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RESEARCH ARTICLE

On sets with unit Hausdorff density in homogeneous groups

Antoine Julia [1] and Andrea Merlo [2]

[1] Département de Mathématiques d'Orsay, Université Paris-Saclay, 91400 Orsay, France;
E-mail: antoine.julia@universite-paris-saclay.fr.
[2] Departamento de Matemáticas, Universidad del País Vasco, Barrio Sarriena s/n, 48940 Leioa, Spain;
E-mail: andrea.merlo@ehu.eus.

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Abstract
It is a longstanding conjecture that given a subset E of a metric space, if E has unit $\mathscr{H}^\alpha \llcorner E$-density almost everywhere, then E is an α-rectifiable set. We prove this conjecture under the assumption that the ambient metric space is a homogeneous group with a *smooth-box* norm.

Contents

1. Introduction

A subset E of a metric space (X, d) is m**-rectifiable** if it can be covered up to an $\mathscr{H}^m$-null set, where here, as usual, $\mathscr{H}^\alpha$ denotes the α-dimensional Hausdorff measure, see [9, Section 2.10.2], by a countable union of Lipschitz images of compact subsets of $\mathbb{R}^m$. It was proved by Kirchheim in [13] that such a set E has **Hausdorff density** equal to one at almost every point, that is:

$$\Theta^m(E, x) := \lim_{r\to 0} \frac{\mathscr{H}^m(E \cap B(x, r))}{(2r)^m} = 1,$$

for $\mathscr{H}^m$-almost all $x \in E$, where $B(x, r)$ denotes the closed unit ball in (X, d) with centre x and radius r. It is natural to ask whether the converse is true. Namely, does the unit Hausdorff density property imply rectifiability of E? A positive answer, assuming that (X, d) is Euclidean space, was given by Mattila in the fundamental work [22], following partial results by Besicovitch [4] and Marstrand [20, 21]. More precisely, the following holds:

Theorem 1.1 (Besicovitch, Marstrand, Mattila). *Fix $\alpha \geq 0$, and let $E \subset \mathbb{R}^n$ be a Borel subset of Euclidean space, such that $\Theta^\alpha(E, x) = 1$ for $\mathscr{H}^\alpha$-almost all $x \in E$. Then α is an integer and E is α-rectifiable.*

This result is surprising, in that one is able to infer a strong geometric property (the existence of flat, α-dimensional tangents to E at almost all points) from an *a priori* much less geometric condition. Note that the unit density condition can be relaxed, to bounds on the lower and upper densities of E, as was shown by Preiss in [28, Corollary 5.4(3)].

In general metric spaces, and for sets of dimension 1, the question was answered by Preiss and Tišer [29]. Their proof uses a variant of the argument by Besicovitch, which is specific to dimension 1 and the fact that a continuum with finite $\mathscr{H}^1$ measure is rectifiable. The extension to higher dimensions is still widely open, even in $\mathbb{R}^n$ with a non-Euclidean norm. A relevant partial result has been achieved by Lorent in [19], where he proved that in l^3_∞, locally uniform measures are rectifiable. More recently, the second named author in [24, 25] has proved analogues of Preiss's rectifiability theorem for 1-codimensional measures in Heisenberg groups and parabolic spaces, see [26], using a notion of rectifiability specific to those spaces. The result in [26] has been obtained in collaboration with Mourgoglou and Puliatti. In this paper, we investigate the structure of Borel sets with unit density for the Hausdorff measure in homogeneous groups (a class of groups that includes Carnot group, see Section 2 for definitions) and we prove:

Theorem 1.2. *Given a homogeneous group $\mathbb{G}$, there is a homogeneous norm $\|\cdot\|$ on $\mathbb{G}$, for which, whenever $\alpha \geq 0$ and $E \subset \mathbb{G}$ is such that $\mathscr{H}^\alpha(E) < +\infty$, the following are equivalent*

(i) *$\alpha \in \mathbb{N}$ and E is covered up to an $\mathscr{H}^\alpha$-null set by countably many Lipschitz images of compact subsets of $\mathbb{R}^\alpha$;*
(ii) *$\Theta^\alpha(E, x) = 1$ for $\mathscr{H}^\alpha$-almost all $x \in E$.*

Let us briefly discuss the result and its connection with the existing literature. In the Heisenberg groups and in the parabolic spaces [24, 25, 26], the objects that solve the 1-codimensional density problem are not Euclidean rectifiable in general. In both cases, they are just $1/2$-Hölder surfaces that can be approximated in the $1/2$-Hölder metric by smooth surfaces. Therefore, what our result is implicitly stating is that the Hausdorff density for such high-dimensional intrinsically rectifiable sets, with respect to the natural norm we construct, must be strictly smaller than 1, and that the sets having unit density for the Hausdorff measure must be low-dimensional. Thus, a more precise assessment of what we prove is the following:

Theorem 1.3. *Given a homogeneous group $\mathbb{G}$, there is a homogeneous norm $\|\cdot\|$ on $\mathbb{G}$, for which, whenever $\alpha \geq 0$ and $E \subset \mathbb{G}$ is a Borel set for which $\Theta^\alpha(E, x) = 1$ for $\mathscr{H}^\alpha$-almost every $x \in E$, then α is an integer and varies among the topological dimensions of the homogeneous subgroups of the first layer V_1 of $\mathbb{G}$ and E is α-rectifiable.*

For instance, in the first Heisenberg group $\mathbb{H}^1$, the only possible values for α are 0 and 1, while the Hausdorff dimension of $\mathbb{H}^1$ is 4. This means, in particular, that on high-dimensional (intrinsic) rectifiable sets in Carnot groups, the Hausdorff measure and the spherical Hausdorff measure do not coincide, which is in stark contrast with what happens in Euclidean space (see, for instance, [9, Theorem 3.2.26]).

Our proof relies on the following observations. First, the Carathéodory construction which yields the measure $\mathscr{H}^\alpha$ is obtained by coverings with *arbitrary closed sets*. This roughly means that if the $\mathscr{H}^\alpha$-measure of a set E is (σ-)finite, given a set S of very small diameter, say $2r$, containing an $x \in E$, one must have $\mathscr{H}^\alpha(E \cap S) \lesssim (2r)^\alpha$ (see Theorem 2.5). On the other hand, item (ii) of Theorem 1.2 implies also $\mathscr{H}^\alpha(B(x,r) \cap E) \gtrsim (2r)^\alpha$. Thus, if there exists a set G strictly containing the unit ball but with diameter no larger than 2, then $\mathscr{H}^\alpha(x * \delta_r(G) \setminus B(x,r))/(2r)^\alpha$ must be very small (this assumption implies, in particular, that the unit ball is not isodiametric). The second observation is that for specific choices of the metric, such a set G exists and can be chosen in a way that the above constraints force the set E with unit Hausdorff density to be concentrated along the horizontal directions, namely, along V_1. More precisely, using the first observation at small scales, we prove that E can be covered by $(1+\varepsilon)$-Lipschitz graphs above V_1 for ε arbitrarily small. Together with the fact that the projection π_1 of $\mathbb{G}$ onto the space V_1 is 1-Lipschitz, this implies that the projection $\pi_1(E)$ has Hausdorff density 1. As the restriction of

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our metric to V_1 is euclidean, we can apply Theorem 1.1 to show that $\pi_1(E)$ is euclidean rectifiable and, using the graph maps above, that E is as well. For the details, see the proof of Theorem 1.3 on page 10.

In the bulk of the paper, we stick to the smooth-box distance, defined by equation (4.1), as it is a natural metric which makes sense in every homogeneous group and makes the construction and computations relatively easy. A similar construction can be achieved in many other cases, such as for the Koranyi norm in the first Heisenberg group, see Proposition 6.1, and similarly to H-type groups endowed with the gauge norm. On the other hand, we stress that our construction does not work for the Carnot-Carathéodory distance in the Heisenberg group, even though the unit ball for this distance is also far from being isodiametric.

