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A Remark on the Moser-Aubin Inequality for Axially Symmetric Functions on the Sphere

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Abstract. Let $\mathcal{S}_r$ be the collection of all axially symmetric functions f in the Sobolev space $H^1(\mathbb{S}^2)$ such that $\int_{\mathbb{S}^2} x_i e^{2f(\mathbf{x})}\,d\omega(\mathbf{x})$ vanishes for $i = 1, 2, 3$. We prove that

$$\inf_{f\in\mathcal{S}_r} \frac{1}{2}\int_{\mathbb{S}^2} |\nabla f|^2\,d\omega + 2\int_{\mathbb{S}^2} f\,d\omega - \log\int_{\mathbb{S}^2} e^{2f}\,d\omega > -\infty,$$

and that this infimum is attained. This complements recent work of Feldman, Froese, Ghoussoub and Gui on a conjecture of Chang and Yang concerning the Moser-Aubin inequality.

Consider the Sobolev space $H^1(\mathbb{S}^2)$ of real functions f on the sphere $\mathbb{S}^2$ with $\int_{\mathbb{S}^2} |\nabla f|^2\,d\omega < \infty$, where ω is Lebesgue measure on $\mathbb{S}^2$ normalized so that $\omega(\mathbb{S}^2) = 1$. Let

$$J_\alpha(f) = \alpha\int_{\mathbb{S}^2} |\nabla f|^2\,d\omega + 2\int_{\mathbb{S}^2} f\,d\omega - \log\int_{\mathbb{S}^2} e^{2f}\,d\omega.$$

Moser [11] has proved that J_1 is bounded below, *i.e.*,

$$\inf_{f\in H^1(\mathbb{S}^2)} J_1(f) > -\infty. \tag{1}$$

It is easy to see that this inequality of Moser trivially implies that

$$J_\alpha(f) > -\infty \quad \text{for all } f \in H^1(\mathbb{S}^2) \text{ and } \alpha \in \mathbb{R}, \tag{2}$$

(because one must have $\int_{\mathbb{S}^2} e^{2f}\,d\omega$ finite for all $f \in H^1(\mathbb{S}^2)$), since were it infinite for some such f, we would have $J_1(f) = -\infty$, contrary to (1)), but of course it says much more. In fact, Moser's inequality (1) is sharp in the sense that J_α is *not* bounded below if $\alpha < 1$. Onofri [12] has shown that the infimum in (1) is actually zero so that $J_1(f) \geq 0$ for all $f \in H^1(\mathbb{S}^2)$.

Aubin [1], on the other hand, was interested in a different kind of improvement over Moser's inequality. Let $\mathcal{S}$ be the set of functions f in $H^1(\mathbb{S}^2)$ for which the centre of mass of e^{2f} is at the origin, *i.e.*, which satisfy

$$\int_{\mathbb{S}^2} e^{2f(\mathbf{x})}\mathbf{x}\,d\omega(\mathbf{x}) = \mathbf{0}. \tag{3}$$

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This condition says that e^{2f} is orthogonal to the eigenspace corresponding to the first non-zero eigenvalue of the Laplacian. Assuming such an orthogonality constraint on the set of functions in Moser's inequality should let us improve the inequality, and indeed Aubin has shown that

$$\inf_{f \in \mathcal{S}} J_\alpha(f) > -\infty, \quad \text{for all } \alpha \in \left(\frac{1}{2}, 1\right], \tag{4}$$

i.e., that the critical exponent 1 for Moser's inequality (1) changes to $\frac{1}{2}$ if we restrict our attention to functions from $\mathcal{S}$. Let

$$C_\alpha = \inf_{f \in \mathcal{S}} J_\alpha(f).$$

Chang and Yang [6, 7] in their study of the question of prescribing Gaussian curvature have shown that if $\alpha \leq 1$ is sufficiently close to 1, then $C_\alpha = 0$. The interesting question now is of what happens in the Aubin inequality (4) for $\alpha = \frac{1}{2}$. It is not known whether $C_{1/2} > -\infty$, although it has been conjectured by Chang and Yang [6, 7] that in fact $C_{1/2} = 0$.

