain all of the points of the plane, so each point lies on exactly one line of each parallel class and each pair of distinct points lies on exactly one line of π. It follows that we do indeed have an affine plane.

THEOREM 3. Let Z be the two-by-two matrix with y_1, y_2 in the first row, x_1, x_2 in the second row. Let A_1 and A_2 be non-singular two-by-two matrices and let A_3 be any two-by-two matrix. Let $Z' = A_1 Z A_2 + A_3$. Then there is a collineation of N such that the general point $(tx_1 + x_2, ty_1 + y_2)$ is carried into $(tx'_1 + x'_2, ty'_1 + y'_2)$ where y'_1, y'_2, x'_1, x'_2 are the elements of Z'.

Proof. The proof is a matter of direct calculation which we will leave to the reader. It may be helpful to consider the following cases:

(1) A_1 and A_2 are the identity

(2) A_2 is the identity and A_3 is the zero matrix

(3) A_1 is the identity and A_3 is the zero matrix.

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In subsequent discussion the cases (1), (2), and (3) above will be referred to as collineations of types (1), (2), or (3) respectively. Note that the collineations of type (1) are translations, while the collineations of type (2) may be written in the form $(x, y) \to (x\alpha + y\beta,\ x\gamma + y\delta)$.

The collineations of type (3) are of the general form $(x, y) \to (x\rho,\ y\rho)$, where ρ is an automorphism of addition in V.

There is one more type of mapping which will give a collineation of N and will be referred to as a collineation of type (4):

$$(tx_1 + x_2,\ ty_1 + y_2) \to (t(x_1\sigma) + x_2\sigma,\ t(y_1\sigma) + y_2\sigma),$$

where σ is an automorphism of F. The next series of Lemmas lead up to the result that every collineation of N is the product of collineations of types (1) - (4).

LEMMA 2. The set of points $(tx_1 + x_2,\ ty_1 + y_2)$ for which $x_1 = y_1 = 0$ is the set of points in a Desarguesian subplane of N. The lines of this subplane have equations of the type $y = x\gamma + \beta$ $\gamma, \beta \in F$ or $x = \alpha$, $\alpha \in F$.

<u>Proof.</u> Let π_o denote the set of points and lines in Lemma 2, then π_o is the Desarguesian plane coordinatized by F. Clearly, π_o is a subplane of N.

LEMMA 3. Let ρ be any collineation of N which fixes each point of π_o and also fixes one additional point. Then ρ is the identity.

<u>Proof.</u> If π_o is pointwise fixed, each parallel class is fixed. Suppose that $(t\alpha_1 + \beta_1,\ t\alpha_2 + \beta_2)$ (α_1 and α_2 not both zero) is fixed. Then the lines $x = t\alpha_1 + \beta_1$, $y = t\alpha_2 + \beta_2$ and their intersections with each of the lines $y = x\gamma + \beta$ $\gamma, \beta \in F$ are fixed. We soon get that all lines in the parallel classes

$x = \text{constant}$ and $y = \text{constant}$ are fixed and finally that all points are fixed.

LEMMA 4. Let ρ be any collineation of N which carries π_o into itself. Then ρ is the product of collineations of types (1) - (4).

Proof. Note that all collineations of types (2) and (4) carry π_o into itself and that the collineations of type (1) include a subgroup transitive on the points of π_o. Furthermore, π_o is Desarguesian and the full collineation group on π_o is induced by collineations of types (1), (2), and (4). Hence there is a collineation σ such that σ is the product of collineations of types (1), (2) and (4) and $\rho\sigma^{-1}$ fixes each point of π_o. Since $\rho\sigma^{-1}$ fixes the line $x = 1$, the image of $(1, t)$ under $\rho\sigma^{-1}$ must be $(1, t\alpha + \beta)$ for some α, β. Furthermore $(1, t\alpha + \beta)$ cannot be in π_o, so $\alpha \neq 0$.

Now consider the collineations of type (3) of the following special form:

$$(tx_1 + x_2, ty_1 + y_2) \to (tx_1\alpha + x_1\beta + x_2, ty_1\alpha + y_1\beta + y_2) .$$

Note that π_o is pointwise fixed, while $(1, t) \to (1, t\alpha + \beta)$. We must have $\alpha \neq 0$ for the transformation to be non-singular. Applying the previous Lemma, we conclude that there is a collineation τ of type (3) such that $\rho\sigma^{-1}\tau^{-1}$ is the identity. This proves Lemma 4.

LEMMA 5. Let P be any point joined to $(0, 0)$ by a line of N. Then there is a collineation of type (3) which carries P into some point of π_o.

Proof. If $P = (tx_1 + x_2, ty_1 + y_2)$ is joined to the origin, then either $x_1 = x_2 = 0$ or there is some λ in F such that $y_1 = x_1\lambda$, $y_2 = x_2\lambda$. Choose α and γ so that

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$x_1 \alpha + x_2 \gamma = 0 = y_1 \alpha + y_2 \gamma$. Then choose the (non-singular) matrix A_2 so that α and γ are the elements of the first column.