The isodiametric problem in a homogeneous group consists in maximising the volume over all compact sets with diameter at most 2. As discussed above, this problem is deeply connected to that of characterising sets with unit Hausdorff density (see Theorem 3.4). In a finite dimensional vector space, it is known that the unit ball is the unique maximiser, up to translations [23]. If $\mathbb{G}$ is a homogeneous group which is not a Banach space, we expect that the unit ball is not isodiametric for any homogeneous norm. Indeed, in homogeneous groups of step 2 or above, there always exists a distance for which the unit ball is not isodiametric, see [30], and there are many examples of nonisodiametric balls in homogeneous groups [18, 27]. The only groups in which we know of a general proof are those in which the diameter of the unit ball is strictly smaller than 2, such as $\mathbb{R}$ with the snowflake distance $|\cdot|^{1/2}$. The problem of checking whether in step higher than 2 every ball is not isodiametric is hard already in the parabolic plane. One can prove that a vast array of metrics have a ball which is not isodiametric. However, to provide an argument working for any norm, one soon finds technical obstructions. One of the first obstacles is that there are norms in the parabolic plane for which every point of the boundary of the unit ball lies at distance 2 from another, so that one cannot 'add a small set' to the ball, as one does for more usual metrics. Examples of this phenomenon are the balls induced by the norms

$$\|(x,t)\|_1 := |x|+|t|^{1/2} \quad \text{(see Remark 3.20 in [27])}$$
$$\|(x,t)\|_\# := \max\{|x|, |t|^{1/2}-\operatorname{sign}(t)x\}.$$

It is worth noting that our result is also relevant when passing from homogeneous groups to the more general context of metric spaces. In [16] and [17, Theorem 1.6], it is proved that if a doubling metric measure space has unique tangents at almost every point, then almost all these tangents are metric Lie groups admitting dilations (and if the space is geodesic, then the Lie groups are Carnot groups). These groups are thus 'universal tangents', and since density and rectifiability are infinitesimal properties of metric spaces, one could expect that if the convergence to the tangent is 'fast enough', our result could be used to prove by blowup the conjecture in a bigger class of metric spaces. The class of metric groups with dilations is larger than that of homogeneous groups, but we prove in Section 5, that any metric group with dilation is biLipschitz equivalent to a group with a slightly more general smooth-box distance for which the unit ball is not isodiametric and the conclusion of Theorem 1.2 holds, meaning that nothing too exotic is to be expected, even though as mentioned above, biLipschitz changes of norm are notoriously difficult to handle for density questions.

The structure of this paper is the following: Section 2 contains notations and basic results on homogeneous groups and measures. Section 3 is devoted to a comparison of the notions of surface measures in Carnot groups. Section 4 contains the proof of Theorem 1.3. In section 5, we address the more general notion of groups with dilations. Finally, in Section 6, we focus on the first Heisenberg group with the Koranyi norm and discuss the full resolution of the density problem in that setting.

2. Preliminaries

We recall that a *positive grading* of a Lie algebra $\mathfrak{g}$ is a direct-sum decomposition $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$, for some integer $s \geq 1$, where $V_s \neq 0$ and $[V_1, V_j] \subseteq V_{j+1}$ for all integers $j \in \{1, \ldots, s\}$ and where we set $V_{s+1} = 0$. A positive grading is said to be a *stratification,* if $[V_1, V_j] = V_{j+1}$ for all $j \in \{1, \ldots, s\}$.

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We also recall that the first layer V_1 of a stratification is usually referred to as the *horizontal layer*. A homogeneous group $\mathbb{G}$ of step s is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a positive grading $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$. See [15], for precise definitions, in particular, it can be decomposed as a direct sum:

$$\mathbb{G} = V_1 \oplus \cdots \oplus V_\kappa,$$

where the finite dimensional vector spaces V_ℓ are called **layers** and κ is the **step** of $\mathbb{G}$. It will always be assumed that $V_1 \neq \{0\}$ (see Remark 2.6). Such a group can be endowed with a family of dilations $(\delta_\lambda)_{\lambda>0}$ defined by:

$$\delta_\lambda(x_1, x_2, \ldots, x_\kappa) = (\lambda x_1, \lambda^2 x_2, \ldots, \lambda^\kappa x_\kappa).$$

By the Baker-Campbell-Hausdorff formula, up to diffeomorphisms, the group operation can be written as follows, for $p, q \in \mathbb{G}$ and $\ell \in \{1, \ldots, \kappa\}$:

$$p * q = p + q + Q(p, q) \quad \text{for all } \; p, q \in \mathbb{R}^n,$$

where $Q = (Q_1, \ldots, Q_s) : \mathbb{R}^n \times \mathbb{R}^n \to V_1 \oplus \ldots \oplus V_s$, and the Q_is have the following properties. For any $i = 1, \ldots s$ and any $p, q \in \mathbb{G}$, we have:

(i) $Q_i(\delta_\lambda p, \delta_\lambda q) = \lambda^i Q_i(p, q)$,
(ii) $Q_i(p, q) = -Q_i(-q, -p)$,
(iii) $Q_1 = 0$ and $Q_i(p, q) = Q_i(p_1, \ldots, p_{i-1}, q_1, \ldots, q_{i-1})$.

Definition 2.1 (Homogeneous norm, homogeneous distance). A distance $d : \mathbb{G} \times \mathbb{G} \to \mathbb{R}$ is **homogeneous** and **left-invariant** if for any $x, y \in \mathbb{G}$, there holds

(i) $d(\delta_\lambda x, \delta_\lambda y) = \lambda d(x, y)$ for any $\lambda > 0$,
(ii) $d(z * x, z * y) = d(x, y)$ for any $z \in \mathbb{G}$.

It is equivalent to define a homogeneous distance d and a **homogeneous norm** $||\cdot||$, as can be seen by writing

$$||x|| := d(0, x), \text{ or conversely } d(x, y) := ||x^{-1} * y||.$$

In the following, d is a left-invariant homogeneous metric on $\mathbb{G}$ and the corresponding closed ball with centre $x \in \mathbb{G}$ and radius $r > 0$ is denoted by $\mathrm{B}_d(x, r)$. We will drop the subscript d when there is no ambiguity. As the first layer V_1 is assumed to be nondegenerate, the balls $B_d(x, r)$ have diameter $2r$. The following result, the elementary proof of which we learned in Metelichenko's thesis [27, Lemma 3.17], will be very useful.

Proposition 2.2. *If $||\cdot||$ is a homogeneous norm on $\mathbb{G}$, then its restriction $|\cdot|$ to V_1 is a vector space norm. Also, if π_1 denotes the projection $\mathbb{G} \to V_1$, $(x_1, \ldots, x_\kappa) \mapsto x_1$, then π_1 is exactly 1-Lipschitz with the norm $|\cdot|$ on V_1.*

Proof. As $\pi_1(x_1, 0, \ldots, 0) = x_1$, the projection π_1 cannot have Lipschitz constant smaller than one. In order to prove the opposite bound, note that π_1 is a group morphism, thus, it suffices to show that $|\pi_1(x)| \leq ||x||$ for $x \in \mathbb{G}$. Fix $x = (x_1, \ldots, x_\kappa) \in \mathbb{G}$. On the one hand, there holds $||\delta_{1/2}(x * x)|| = ||x * x||/2 \leq ||x||$, and on the other hand:

$$\delta_{1/2}(x * x) = (x_1, 2^{-1} x_2, \ldots, 2^{1-\kappa} x_\kappa).$$

Iterating this operation, and noting that $\delta_{1/2^n}((x * x) * \cdots * (x * x)) \to (x_1, 0, \ldots, 0)$, one gets $||(x_1, 0, \ldots, 0)|| \leq ||x||$. Finally, given x_1 and y_1 in V_1, there holds

$$|x_1 + y_1| = ||\pi_1((x_1, 0, \ldots, 0) * (y_1, 0, \ldots, 0))|| \leq ||(x_1, 0, \ldots, 0)|| + ||(y_1, 0, \ldots, 0)|| = |x_1| + |y_1|,$$

and thus, $|\cdot|$ is a (vector-space) norm on V_1, since homogeneity and positivity are immediate. □

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Since one of the themes of the paper is the distinction between the various types of Hausdorff measures, we introduce them here.