The investigation of Aubin's inequality was motivated by the Nirenberg problem on the sphere: If R is the scalar curvature of the sphere and F a function, then find a conformal metric g' with with scalar curvature $R' = R + F$. It was shown by Kazdan and Warner [9, 10] that no solution exists if F is in the eigenspace corresponding to the first non-zero eigenvalue of the Laplacian, which condition for $F = e^{2f}$ just boils down to (3). Aubin's examination [1, Theorem 8 and Corollary 3] of the Nirenberg problem transformed the issue to an optimization problem and used (4) in the investigation of the latter. See [1] for more information on these topics.

Now, we may write functions on the sphere in terms of the angular θ and ϕ coordinates and put $x_1 = \cos\theta$. We say that a function f on $\mathbb{S}$ is *axially symmetric* if it depends only on x_1. The original Moser inequality [11] was proved by means of spherical symmetrization which replaces a general function $f \in H^1(\mathbb{S}^2)$ by a certain equimeasurable axially symmetric function f^* whose Dirichlet integral $\int_{\mathbb{S}^2} |\nabla f^*|^2\, d\omega$ does not exceed the Dirichlet integral of f. Perhaps motivated by this, Feldman, Froese, Ghoussoub and Gui [8] have considered the Aubin inequality (4) for axially symmetric functions f. Let $\mathcal{S}_r$ be the set of all axially symmetric functions f in $\mathcal{S}$. By some quite original methods they have shown [8] that

$$\inf_{f \in \mathcal{S}_r} J_\alpha(f) = 0$$

for $\frac{16}{25} - \varepsilon \leq \alpha \leq 1$, where ε is some unknown but strictly positive constant.

In this note we continue the investigation of the Aubin inequality for the axially symmetric functions from $\mathcal{S}$. We show that

$$\inf_{f \in \mathcal{S}_r} J_{1/2}(f) > -\infty$$

and that this infimum is attained at some $f \in \mathcal{S}_r$. It is however still not known whether the infimum is equal to 0, nor is it known if our result remains true if $\mathcal{S}_r$ is replaced by $\mathcal{S}$. Our work provides further evidence for the conjecture that $C_{1/2} > -\infty$. One can also hope that

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the knowledge that $J_{1/2}$ attains its infimum on $\mathcal{S}_r$ might allow one to prove various properties of the functions at which the infimum is attained (see [8] for a variational equation for such functions) and perhaps to prove that such functions must be constant.

We also recall here an inequality of Osgood, Phillips and Sarnak [13, Corollary 2.2] which will be useful to us:

$$J_{1/2}(f) \geq 0 \quad \text{for all antipodally symmetric } f \in H^1(\mathbb{S}^2), \tag{5}$$

where we say that f is *antipodally symmetric* if $f(\mathbf{x}) = f(-\mathbf{x})$ for all $\mathbf{x} \in \mathbb{S}^2$.

The reader interested in other work related to the Moser inequality may want to see, *e.g.*, [2, 3, 4, 5].

Recall that axially symmetric functions f depend only on x_1. Write $g(x) = f(x, 0, 0)$ for $x \in (-1, 1)$ and $f \in \mathcal{S}_r$. Then, following [8], note that

$$J_\alpha(f) = \frac{\alpha}{2}\int_{-1}^{1}(1-x^2)|g'(x)|^2\,dx + \int_{-1}^{1} g(x)\,dx - \log\frac{1}{2}\int_{-1}^{1} e^{2g(x)}\,dx.$$

Let $\mathcal{G}_r$ be the set of functions $g(x)$ of the form $f(x, 0, 0)$ where $f \in \mathcal{S}_r$. The set $\mathcal{G}_r$ is then equal to the set of functions g in $H^1(-1, 1)$ with $\int_{-1}^{1} xe^{2g(x)}\,dx = 0$, where $H^1(-1, 1)$ is the Sobolev space of real functions g on $(-1, 1)$ with

$$\|g\|^2 \stackrel{\text{def}}{=\!=} \int_{-1}^{1}(1-x^2)\big(g'(x)\big)^2\,dx < \infty$$

(see [8]). We shall write $H^1 = H^1(-1, 1)$ for short.

Let

$$I(g) = \frac{1}{4}\int_{-1}^{1}(1-x^2)\big(g'(x)\big)^2\,dx + \int_{-1}^{1} g(x)\,dx - \log\frac{1}{2}\int_{-1}^{1} e^{2g(x)}\,dx,$$

for $g \in H^1$. Then, $J_{1/2}(f) = I(g)$ if g is defined by $g(x) = f(x, 0, 0)$. The purpose of this note is to prove the following result.