THEOREM 4. Every collineation of N is the product of collineations of types (1) - (4).

Proof. Let ρ be any collineation of N. Let ρ carry the origin into the point P, (0, 1) into the point Q. Then there is a collineation σ which is the product of collineations of types (1) - (3) such that $\rho\sigma^{-1}$ fixes (0, 0) and (0, 1). Since π_0 is the only subplane of order q which contains (0, 0) and (0, 1), $\rho\sigma^{-1}$ carries π_0 into itself. Hence $\rho\sigma^{-1}$ is the product of collineations of types (1) - (4). Finally, ρ itself is the product of collineations of types (1) - (4).

Representation in Terms of Spreads and Congruences

The following representation is essentially due to Bruck. We shall give what amounts to a restatement of some of the material in his Saskatoon lectures [2] so as to clarify the relations to the specific form of representation of derivable nets as given earlier in this paper. As an example of the advantages of looking at things in more than one way, we note that Theorems 2 and 3 were discovered by the author in studying the representation to follow.

Let V_4 denote the vector space consisting of quadruples of elements of F. Let W be the two dimensional subspace of V_4 generated by (1, 0, 0, 0) and (0, 1, 0, 0). Let us set up a 1 - 1 correspondence between the points of N and two dimensional subspaces of V_4 independent of W as follows:

The point $(tx_1 + x_2, ty_1 + y_2)$ corresponds to the two-space generated by $(x_1, x_2, 0, 1)$ and $(y_1, y_2, 1, 0)$. These two-spaces independent of W will be called "point-spaces".

Consider the set of points on the line $y = x\gamma + b$, where $b = t\beta_1 + \beta_2$. This amounts to having $y_1 = x_1\gamma + \beta_1$, $y_2 = x_2\gamma + \beta_2$. The point-spaces corresponding to points on $y = x\gamma + b$ are precisely those which contain the one-dimensional subspace of V_4 generated by $(\beta_1, \beta_2, 1, -\gamma)$. The set of points on the line $x = t\alpha_1 + \alpha_2$ corresponds to a set of point-spaces intersecting in the one-space with basis $(\alpha_1, \alpha_2, 0, 1)$. Note that every one-space independent of W has a basis of the form $(\beta_1, \beta_2, 1, -\gamma)$ or $(\alpha_1, \alpha_2, 0, 1)$.

We have established the following theorem:

THEOREM 5. Let W be a fixed two-dimensional subspace of V_4. Let the two-spaces independent of W be called point-spaces; let the one-spaces independent of W be called line-spaces. Then there is a one-to-one correspondence between the points of N and the point-spaces, between the lines of N and the line spaces. The correspondence is such that the set of points on a line corresponds to the set of point-spaces including a line-space.

COROLLARY 1. Two points of N are not joined by a line of N if and only if the corresponding point-spaces are independent.

Proof. Two distinct two-dimensional subspaces of V_4 either are independent (i.e., have only the identity in common) or intersect in a one-space. In the latter case, the one-space must be independent of W (and thus a line-space) if the two-spaces are independent of W.

COROLLARY 2. Corresponding to each transversal of N, there is a set of $q^2 + 1$ two-dimensional subspaces of V_4 which are pairwise independent.

Proof. A transversal to N is a set of $n = q^2$ points of N, no two of which are joined by a line of N. The corresponding point-spaces, together with W, form a set of $q^2 + 1$ two-spaces which are pairwise independent.

We now wish to look at Andre' s "congruence" [1] and the "spreads" of Bruck and Bose [2, 3].

Definition (Andre'). A congruence of a group G is a class of subgroups of G (called the components of the congruence) such that (1) Each element of G distinct from the identity belongs to exactly one component and (2) Each pair of distinct components generates the whole group.

Definition. (Bruck and Bose) [3]. Let Σ be a projective space of dimension $2r - 1$. Let S be a collection of $(r-1)$-dimensional projective subspaces of Σ. Then S is a spread if each point of Σ is contained in exactly one member of S.

Now for $r \geqq 2$, Σ can be represented by a vector space of dimension $2r$. The points of Σ are represented by one-dimensional subspaces of the vector space, the $(r-1)$-dimensional projective subspaces are represented by r-dimensional vector spaces. Thus disjoint $(r-1)$-dimensional projective spaces correspond to vector spaces having only the zero vector in common. Two r-dimensional vector spaces which have only the zero vector in common will be independent and will generate the whole vector space. On the other hand, Andre' has shown that any group which admits a congruence must be isomorphic to a vector space of even dimension over some skew field. Thus the concepts of spread and congruence are exactly equivalent.

COROLLARY 3. The two-spaces of Corollary 2 are the components of a congruence on V_4. Each transversal of N corresponds to a spread in three-dimensional projective space over F.

THEOREM 6. Associated with each transversal T to N is a translation plane π. The points of π are the elements of V_4. The lines of π consist of the subspace W, the point-spaces corresponding to the points of T, and the cosets of these subspaces.