Definition 2.3 (Hausdorff and spherical measures). For $\alpha \geq 0$, define the α-dimensional **Hausdorff measure** relative to the left-invariant homogeneous distance d

$$\mathscr{H}_d^\alpha(E) = \sup_{\delta>0} \mathscr{H}_{d,\delta}^\alpha(E) := \sup_{\delta>0} \inf\left\{ \sum_{j=1}^\infty (\operatorname{diam} S_j)^\alpha : E \subseteq \bigcup_{j=1}^\infty S_j,\ S_j \text{ closed},\ \operatorname{diam} S_j \leq \delta \right\}.$$

Along with the Hausdorff measure, it is useful to study other surface measures, we will focus on two of them: the α-dimensional **spherical measure**:

$$\mathscr{S}^\alpha(A) := \sup_{\delta>0} \inf\left\{ \sum_{j=1}^\infty (2r_j)^\alpha : A \subseteq \bigcup_{j=1}^\infty \mathrm{B}_d(x_j, r_j),\ r_j \leq \delta \right\}$$

and the α-dimensional **centred spherical measure**:

$$\mathscr{C}^\alpha(A) := \sup_{E \subseteq A} \mathscr{C}_0^\alpha(E),$$

where

$$\mathscr{C}_0^\alpha(E) := \sup_{\delta>0} \inf\left\{ \sum_{j=1}^\infty (2r_j)^\alpha : E \subseteq \bigcup_{j=1}^\infty B_d(x_j, r_j),\ x_j \in E,\ r_j \leq \delta \right\}.$$

These three measures are Borel regular outer measures and are mutually absolutely continuous, more precisely, $\mathscr{H}^\alpha \leq \mathscr{S}^\alpha \leq \mathscr{C}^\alpha \leq 2^\alpha \mathscr{H}^\alpha$.

Definition 2.4 (Lower and upper densities). If ϕ is a Radon measure on $\mathbb{G}$, and $\alpha > 0$, define the **lower** and **upper α-densities of ϕ at** x by

$$\Theta_*^\alpha(\phi, x) := \liminf_{r\to 0} \frac{\phi(B(x,r))}{(2r)^\alpha}, \qquad \text{and} \qquad \Theta^{\alpha,*}(\phi, x) := \limsup_{r\to 0} \frac{\phi(B(x,r))}{(2r)^\alpha}.$$

If the upper and lower densities coincide, their value is called the **density of ϕ at** x. If $\phi = \mathscr{H}^\alpha \llcorner E$, where E is a subset of $\mathbb{G}$, its density (respectively, upper, lower densities) is also called **Hausdorff density of** E and denoted $\Theta^\alpha(E, x)$.

The following fundamental result on Hausdorff measures lies at the core of this paper. It is a consequence of [9, Theorems 2.10.17 and 2.10.18] (see also Theorem 2.1 and Lemma 3.1 in [12]).

Proposition 2.5. *Suppose that $E \subset \mathbb{G}$ is a Borel set of $\mathscr{H}^\alpha$-finite measure. Then for $\mathscr{H}^\alpha$-almost every $x \in E$, there holds:*

$$\lim_{\delta\to 0} \sup\left\{ \frac{\mathscr{H}^\alpha(E \cap S)}{(\operatorname{diam} S)^\alpha},\ x \in S,\ \operatorname{diam} S < \delta \right\} \leq 1. \tag{2.1}$$

Moreover, in the case $E = \mathbb{G}$, denoting by Q the homogeneous dimension of $\mathbb{G}$, for $x \in \mathbb{G}$, there holds:

$$\lim_{\delta\to 0} \sup\left\{ \frac{\mathscr{H}^Q(S)}{(\operatorname{diam} S)^Q},\ x \in S,\ \operatorname{diam} S < \delta \right\} = 1.$$

There is a degenerate case, which needs to be ruled out:

Proposition 2.6. *If $V_1 = \{0\}$, then sets with unit Hausdorff density are necessarily of dimension* 0.

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Proof. First, let us show that a ball of radius r in such a group necessarily has diameter strictly less than $2r$.

In order to see this, let ℓ be the index of the first nontrivial layer and denote by $\mathbb{G}'$ the homogeneous group obtained from $\mathbb{G}$ by dividing by ℓ the homogeneity of each layer of $\mathbb{G}$. It is immediate to see that $||\cdot||^{\ell}$ is a norm on the homogeneous group $\mathbb{G}'$. Since the unit ball $B(0,1)$ is the same in $\mathbb{G}$ and $\mathbb{G}'$, on the one hand, thanks to Proposition 2.2, we know that the diameter of $B(0,1)$ in $\mathbb{G}'$ is 2 because the group has nontrivial first layer, on the other, in $\mathbb{G}$, its diameter is $2^{1/\ell} < 2$. By homogeneity, there exists $\varepsilon > 0$, such that $\mathrm{diam}(B(x,r)) \leq (1-\varepsilon)(2r)$ for all $x \in \mathbb{G}$ and all $r > 0$. Given $E \subset \mathbb{G}$ with Hausdorff α-density $1\mathscr{H}^{\alpha}$-almost everywhere, using the first statement of Proposition 2.5 at a point where it holds, for small enough $r > 0$, one gets

$$\mathscr{H}^{\alpha}(E \cap B(x,r)) \leq (1+\varepsilon)^{\alpha}\mathrm{diam}B(x,r)^{\alpha} < (1+\varepsilon)^{\alpha}(1-\varepsilon)^{\alpha}(2r)^{\alpha} = (1-\varepsilon^2)^{\alpha}(2r)^{\alpha}.$$

Dividing by $(2r)^{\alpha}$ and letting r go to zero contradicts the Hausdorff density hypothesis if $\alpha > 0$. □

3. Comparison of surface measures in Carnot groups

Throughout this section, $\mathbb{G}$ will be a fixed Carnot group and d a fixed homogeneous left-invariant distance on $\mathbb{G}$.

Definition 3.1 ($C^1_{\mathrm{H}}(\mathbb{G},\mathbb{G}')$-rectifiable sets). Let $\mathbb{G}'$ be another Carnot group endowed with the left-invariant homogeneous distance d'. Let $\Omega \subseteq \mathbb{G}$ be open, and let $f : \Omega \to \mathbb{G}'$ be a map. We say that f is **Pansu differentiable at** $x \in \Omega$ if there exists a homogeneous homomorphism $Df_x : \mathbb{G} \to \mathbb{G}'$, such that

$$\lim_{y \to x} \frac{d'(f(x)^{-1} \cdot f(y), Df_x(x^{-1} \cdot y))}{d(x,y)} = 0.$$

Moreover, we say that f is **of class C^1_{H} in** Ω if the map $x \mapsto Df_x$ is continuous from Ω to the space of homogeneous homomorphisms from $\mathbb{G}$ to $\mathbb{G}'$. Given an arbitrary Carnot group $\mathbb{G}$, we say that $\Gamma \subseteq \mathbb{G}$ is a **C^1_{H}-submanifold** of $\mathbb{G}$ if there exists a Carnot group $\mathbb{G}'$, such that for every $p \in \Gamma$, there exists an open neighbourhood Ω of p and a function $f \in C^1_{\mathrm{H}}(\Omega;\mathbb{G}')$, such that

$$\Gamma \cap \Omega = \{g \in \Omega : f(g) = 0\}, \tag{3.1}$$

and $Df_p : \mathbb{G} \to \mathbb{G}'$ is surjective and $\mathrm{K}er(Df_p)$ admits a complementary subgroup. In this case, we say that Γ is a **$C^1_{\mathrm{H}}(\mathbb{G},\mathbb{G}')$-submanifold**. Finally, we say that $\Gamma \subseteq \mathbb{G}$ is a **$(\mathbb{G},\mathbb{G}')$-rectifiable set** if $\mathscr{H}^{Q-Q'}(\Gamma) < +\infty$ and there exist countably many subsets Γ_i of $\mathbb{G}$ that are $C^1_{\mathrm{H}}(\mathbb{G},\mathbb{G}')$-submanifolds, such that

$$\mathscr{H}^{Q-Q'}\left(\Gamma \setminus \bigcup_{i=1}^{+\infty} \Gamma_i\right) = 0.$$

At $\mathscr{H}^{Q-Q'} \llcorner \Gamma$-almost every point x, Γ admits a tangent $\mathbb{V}(x)$, which is a homogeneous subgroup of $\mathbb{G}$ with homogeneous dimension $Q - Q'$.