Theorem 1 *There exists $g_0 \in \mathcal{G}_r$ such that*

$$\inf_{g\in\mathcal{G}_r} I(g) = I(g_0) > -\infty.$$

Remark 1 In fact, we shall show a little more. What we shall show is that every minimizing sequence $h_n \in \mathcal{G}_r$ for I has a subsequence h_{n_k} converging weakly in H^1 to a function h_0 such that $I(h_{n_k}) \to I(h_0)$ perhaps unless the minimum of I over $\mathcal{G}_r$ is zero.

Remark 2 The correspondence between $\mathcal{G}_r$ and $\mathcal{S}_r$ shows that our Theorem is equivalent to the assertion that $\inf_{f\in\mathcal{S}_r} J_{1/2}(f) = J_{1/2}(f_0) > -\infty$.

To prove our Theorem, first let $\mathcal{G}_s$ be the set of even functions in $H^1(-1,1)$. Let Γ_+ be the functional on $H^1(-1,1)$ given by

$$\Gamma_+(g) = \frac{1}{4}\int_0^1 (1-x^2)\big(g'(x)\big)^2\,dx + \int_0^1 g(x)\,dx.$$

Given any function $g \in H^1(-1,1)$, write $\hat{g}(x) = g(-x)$. Let $\Gamma_-(g) = \Gamma_+(\hat{g})$. Let

$$\Lambda_+(g) = \int_0^1 e^{2g(x)}\,dx.$$

Put $\Lambda_-(g) = \Lambda_+(\hat{g})$ and define

$$\tilde{I}(g) = \frac{1}{4}\int_{-1}^1 (1-x^2)\big(g'(x)\big)^2\,dx + \int_{-1}^1 g(x)\,dx - \log\min\big(\Lambda_+(g),\Lambda_-(g)\big).$$

Note that

$$\tilde{I}(g) = \Gamma_+(g) + \Gamma_-(g) - \log\min\big(\Lambda_+(g),\Lambda_-(g)\big).$$

Put $\Gamma(g) = \Gamma_+(g) + \Gamma_-(g)$ and $\Lambda(g) = \Lambda_+(g) + \Lambda_-(g)$.

Lemma 1 *We have* $\inf_{g\in H^1}\tilde{I}(g) \geq \inf_{g\in\mathcal{G}_s} I(g)$.

Recall that $\inf_{g\in\mathcal{G}_s} I(g) \geq 0$ by (5) (since if we define $f(x_1,x_2,x_3) = g(x_1)$ and if $g \in \mathcal{G}_s$, then f is an antipodally symmetric function in $H^1(\mathbb{S})$, with $J_{1/2}(f) = I(g)$). Since $\tilde{I}(0) = 0$, the following result immediately follows from this observation and Lemma 1.

Corollary 1 *We have* $\inf_{g\in H^2}\tilde{I}(g) = 0$.

We shall also need another lemma.

Lemma 2 *Suppose g_n is a sequence of functions in $\mathcal{G}_r$ such that $\|g_n\| \to \infty$. Then* $\liminf_n \tilde{I}(g_n) \geq 0$.

Proof of Lemma 1 Fix $g \in H^1$. Replacing g by $\hat{g}$ if necessary, assume $\Lambda_+(g) \geq \Lambda_-(g)$. Then,

$$\tilde{I}(g) = \Gamma_+(g) + \Gamma_-(g) - \log\Lambda_-(g).$$

Suppose first that $\Gamma_+(g) \geq \Gamma_-(g)$. Then define $h(x) = g(x)$ for $x \in (-1,0]$ and put $h(x) = g(-x)$ for $x \in [0,1)$. Clearly $h \in \mathcal{G}_s$. Moreover,

$$\Gamma_+(h) = \Gamma_-(h) = \Gamma_-(g)$$

and

$$\Lambda_+(h) = \Lambda_-(h) = \Lambda_-(g).$$

Thus

$$I(h) = 2\Gamma_-(g) - \log\Lambda_-(g) \leq \tilde{I}(g),$$

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since $\Gamma_+(g) \geq \Gamma_-(g)$.