Proof. Theorem 6 comes immediately from Andre''s connection between translation planes and congruences [1]. We note in passing that the relation between spreads and trans-

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lation planes is essentially equivalent to the relation between congruences and translation planes.

THEOREM 7. The translation plane of Theorem 6 is dual to the plane of Theorem 2.

Proof. Let π_1 be the plane of Theorem 2 and let π_2 be the plane of Theorem 6. (Strictly speaking, what we have is a duality between the projective versions of these two planes.)

We shall set up a 1 - 1 mapping between the q^4 affine points of π_2 and a set of q^4 lines of π_1. We shall show that this mapping preserves incidence in the sense that collinear affine points map into concurrent (or parallel) lines. We shall then be able to extend our mapping so that the points at infinity of π_2 are mapped into the remaining lines (including the line at infinity) of π_1.

First, however, we must obtain a more explicit representation for the lines of π_2. As in Theorem 2, let the transversal T be represented by $y = f(x)$ and let $f(x) = tf_1(x) + f_2(x)$, where f_1 and f_2 belong to F. Let $c = t\gamma_1 + \gamma_2$. Consider the component of the congruence (line of π_2) corresponding to the point $(c, f(c))$ of π_1. This is the two-space generated by $(\gamma_1, \gamma_2, 0, 1)$ and $(f_1(c), f_2(c), 1, 0)$. It is readily established that an element $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of V_4 belongs to this component if and only if $t\alpha_1 + \alpha_2 = c\alpha_4 + f(c)\alpha_3$. For each fixed $(\delta_1, \delta_2, 0, 0)$ there will be a coset such that $t\alpha_1 + \alpha_2 = c\alpha_4 + f(c)\alpha_3 + d$, where $d = t\delta_1 + \delta_2$. Thus, for each c, d the set of vectors $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ such that $t\alpha_1 + \alpha_2 = c\alpha_4 + f(c)\alpha_3 + d$ is a line of π_2. For each fixed c, the q^2 possible choices for d will determine a complete set of q^2 lines in a parallel class.

Besides this, we have the lines consisting of the component

W and its cosets. W consists of those vectors in V_4 for which $\alpha_3 = \alpha_4 = 0$. We will have a coset (parallel line) for each fixed choice of α_3 and α_4.

We now look at the following mapping from the affine points of π_2 to the lines of π_1:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \to y = f(x)\,\alpha_3 + x\alpha_4 - (t\alpha_1 + \alpha_2) .$$

The set of points of π_2 such that $t\alpha_1 + \alpha_2 = c\alpha_4 + f(c)\,\alpha_3 + d$ thus maps into a set of lines of π_1 having the general form

$$y = f(x)\,\alpha_3 + \alpha_4 - [c\alpha_4 + f(c)\,\alpha_3 + d] .$$

These lines all go through the point $(c, -d)$ in π_1. With the exception of W and its cosets, we have verified that collinear points do indeed map into concurrent lines.

Each point $(\delta_1, \delta_2, 0, 0)$ in W maps into a line $y = -(t\delta_1 + \delta_2)$. The set of points in W maps into the set of parallel lines of the form y = constant, which may be thought of as intersecting in a point at infinity.

If $(\delta_1, \delta_2, \lambda_1, \lambda_2)$ and $(\gamma_1, \gamma_2, \lambda_1, \lambda_2)$ are two points in the same coset of W, their images are the parallel lines

$$y = f(x)\,\lambda_1 + x\lambda_2 - (t\delta_1 + \delta_2)$$

and

$$y = f(x)\,\lambda_1 + x\lambda_2 - (t\gamma_1 + \gamma_2) .$$

Thus again, the set of points on any line represented by a coset of W will map into a parallel class - i.e., the set of lines intersecting in some point at infinity.

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In effect, we have shown that our mapping induces a mapping from the affine lines of π_2 to the points of π_1. Notice that the parallel class of the line of π_2 given by $t\alpha_1 + \alpha_2 = c\alpha_4 + f(c)\,\alpha_3 + d$ is determined by c. Thus we may map the corresponding point at infinity into the line $x = c$ in π_1. Finally, we map the line at infinity of π_2 into the point at infinity common to all of the lines $x = \text{constant}$ in π_1 and the point at infinity common to W and its cosets into the line at infinity of π_1. Our duality is then completely established.

COROLLARY. The plane of Theorem 2 is a dual translation plane.

Remarks: While we have shown that each transversal to N can be embedded as a line in a dual translation plane which contains N, it is by no means true that every plane containing N is a dual translation plane. Indeed, it follows from a construction due to the author [4] that every transversal to N can also be embedded in a semi-translation plane which contains N.

The translation plane of Theorem 6 is actually of dimension two over what Andre' [1] calls the kernel (Kern) and what Bruck and Bose [3] call the left operator skewfield. We do have, then, a complete equivalence between these two problems: (1) Determine all transversals of a derivable net, (2) Determine all translation planes which are of dimension two over their respective kernels. The equivalence between transversals arising out of the collineations of N may be more convenient to handle than the related problem of different coordinate systems for the same translation plane.

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