Definition 3.2 (Tangent measures). Let ϕ be a Radon measure on $\mathbb{G}$. For any $x \in \mathbb{G}$ and any $r > 0$, we define the rescaled measure

$$T_{x,r}\phi(E) := \phi(x * \delta_r(E)), \qquad \text{for any Borel set } E \subset \mathbb{G}.$$

The *α-dimensional tangents* to ϕ at x are the Radon measures ν, which are weak* limits of a sequence $r_i^{-\alpha} T_{x,r_i}\phi$ for $r_i \to 0$. The set of these measures is denoted by $\mathrm{Tan}_{\alpha}(\phi, x)$.

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Definition 3.3. Let ϕ be a Radon measure in $\mathbb{G}$, and let $h \in \mathbb{N}$. We say that ϕ is $\mathscr{P}_h$-rectifiable if

(i) $0 < \Theta^h_*(\phi, x) \le \Theta^{h,*}(\phi, x) < \infty$ for ϕ-almost every $x \in \mathbb{G}$;
(ii) for ϕ-almost every $x \in \mathbb{G}$, there exists a homogeneous subgroup $\mathbb{V}(x)$ of $\mathbb{G}$ of Hausdorff dimension $h \in \mathbb{N}$, such that $\mathrm{Tan}_h(\phi, x) \subseteq \{\lambda \mathscr{H}^h \llcorner \mathbb{V}(x) : \lambda > 0\}$.

The various surface measures on $(\mathbb{G}, \mathbb{G}')$-rectifiable sets are now well understood, as summed up in the following statement which is a combination of [1, Theorem 1.1], [2, Proposition 5.2], as well as the results of [12].

Theorem 3.4. *Suppose that $\Gamma \subseteq \mathbb{G}$ is a Borel, $(\mathbb{G}, \mathbb{G}')$-rectifiable set. Then $\mathscr{H}^{Q-Q'} \llcorner \Gamma$, $\mathcal{S}^{Q-Q'} \llcorner \Gamma$ and $\mathscr{C}^{Q-Q'} \llcorner \Gamma$ are $\mathscr{P}_{Q-Q'}$-rectifiable. Furthermore, at almost every point $x \in \Gamma$*

$$\Theta^{Q-Q'}(\mathscr{C}^{Q-Q'} \llcorner \Gamma, x) = 1. \tag{3.2}$$

Finally, letting $\mathbb{V}(x)$ be the $Q - Q'$ subgroup for which part (ii) of Definition 3.3 is satisfied

$$\mathscr{H}^{Q-Q'} \llcorner \Gamma = \mathcal{A}(\mathscr{C}^{Q-Q'} \llcorner \mathbb{V}(\cdot))\mathscr{C}^{Q-Q'} \llcorner \Gamma, \quad \text{and} \quad \mathscr{H}^{Q-Q'} \llcorner \Gamma = \mathcal{A}(\mathcal{S}^{Q-Q'} \llcorner \mathbb{V}(\cdot))\mathcal{S}^{Q-Q'} \llcorner \Gamma,$$

where for a homogeneous subgroup $\mathbb{V}$ of $\mathbb{G}$ with dimension $Q - Q'$ and a Haar measure μ on $\mathbb{V}$, we write:

$$\mathcal{A}(\mu) := \sup\{(\mathrm{diam} S)^{-(Q-Q')}\mu(S),\ 0 < \mathrm{diam} S < +\infty\}.$$

The above result tells us that the coefficients that allow to express the Hausdorff measure on smooth surfaces in terms of the centred Hausdorff measure or the spherical Hausdorff measure are almost everywhere the volume of the isodiametric is set in the tangents. (with respect to the centred or the spherical measures)

Corollary 3.5. *Under the hypotheses of Theorem 3.4 and at a point x where its conclusions holds, suppose that*

$$\Theta^{Q-Q'}(\mathscr{H}^{Q-Q'} \llcorner \Gamma, x) = 1. \tag{3.3}$$

Then the intersection of $B(0,1)$ with $\mathbb{V}(x)$ is an isodiametric shape in the homogeneous group $\mathbb{V}(x)$ with the metric given by the restriction of d.

Proof. Combining the Hausdorff density hypothesis (3.3) and the centred spherical density property (3.2) yields

$$1 = \mathcal{A}(\mathscr{C}^{Q-Q'} \llcorner \mathbb{V}(x)) = \sup\left\{\frac{(\mathscr{C}^{Q-Q'} \llcorner \mathbb{V}(x))(S)}{(\mathrm{diam} S)^{(Q-Q')}},\ 0 < \mathrm{diam} S < +\infty\right\}. \tag{3.4}$$

However, since $\mathscr{C}^{Q-Q'} \llcorner \Gamma$ is $\mathscr{P}^{Q-Q'}$-rectifiable and thanks to (3.2), there also holds

$$\frac{(\mathscr{C}^{Q-Q'} \llcorner \mathbb{V}(x))(B(0,1))}{2^{(Q-Q')}} = \lim_{r \to 0} \frac{(\mathscr{C}^{Q-Q'} \llcorner \Gamma)(B(x,r))}{(2r)^{(Q-Q')}} = 1,$$

which implies that $B(0,1)$ realises the supremum of the right-hand side of (3.4). Therefore, the restriction of the unit ball $B(0,1) \cap \mathbb{V}(x)$ is isodiametric in the group $\mathbb{V}(x)$ endowed with the restriction of the metric d. □

4. Sets with Hausdorff density 1 in homogeneous groups

One can endow a homogeneous group $\mathbb{G}$ as above with a smooth-box homogeneous norm of the form

$$||(x_1, \ldots, x_\kappa)|| := \max\{\varepsilon_\ell |x_\ell|^{1/\ell},\ \ell = 1, \ldots, \kappa\}, \tag{4.1}$$

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where $|x_\ell|$ stands for the euclidean norm of the vector $x_\ell \in V_\ell$ and where $\varepsilon_1 = 1$. In all the following, we suppose $\kappa \geq 2$. Throughout the rest of this section, we will denote by $B(x, r)$ the ball of centre x and radius $r > 0$ relative to the metric induced by the norm in (4.1). For a proof of the following lemma, we refer to the appendix of [11].

Lemma 4.1. *Provided the ε_ℓ are small enough with respect to $\varepsilon_1, \ldots, \varepsilon_{\ell-1}$, then the function (4.1) defines a homogeneous norm, and thus a metric, on $\mathbb{G}$.*

The unit ball of this homogeneous norm is not isodiametric, indeed, it is easy to find a strictly larger set diameter 2. Fix $19/10 < \xi < 2$. Throughout the rest of the section, we choose $\varepsilon_2, \ldots, \varepsilon_\kappa$ in such a way that the conclusion of Lemma 4.1 holds and

$$2\xi^{\ell-1} + 16^{\ell+1}\varepsilon_\ell^\ell \tilde{c}_\ell \leq 2^\ell \text{ for any } \ell \geq 2. \tag{4.2}$$

Proposition 4.2. *Let ξ be the parameter chosen above, and define the set $G(0, 1)$ as the family of those $(x_1, x_2, \ldots, x_\kappa) = x \in \mathbb{G}$ for which*

$$|x_1| \leq 1 \qquad \textit{and} \qquad \varepsilon_\ell^\ell |x_\ell| \leq \xi^{\ell-1} \textit{ for } \ell \geq 2.$$

If the $\varepsilon_1, \ldots, \varepsilon_\kappa$ are chosen small enough with respect to the $\tilde{c}_\ell$, then the diameter of the set $G(0, 1)$ is 2 and $B(0, 1) \subseteq G(0, 1)$.