Suppose now that $\Gamma_+(g) \leq \Gamma_-(g)$. Define $h(x) = g(x)$ for $x \in [0,1)$ and put $h(x) = g(-x)$ for $x \in (-1,0]$. Again, clearly $h \in \mathcal{G}_s$, and

$$I(h) = 2\Gamma_+(g) - \log \Lambda_+(g).$$

But $-\log \Lambda_+(g) \leq -\log \Lambda_-(g)$ and $\Gamma_+(g) \leq \Gamma_-(g)$, so that once again

$$I(h) \leq \tilde{I}(g).$$

Hence for any $g \in H^1$ we may construct an $h \in \mathcal{G}_s$ with $I(h) \leq \tilde{I}(g)$, and the desired result follows. ■

Proof of Lemma 2 Let $g_n \in \mathcal{G}_r$ be a sequence with the specified properties. Without loss of generality assume that $g_n(0) = 0$ for all n. Passing to a subsequence we may assume that $I(g_n)$ converges to some limit in $[-\infty, \infty)$. (The case of it tending to $+\infty$ is trivial.) Fix $0 < \varepsilon < \frac{1}{2}$. Moreover, fix $\delta > 0$, the choice of which will later be more clearly explained and will depend on ε. Put $M(g) = \int_{-(1-\varepsilon)}^{1-\varepsilon} e^{2g(x)}\,dx$. Fix n. Suppose first that $M(g_n) \geq \delta\Lambda(g_n)$. Then,

$$I(g_n) = \Gamma(g_n) - \log \frac{1}{2}\Lambda(g_n) \geq \Gamma(g_n) - \log[(2\delta)^{-1} M(g_n)].$$

By [8, inequality (2.3)], we have

$$|g(x)| \leq \|g\| \operatorname{arctanh}^{\frac{1}{2}}(|x|),$$

for any $g \in H^1$. It follows that

$$\log[(2\delta^{-1})M(g_n)] \leq \log[(2\delta)^{-1} \cdot 2e^{2c_1\|g_n\|}] \leq -\log\delta + 2c_1\|g_n\|,$$

where $c_1 = \operatorname{arctanh}^{\frac{1}{2}}(1-\varepsilon)$. Moreover, it also follows that

$$\int_{-1}^{1} g_n(x)\,dx \geq -\int_{-1}^{1} |g_n(x)|\,dx \geq -c_2\|g\|,$$

where $c_2 = \int_{-1}^{1} \operatorname{arctanh}^{\frac{1}{2}}(|x|)$. Hence,

$$I(g_n) \geq \frac{1}{4}\|g_n\|^2 - c_2\|g\| - 2c_1\|g\| + \log\delta.$$

But the right hand side of this expression tends to infinity as $n \to \infty$ providing $\|g_n\| \to \infty$, and we have assumed the left hand side does not. Hence, the expression $M(g_n) \geq \delta\Lambda(g_n)$ can only hold for finitely many n.

Thus, for n sufficiently large (with size depending on δ) we have $M(g_n) < \delta\Lambda(g_n)$. We shall now always assume we are given such an n. Let $N_+(g) = \int_{1-\varepsilon}^{1} e^{2g(x)}\,dx$ and $N_-(g) =$

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$N_+(\hat{g})$. I claim that $N_+(g_n) \geq (1-2\varepsilon)N_-(g_n)$ providing we have taken δ sufficiently small. For suppose that $N_+(g_n) < (1-2\varepsilon)N_-(g_n)$ for some $0 < \delta < 1$. Then,

$$\begin{aligned} 0 &= \int_{-1}^{1} x e^{2g_n(x)}\, dx \\ &\leq (\varepsilon - 1)N_-(g_n) + M(g_n) + N_+(g_n) \\ &< (\varepsilon - 1)N_-(g_n) + M(g_n) + (1-2\varepsilon)N_-(g_n) \\ &= -\varepsilon N_-(g_n) + M(g_n). \end{aligned} \tag{6}$$

But we have an n such that

$$\begin{aligned} \delta^{-1}M(g_n) &< \Lambda(g_n) \\ &= N_-(g_n) + N_+(g_n) + M(g_n) \\ &< (2-2\varepsilon)N_-(g_n) + M(g_n), \end{aligned}$$

so that $M(g_n) < \frac{\delta(2-2\varepsilon)}{1-\delta}N_-(g_n)$. Hence, (6) implies that

$$0 < \left[-\varepsilon + \frac{\delta(2-2\varepsilon)}{1-\delta}\right]N_-(g_n).$$