Proof. Checking layer by layer, one sees that $B(0, 1) \subset G(0, 1) \subset B(0, 2)$. It remains to prove that if x, y are points in $G(0, 1)$, there holds $||x^{-1} * y|| \leq 2$. This is also done layer by layer: for the first layer, one immediately has $|y_1 - x_1| \leq 2$. From the second layer onward, that is, for $\ell \geq 2$, we have

$$\begin{aligned}\varepsilon_\ell^\ell |\pi_\ell(x^{-1} * y)| &= \varepsilon_\ell^\ell |-x_\ell + y_\ell + Q_\ell(-x_1, \ldots, -x_{\ell-1}, y_1, \ldots, y_{\ell-1})| \\ &\leq \varepsilon_\ell^\ell |x_\ell| + \varepsilon_\ell^\ell |y_\ell| + \varepsilon_\ell^\ell |Q_\ell(-x_1, \ldots, -x_{\ell-1}, y_1, \ldots, y_{\ell-1})| \\ &\leq 2\xi^{\ell-1} + \varepsilon_\ell^\ell |Q_\ell(-x_1, \ldots, -x_{\ell-1}, y_1, \ldots, y_{\ell-1})|.\end{aligned} \tag{4.3}$$

Recall that each entry of $Q_\ell : (V_1 \oplus \ldots \oplus V_{\ell-1}) \times (V_1 \oplus \ldots \oplus V_{\ell-1}) \to V_\ell$ is a polynomial in the components of x and y of order lower than ℓ, and is ℓ homogeneous with respect to the group dilations. Hence, following the footsteps of [11], we can find positive real numbers $\tilde{c}_\ell > 0$, such that

$$\begin{aligned}|Q_\ell(x, y)| &\leq \sum_{j=1}^{\ell-1} c_{j,\ell} ||x||^j ||y||^{\ell-j} \leq \tilde{c}_\ell \sum_{j=1}^{\ell-1} \binom{\ell}{j} ||x||^j ||y||^{\ell-j} \\ &\leq \tilde{c}_\ell (||x|| + ||y||)^\ell \leq 2^{\ell-1}\tilde{c}_\ell (||x||^\ell + ||y||^\ell),\end{aligned} \tag{4.4}$$

where the last inequality above comes from Jensen's inequality. The above chain of inequalities together with (4.2) and (4.3) implies

$$\begin{aligned}\varepsilon_\ell^\ell |\pi_\ell(x^{-1} * y)| &= \varepsilon_\ell^\ell |-x_\ell + y_\ell + Q_\ell(-x_1, \ldots, -x_{\ell-1}, y_1, \ldots, y_{\ell-1})| \\ &\leq 2\xi^{\ell-1} + 2^{\ell-1}\varepsilon_\ell^\ell \tilde{c}_\ell (||x||^\ell + ||y||^\ell) \\ &\leq 2\xi^{\ell-1} + 2^{\ell-1}\varepsilon_\ell^\ell \tilde{c}_\ell \cdot 2 \leq 2^\ell,\end{aligned}$$

which concludes the proof. □

Definition 4.3. For any $x \in \mathbb{G}$ and any $r > 0$, we define $G(x, r) := x * \delta_r(G(0, 1))$.

Before passing to the proof of our main theorem, a technical result is needed: it quantifies how far $G(0, 1)$ is far from $B(0, 1)$, in the sense that $G(0, 1) \backslash B(0, 1)$ contains large balls.

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Proposition 4.4. *Define the functions* $r, s : \mathbb{G} \to (0, +\infty)$ *by*

$$r(x) := \max\left\{\frac{\|x\|+|x_1|}{2}, \|x\|\Big(\frac{(1+10^{-1})(1+2^\ell \varepsilon_\ell^\ell c_\ell)}{\xi^{\ell-1}}\Big)^{1/\ell} \text{ for } \ell = 2,\dots,\kappa\right\},$$

and

$$s(x) := \frac{\|x\| - r(x)}{10}.$$

Then if $x \in \mathbb{G}$ *satisfies* $|x_1| < ||x||$, *there holds*

$$B(x, s(x)) \subset G(0, r(x)) \backslash B(0, r(x)).$$

Proof. By homogeneity of the statement, it suffices to handle the case $\|x\| = 1$. Fix $x \in \mathbb{G}$ with $||x|| = 1$ and $|x_1| < 1$. Notice that by (4.2), $r(x) < \|x\|$ and $0 < s(x) < 10^{-1}$. For $y \in B(x, s(x))$, there holds

$$\|y\| \geq \|x\| - s(x) = (1 - 10^{-1})\|x\| + 10^{-1} r(x) > r,$$

so that $y \notin B(0, r(x))$. On the other hand, we claim that $y \in G(0, r(x))$, let us prove this layer by layer. Clearly

$$|y_1| \leq |x_1| + s(x) \leq \frac{10^{-1}}{2}\|x\| + (1 - 10^{-1}/2)|x_1| < \frac{\|x\|+|x_1|}{2} \leq r.$$

Now consider a layer $\ell \geq 2$, there holds

$$\begin{aligned}
\varepsilon_\ell^\ell |y_\ell| &\leq \epsilon_\ell^\ell |x_\ell| + s(x)^\ell + 2^{\ell-1}\varepsilon_\ell^\ell \tilde{c}_\ell(||x||^\ell + s(x)^\ell)\\
&\leq 1 + s(x)^\ell + 2^{\ell-1}\varepsilon_\ell^\ell \tilde{c}_\ell(1 + s(x)^\ell)\\
&\leq (1 + 2^{\ell-1}\varepsilon_\ell^\ell \tilde{c}_\ell)(1 + s(x)^\ell).
\end{aligned}$$

Recalling the definition of $r(x)$, we have

$$\varepsilon_\ell^\ell |y_\ell| \leq \frac{\xi^{\ell-1} r(x)^\ell}{1 + 10^{-1}}(1 + s(x)^\ell) \leq \xi^{\ell-1} r(x)^\ell.$$

□

Proposition 4.5. *Let* $0 < \varepsilon < 1$. *Assume* E *is a set in* $\mathbb{G}$, *such that*

$$E \cap x * \{y \in \mathbb{G} : \|y\| > (1 - \varepsilon)^{-1}|y_1|\} = \emptyset \qquad \text{for any } x \in E. \tag{4.5}$$

Then, there exists a continuous function $\varphi : \pi_1(E) \to \mathbb{G}$, *such that* $\varphi(\pi_1|_E) = \mathrm{Id}$ *on* E *and that for any* $a, b \in \pi_1(E)$, *we have*

$$\|\varphi(a)^{-1}\varphi(b)\| \leq (1 - \varepsilon)^{-1}|b - a|. \tag{4.6}$$

Proof. First, note that the projection π_1 is injective on E. To see this, pick two points $x, y \in E$, thanks to (4.5), there holds

$$0 < \|x^{-1} * y\| \leq (1 - \varepsilon)^{-1}|y_1 - x_1|. \tag{4.7}$$

Thus, π_1 has an inverse on its image, denote it by φ. Estimate (4.6) follows from (4.7). □

Remark 4.6. By definition of r and s, it is clear that given positive parameters $\varepsilon > 0$ and $\alpha > 0$, there exists a $k = k(\varepsilon, \alpha) \in \mathbb{N}$, such that if $s(x)^\alpha \leq 2\|x\|^\alpha/(k - 1)$ for some $x \in \mathbb{G}$, then $|x_1| \geq (1 - \varepsilon)\|x\|$.

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With these tools at hand, we can now state and prove the main step in the proof of Theorem 1.3:

Proposition 4.7. *Fix $\alpha, \varepsilon > 0$, and choose $k = k(\varepsilon, \alpha) \geq 2$ as in Remark 4.6. Let E be a Borel set, such that*

$$\lim_{r\to 0} \frac{\mathscr{H}^\alpha \llcorner E(B(x,r))}{(2r)^\alpha} = 1, \qquad \text{for } \mathscr{H}^\alpha\text{-almost every } x \in E.$$

For each $j \in \mathbb{N}$, let $E_{j,k}$ be the set of those $x \in E$, such that

$$1 - \frac{1}{k} \leq \frac{\mathscr{H}^\alpha(B(x,r)\cap E)}{(2r)^\alpha} \quad \text{and} \quad \frac{\mathscr{H}^\alpha(G(x,r)\cap E)}{(2r)^\alpha} \leq 1 + \frac{1}{k}, \qquad \text{for all } 0 < r < 1/j. \tag{4.8}$$

Then $\mathscr{H}^\alpha(E \setminus \cup_{j\in\mathbb{N}} E_{j,k}) = 0$ and the set $E_{j,k}$ can be covered by images of $(1-\varepsilon)^{-1}$-biLipschitz maps from a subset of V_1 endowed with the euclidean metric to $\mathbb{G}$.

Proof. By Propositions 2.5 and 4.2 and the properties of E for any $k \in \mathbb{N}$, we have $\mathscr{H}^\alpha(E \setminus \cup_{j\in\mathbb{N}} E_{j,k}) = 0$. Fix two indices $j, k \in \mathbb{N}$ and cover $E_{j,k}$ by countably many balls $B(z_i, r_i)$ with $r_i < (10j)^{-1}$. For $i \in \mathbb{N}$, we will show that $E_{j,k} \cap B(z_i, r_i)$ is the image of a 1-bi-Lipschitz map from a subset of the vector space V_1 endowed with the Euclidean metric to $\mathbb{G}$.

Let $x, y \in E_{j,k} \cap B(z_i, r_i)$ and without loss of generality, since left translations are isometries, suppose that $x = 0$. Suppose that $|y_1| < \|y\|$. By Proposition 4.4, we know that $B(y, s(y)) \subset G(0,r)\backslash B(0,r)$ for some $r \leq \|y\| < (10j)^{-1}$. However, by definition of $E_{j,k}$, there holds

$$\mathscr{H}^\alpha(G(0,r)\backslash B(0,r)\cap E) \leq \mathscr{H}^\alpha(G(0,r)\cap E) - \mathscr{H}^\alpha(B(0,r)\cap E) \leq 2k^{-1}(2r)^\alpha, \tag{4.9}$$

and

$$\mathscr{H}^\alpha\big(B(y, s(y))\cap E\big) \geq \Big(1 - \frac{1}{k}\Big)(2s(y))^\alpha, \tag{4.10}$$

since $s(y) < \|y\| \leq (10j)^{-1}$. In particular, combining (4.9) and (4.10) and the choice of r, we infer that

$$\begin{aligned}\Big(1 - \frac{1}{k}\Big)\Big(\frac{s(y)}{\|y\|}\Big)^\alpha &\leq \frac{\mathscr{H}^\alpha \llcorner E(B(y, s(y)))}{(2\|y\|)^\alpha} \\ &\leq \frac{\mathscr{H}^\alpha \llcorner E(G(0,r)\backslash B(0,r))}{(2\|y\|)^\alpha} \leq \frac{2k^{-1}(2r)^\alpha}{(2\|y\|)^\alpha} \leq 2k^{-1}.\end{aligned} \tag{4.11}$$

By the choice of k and Remark 4.6, we conclude that for $i \in \mathbb{N}$ and $x, y \in B(z_i, r_i) \cap E_{j,k}$, we have

$$\|x^{-1} * y\| \leq (1-\varepsilon)^{-1} |(x^{-1} * y)_1|$$

and Proposition 4.5 concludes the proof. □

Proof of Theorem 1.3. As π_1 is 1-Lipschitz, $\pi_1(E)$ has σ-finite Hausdorff measure, in particular, it has upper Hausdorff density at most 1, $\mathscr{H}^\alpha$-almost everywhere. Fix $\eta > 0$, by Proposition 4.7, for every $\eta > 0$, E can be covered up to an $\mathscr{H}^\alpha$-null set by countably many images of $(1+\eta)$-biLipschitz maps f_n^η defined on subsets A_n^η of V_1. As E has unit Hausdorff density $\mathscr{H}^\alpha$-almost everywhere, so do its subsets $f_n^\eta(A_n^\eta)$ at almost every point (see, for instance, [2, Proposition 2.2]). Pick such a point $x \in f_n^\eta(A_n^\eta)$. Writing $f := f_n^\eta$ and $A := A_n^\eta$ for simplicity, there holds

$$\lim_{r\to 0} \frac{\mathscr{H}^\alpha(f(A)\cap B(x,r))}{(2r)^\alpha} = 1.$$

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On the one hand, since π_1 is 1-Lipschitz, there holds

$$\pi_1(f(A) \cap B(x,r)) \subset A \cap B(\pi_1(x), r),$$

and on the other hand, as f is $(1+\eta)$-Lipschitz, we have

$$\mathscr{H}^\alpha(f(A) \cap B(x,r)) \leq (1+\eta)^\alpha \mathscr{H}^\alpha(A \cap B(\pi_1(x), r)).$$

This implies

$$\frac{\mathscr{H}^\alpha(\pi_1(E) \cap B(x,r))}{(2r)^\alpha} \geq \frac{\mathscr{H}^\alpha(\pi_1(A) \cap B(\pi_1(x),r))}{(2r)^\alpha} \geq \frac{1}{(1+\eta)^\alpha} \frac{\mathscr{H}^\alpha(f(A) \cap B(x,r))}{(2r)^\alpha}.$$

Letting r go to 0, we obtain a bound on the lower density of $\pi_1(E)$:

$$\Theta^\alpha_*(\pi_1(E), x) \geq (1+\eta)^{-\alpha}.$$

Thus, for every $\eta > 0$, there is a subset of $\pi_1(E)$ of full $\mathscr{H}^\alpha$-measure with lower Hausdorff density at least $(1+\eta)^{-\alpha}$. Taking a countable intersection for a sequence of parameters η going to zero yields a subset of $\pi_1(E)$ of full measure with unit Hausdorff density. Note that the Hausdorff measure and the density on V_1 are computed with respect to the euclidean norm. By Marstrand's Theorem (see [8, Theorem 3.1], we have $\alpha \in \{1, \dots, n_1\}$. Therefore, the main result of [22] applies and $\pi_1(E)$ is euclidean rectifiable. Using the biLipschitz maps of Proposition 4.7, one therefore sees that E itself is α-rectifiable (in the metric sense of Federer, see [9, Section 3.2.14]). In addition, by the Pansu-Rademacher Theorem, it is possible to show that the tangent measures of $\mathscr{H}^\alpha \llcorner E$ must be almost everywhere the Haar measure of some subgroup of V_1, and hence, α must be the dimension of one of those horizontal subgroups. □

5. The case of metric Lie groups admitting dilations

In this section, we discuss metric Lie groups admitting dilations, and show that they are not too different from homogeneous groups, in the sense that they admit biLipschitz equivalent 'homogeneous' distances for which Theorem 1.2 holds, with the same proof as above.