This leads to a contradiction if $\delta > 0$ is sufficiently small, and hence we see that indeed if $\delta > 0$ is chosen sufficiently small, then $N_+(g_n) \geq (1-2\varepsilon)N_-(g_n)$ (for n sufficiently large that $M(g_n) < \delta\Lambda(g_n)$). Applying the same argument to $\hat{g}_n$ we see that $N_-(g_n) \geq (1-2\varepsilon)N_+(g_n)$, for the same small δ. Thus,

$$N_+(g_n) + N_-(g_n) \leq 2(1-2\varepsilon)^{-1}\min\bigl(N_+(g_n), N_-(g_n)\bigr).$$

But $N_+(g_n) + N_-(g_n) = \Lambda(g_n) - M(g_n) > (1-\delta)\Lambda(g_n) \geq (1-2\varepsilon)\Lambda(g_n)$ providing we choose δ smaller than 2ε, and $\Lambda_\pm(g_n) \geq N_\pm(g_n)$, so that

$$\Lambda(g_n) \leq 2(1-2\varepsilon)^{-1}(1-2\varepsilon)^{-1}\min\bigl(\Lambda_-(g_n), \Lambda_+(g_n)\bigr).$$

Let $A = (1-2\varepsilon)^{-2}$. Then,

$$\begin{aligned} I(g_n) &= \Gamma(g_n) - \log\frac{1}{2}\Lambda(g_n) \\ &\geq \Gamma(g_n) - \log\bigl[A\min\bigl(\Lambda_-(g_n), \Lambda_+(g_n)\bigr)\bigr] \\ &= \tilde{I}(g_n) - \log A. \end{aligned}$$

But $\tilde{I}(g_n) \geq 0$ by Corollary 1. Hence, $I(g_n) \geq -\log A = 2\log(1-2\varepsilon)$. This holds for all sufficiently large n, the size of n depending on δ which in turn depends on ε. Since $\varepsilon \in (0, \frac{1}{2})$ was arbitrary, it follows that $\liminf I(g_n) \geq 0$ as desired. ■

Proof of Theorem Let $g_n \in \mathcal{G}_r$ be a minimizing sequence for I, *i.e.*, suppose that $\lim I(g_n) = \inf_{g\in\mathcal{G}_r} I(g)$. Passing to a subsequence, assume $\|g_n\|$ converges to some number

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in $[0,\infty]$. If $\|g_n\| \to \infty$ then $\liminf I(g_n) \geq 0$ by Lemma 2, and so $\inf_{g\in\mathcal{G}_r} I(g) \geq 0$. But $I(0) = 0$, so that by letting $g_0 = 0$ we are done. Suppose now that $\|g_n\|$ is a bounded sequence. Then, passing to a subsequence if necessary, we may assume g_n converges weakly to some g_0.

Weak convergence in $H^1(-1,1)$ clearly entails almost everywhere convergence on $(-1,1)$. Moser's inequality (1) applied to $2g_n$ and the fact that $\|g_n\|$ is a bounded sequence then imply that

$$\sup_n \int_{-1}^{1} e^{4g_n(x)}\,dx < \infty$$

(*cf.* [8, Proof of Theorem 1.1]). Hence the e^{2g_n} are uniformly bounded in $L^2(-1,1)$, and thus are uniformly integrable, so that

$$\Lambda(g_n) \to \Lambda(g_0),$$

and, *a fortiori*,

$$\int_{-1}^{1} g_n(x)\,dx \to \int_{-1}^{1} g_0(x)\,dx.$$

The functions $x \mapsto xe^{2g_n(x)}$ are also of course uniformly bounded in $L^2(-1,1)$ and thus uniformly integrable, so that

$$\int_{-1}^{1} xe^{2g_n(x)}\,dx \to \int_{-1}^{1} xe^{2g_0(x)}\,dx,$$

which implies that $g_0 \in \mathcal{G}_r$ since each g_n lies in $\mathcal{G}_r$. Weak convergence, on the other hand, implies that $\liminf \|g_n\| \geq \|g_0\|$. We thus see that $I(g_0) \leq \liminf I(g_n) = \inf_{g\in\mathcal{G}_r} I(g)$, and since $g_0 \in \mathcal{G}_r$ it follows that the inequality must be an equality. And of course by (2) we must have $I(g_0) > -\infty$. ■

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