Definition 5.1. If $\mathbb{G}$ is a Lie group with Lie algebra $\mathfrak{g}$, a linear map $A : \mathfrak{g} \to \mathfrak{g}$ is a **derivation** if it obeys the Leibniz law $A([x,y]) = [A(x), y] + [x, A(y)]$. A derivation determines a one parameter family of dilations $(0, +\infty) \to \mathrm{Aut}(\mathbb{G})$:

$$\lambda \mapsto \delta_\lambda := \lambda^A = e^{(\log \lambda)A}.$$

A left-invariant distance d on $\mathbb{G}$ is A-homogeneous if for every $\lambda \in (0, \infty)$, we have $d(\delta_\lambda x, \delta_\lambda y) = \lambda d(x,y)$.

Note that if A is diagonalisable in $\mathbb{R}$ with integer eigenvalues, then $\mathbb{G}$ admits an A-homogeneous distance and is in fact a homogeneous group. If A is diagonalisable in $\mathbb{R}$, but not all eigenvalues are integers, we can construct an equivalent homogeneous distance on $\mathbb{G}$, such that the results of the previous section apply. This is relevant because combining Theorem 1.2, Corollary 2.6 and Lemma 6.1 from [17], yields the following:

Proposition 5.2. *A Lie group endowed with an A-homogeneous distance for some derivation A of its Lie algebra is biLipschitz equivalent to a Lie group with an A-homogeneous distance, where A has real spectrum* $1 \leq t_1 \leq \cdots \leq t_\kappa < \infty$. *Moreover, denoting by* V_{t_j} *the eigenspaces of A,* $A|_{V_1}$ *is diagonalisable.*

The authors of [17] construct an A-homogeneous distance on such groups, which is similar to the Koranyi norm. It is more convenient for us to define a smooth-box distance instead. The following statement contains the properties of this distance, which suffice to prove an analogue of Theorem 1.2.

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Theorem 5.3. *If* $(\mathbb{G}, A)$ *is a Lie group endowed with an A-homogeneous distance for some derivation A of its Lie algebra, then*

(i) *for each j, there exists an A-homogeneous distance* $||\cdot||$ *on* $\mathbb{G}$ *defined by*

$$||(x_{t_1}, \dots, x_{t_\kappa})|| := \max\{\varepsilon_{t_j} ||x_{t_j}||_{V_{t_j}}^{1/t_j}, j = 1, \dots, \kappa\}, \tag{5.1}$$

where $\varepsilon_{t_1} = 1$ *if* $V_{t_1} \neq \{0\}$ *and the other* ε_{t_j} *are chosen suitably small with respect to the coefficients of the expression for the operation given by the Baker-Campbell-Hausdorff formula. Here, the norms* $\|\cdot\|_{V_{t_j}}$ *can be taken to be the euclidean norm on* V_{t_j}.

(ii) *The projection* $\pi_1 : x \mapsto x_{t_1} \in V_{t_1}$ *is 1-Lipschitz continuous.*

(iii) *If* $B(x, r)$ *denotes the balls relative to the norm* $||\cdot||$*, one can define* $G(0, 1)$ *in much the same way as Proposition 4.2 and one has* $G(0, 1) \supset B(0, 1)$ *as in Definition 4.3, and* $G(0, 1)$ *has diameter* 2.

Proof. The proof of (i) is a modification of that of [17, Theorem 1.2]. One first choses a basis of $\mathfrak{g}$ in which A is in Jordan normal form. Considering the spaces V_{t_j} one by one and rescaling their bases yields an euclidean norm on each V_{t_j}, such that for $\lambda \in (0, 1\,]$, the operator norm of $\lambda^A|_{V_{t_j}}$ is bounded from above by $\lambda^{t_j - \theta_j}$ for a $\theta_j > 0$, which can be chosen arbitrarily small. As $A|_{V_1}$ is diagonalisable in $\mathbb{R}$, with eigenvalues at least 1, the operator norm of $\lambda^A|_{V_1}$ is necessarily λ. Writing $||\cdot||$ as in (5.1), it suffices to prove that the 'unit ball' $B := \{x, ||x|| \leq 1\}$ is A-convex[1], that is, for $\lambda \in (0, 1)$ and x, y in B, there holds

$$||(\lambda^A x) \cdot ((1-\lambda)^A y)|| \leq 1.$$

This is done iteratively over the subspaces V_{t_j}. For V_1 it is clear. For V_{t_j} with $1 < t_j < 2$, one proceeds as in Lemma 6.5 of [17] (the key remark being that the components of all commutators vanish in the layers of order less than 2). For $t_j = 2$, the reasoning of Lemma 6.7 in [17] works: for $x, y \in \mathbb{G}$ and $\lambda \in (0, 1)$, by the Baker-Campbell-Hausdorff formula, there holds

$$((\lambda^A x) \cdot ((1-\lambda)^A y))_2 = (\lambda^A x)_2 + ((1-\lambda)^A y)_2 + \frac{\lambda(1-\lambda)}{2}[x_1, y_1],$$

So that

$$\|((\lambda^A x) \cdot ((1-\lambda)^A y))_2\|_2 \leq \\ \lambda^2(\|x_2\|_2 + f(\lambda)) + (1-\lambda)^2(\|y_2\|_2 + f(1-\lambda)) + C\lambda(1-\lambda)\|x_1\|_1\|y_1\|_1,$$

where f is an error term coming from the dilations and is subpolynomial in $|\log \lambda|$. The whole controlled by a constant independent of x, y. It then suffices to take ε_2 small enough.

For $t_j > 2$, one needs to use the Baker-Campbell-Hausdorff formula, as in the proof that the smooth-box norm on a Carnot group is a metric (see the appendix of [11]), but as for $t_j = 2$, it is slightly more subtle (see Proposition 6.8 in [17]).

In order to prove (ii), pick $x = (x_1, x_{t_2}, \dots, x_{t_\kappa}) \in \mathbb{G}$. It suffices to note that on the one hand

$$||((1/2)^A x) \cdot ((1/2)^A y)|| \leq 1,$$

which follows from convexity, and on the other hand, that iterating the procedure yields a sequence which tends to $(x_1, 0, \dots)$ as in Proposition 2.2. The fact that the components of order $t_j > 1$ tend to 0 is a consequence of $2(1/2)^{t_j} < 1$. And the fact that the first component remains constant is clear because $A|_{V_1}$ is diagonalisable over $\mathbb{R}$ with eigenvalues 1 (in fact, it is the identity). The proof of (iii) is straightforward using the properties of $||\cdot||$ and the method of the previous section. □

[1]This condition is the analogue of convexity in vector spaces, combining it with homogeneity, one can easily infer the triangle inequality.

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Corollary 5.4. *Suppose $\mathbb{G}$ is a Lie group endowed with an A-homogeneous distance for some derivation A of its Lie algebra. Then, there exists a biLipschitz equivalent A-homogeneous distance $\|\cdot\|$ on $\mathbb{G}$, such that the following holds. Whenever $\alpha \geq 0$ and $E \subset \mathbb{G}$ is a Borel set for which $\Theta^\alpha(E,x) = 1$ for $\mathscr{H}^\alpha$-almost every $x \in E$, then α is not larger than the dimension of the first layer of the Lie algebra of $\mathbb{G}$, α is an integer and E is α-rectifiable.*

6. Discussion of the results in the first Heisenberg group

The point of this section is to compare the results of this work with the existing literature in the particular case of the first Heisenberg group. The group law of $\mathbb{H}^1$ is given by

$$(x_1, x_2) * (y_1, y_2) = (x_1 + y_1, x_2 + y_2 + 2\langle x_1, Jy_1\rangle),$$

where J is the standard symplectic 2×2 matrix. Throughout this section, we will endow $\mathbb{H}^1$ with the Koranyi metric, which is the following homogeneous left-invariant distance

$$d_{\mathscr{K}}(x,y) := (|x_1 - y_1|^4 + |x_2 - y_2 - 2\langle x_1, Jy_1\rangle|^2)^{1/4}. \tag{6.1}$$

The fact that this formula defines a distance was first proved in [7].

Before delving into the discussion, we recall that whenever we endow $\mathbb{H}^1$ with a left-invariant and homogeneous metric d, the metric space $(\mathbb{H}^1, d)$ is a 2, 3, 4-purely unrectifiable metric space of Hausdorff dimension 4 (see [14]). Another property of the Koranyi metric is that its ball, like that of the smooth-box metric, is not isodiametric (see [30, Theorem 3.5] and [31, Theorem 38]).

Proposition 6.1. *Let $G(0,1)$ be the subset of those $x = (x_1, x_2) \in \mathbb{H}^1$, such that*

$$|x_1| \leq 1 \qquad \text{and} \qquad |x_2|^2 \leq 2(1 - |x_1|^4).$$

Then $B(0,1) \subsetneq G(0,1)$ and $\operatorname{diam}(G(0,1)) \leq 2$.

Remark 6.2. The proof follows the argument of Cygan's proof of the subadditivity of the Koranyi norm [7]. In particular, it works in any H-type group endowed with the gauge distance.

Proof. Let $p := (x_1, x_2)$ and $p' := (x_1', x_2')$ be points of $G(0,1)$. Using the symmetry of G, and to avoid annoying signs, we compute the distance between p^{-1} and p', we can write

$$\begin{aligned}
\|p * p'\|^4 &= (|x_1|^2 + |x_1'|^2 + 2\langle x_1, x_1'\rangle)^2 + (x_2 + x_2' + 2\langle x_1, Jx_1'\rangle)^2\\
&= |x_1|^4 + |x_1'|^4 + 4\langle x_1, x_1'\rangle^2 + 4(|x_1|^2 + |x_2|^2)\langle x_1, x_1'\rangle + 2|x_1|^2|x_1'|^2\\
&\quad + |x_2|^2 + |x_2'|^2 + 4\langle x_1, Jx_1'\rangle^2 + 4(x_2 + x_2')\langle x_1, Jx_1'\rangle + 2x_2x_2'\\
&= (|x_1|^4 + |x_2|^2) + (|x_1'|^4 + |x_2'|^2)\\
&\quad + 2|x_1|^2|x_1'|^2 + 2x_2x_2'\\
&\quad + 4\langle x_1, x_1'\rangle^2 + 4\langle x_1, Jx_1'\rangle^2\\
&\quad + 4|x_1|^2\langle x_1, x_1'\rangle + 4x_2\langle x_1, Jx_1'\rangle\\
&\quad + 4|x_1'|^2\langle x_1, x_1'\rangle + 4x_2'\langle x_1, Jx_1'\rangle.
\end{aligned}$$

Now we proceed as in [7], using Cauchy-Schwarz and the inequality $\langle z, z'\rangle^2 + \langle z, Jz'\rangle^2 \leq |z|^2|z'|^2$. Noting that $|x_2|^2 \leq 2(1 - |x_1|^4)$, we obtain

$$\begin{aligned}
\|p * p'\|^4 \leq 4 &- |x_1|^4 - |x_1'|^4\\
&+ 4|x_1|^2|x_1'|^2 + 2(2 - |x_1|^4)^{1/2}(2 - |x_1'|^4)^{1/2}\\
&+ 4\big((2 - |x_1|^4)^{1/2} + (2 - |x_1'|^4)^{1/2}\big)|x_1||x_1'|.
\end{aligned}$$

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Using Young's inequality for the terms on the second line yields

$$\|p * p'\|^4 \leq \ 8 + 4\big((2 - |x_1|^4)^{1/2} + (2 - |x_1'|^4)^{1/2}\big)|x_1||x_1'|.$$

We need to show that the whole is smaller than 16, so it suffices to show:

$$\forall a, b \in [\,0, 1\,], \quad \big((2 - a^4)^{1/2} + (2 - a^4)^{1/2}\big)ab \leq 2.$$

Since $\sqrt{1 + s} \leq 1 + s/2$, it suffices that there holds

$$\forall a, b \in [\,0, 1\,], \quad \big(3 - a^4/2 - b^4/2\big)ab \leq 2.$$

Freezing the variable a, and differentiating, we remark that the maximum is reached at $b = 1$. So we just need to check that $(5/2 - a^4/2)a \leq 2$ for $a \in [\,0, 1\,]$, which we leave to the reader. □

It is not hard to see that one can then obtain an analogue of Proposition 4.4, taking, for instance, $r(x)^4 := |x_1|^4 + \frac{3}{2}|x_2|^2$ and $s(x)^4 := c^4|x_2|^2$, for some $c > 0$ small enough. This, in turn, implies that the proof of Theorem 1.3 can be adapted to this setting, and we get:

Theorem 6.3. *Suppose E is a Borel subset of $\mathbb{H}^1$ for which there exists an $\alpha > 0$, such that*

$$\lim_{r \to 0} \frac{\mathscr{H}^\alpha(B(x, r) \cap E)}{(2r)^\alpha} = 1, \qquad \text{for } \mathscr{H}^\alpha\text{-almost all } x \in E, \tag{6.2}$$

and where $B(x, r)$ is the ball relative to the Koranyi distance (see (6.1)). Then $\alpha = 1$ and E can be covered with countably many Lipschitz images of compact subsets of $\mathbb{R}$ up to an $\mathscr{H}^1$-null set.

Putting together [13] and Theorem 6.3, we obtain, as in Theorem 1.2, that rectifiable sets are characterised by having unit density for the Hausdorff measure almost everywhere. Furthermore, thanks to the work of Antonelli, Chousionis, Magnani, Tyson and the second named author, the *density problem*, see [24] for a proper definition, has a complete and satisfactory solution in $\mathbb{H}^1$ endowed with the Koranyi metric

Theorem 6.4 [3, 5, 6, 24, 25]. *Let ϕ be a Radon measure on $(\mathbb{H}^1, d)$, such that the limit*

$$\lim_{r \to 0} r^{-\alpha}\phi(B(x, r))$$

exists and is positive and finite ϕ-almost everywhere, then $\phi \ll \mathscr{H}^\alpha$, $\alpha \in \{0, 1, \ldots, 4\}$ and ϕ is $\mathscr{P}_\alpha$-rectifiable, or in other words

(i) *if $\alpha = 0$, then ϕ is a sum of countably many multiples of Dirac masses;*
(ii) *if $\alpha = 1$, then $\mathbb{H}^1$ can be covered ϕ-almost all by countably many Lipschitz images of $\mathbb{R}$;*
(iii) *if $\alpha = 2$, then ϕ-almost everywhere the blowups of the measure ϕ coincide with the vertical line $\mathscr{V} := \mathrm{span}\{e_3\}$, for a precise statement, we refer to the introduction of [3];*
(iv) *if $\alpha = 3$, then $\mathbb{H}^1$ can be covered up to a ϕ-null set by countably many $C^1_{\mathbb{H}^1}$ surfaces (see Definition 3.1).*

For the last point, recall that $C^1_{\mathbb{H}^1}$-surfaces, introduced in [10], can be fractals from the euclidean perspective and that at the present time, it is unknown if they are 'parabolic rectifiable'. This would mean that they can be covered up to a null set by countably many Lipschitz images of compact subsets of the parabolic plane, which is the model for homogeneous 1-codimensional subgroups of $\mathbb{H}^1$.

The above result is the analogue of the celebrated Preiss' rectifiability theorem, see [28], in the first Heisenberg group $\mathbb{H}^1$. Theorems 6.3 and 6.4 show that there are both great similarities and differences between the general homogeneous groups and euclidean space. On the one hand, the density problem has a natural solution in the form of Theorem 6.4. On the other, Theorem 6.3 tells us that $C^1_{\mathbb{H}^1}$-regular surfaces *cannot* have unit density for the Hausdorff measure. This phenomenon, however, should be

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expected in view of Theorem 3.4, which tells us that in order to have unit density for the Hausdorff measure at a point for a $C^1_{\mathbb{H}^1}$-surface, we need the intersection of the ball with the tangent plane at that point to be an isodiametric set.

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