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SCALAR EXTENSION OF QUADRATIC LATTICES II

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Let k be a totally real algebraic number field, $\mathfrak{O}$ the maximal order of k, and let L (resp. M) be a $\boldsymbol{Z}$-lattice of a positive definite quadratic space U (resp. V) over the field $\boldsymbol{Q}$ of rational numbers. Suppose that there is an isometry σ from $\mathfrak{O}L$ onto $\mathfrak{O}M$. We have shown that the assumption implies $\sigma(L) = M$ in some cases in [2]. Our aim in this paper is to improve the results of [2]. In §1 we introduce the notion of E-type: Let L be a positive definite quadratic lattice over $\boldsymbol{Z}$. If any minimal vector of $L \otimes M$ is of the form $x \otimes y$ $(x \in L, y \in M)$ for any positive definite quadratic lattice M over $\boldsymbol{Z}$, then we say that L is of E-type. Some sufficient conditions for E-type are given in §1 and they are applied to our aim in §2.

NOTATIONS. As usual $\boldsymbol{Z}$ (resp. $\boldsymbol{Q}$) is the ring (resp. the field) of rational integers (resp. of rational numbers). By a positive definite quadratic lattice L over $\boldsymbol{Z}$ we mean a $\boldsymbol{Z}$-lattice L of a positive definite quadratic space V over $\boldsymbol{Q}$ (rank $L = \dim V$). For a positive definite quadratic lattice L we denote $\min Q(x)$ by $m(L)$ where Q is the quadratic form of L and x runs over non-zero elements of L, and we call an element x of L a minimal vector of L if $Q(x) = m(L)$. $Q(x), B(x, y)$ denote quadratic forms and corresponding bilinear forms $(2B(x, y) = Q(x + y) - Q(x) - Q(y))$.

§1. Let L, M be positive definite quadratic lattices over $\boldsymbol{Z}$ with bilinear forms B_L, B_M respectively. Then the tensor product $L \otimes M$ over $\boldsymbol{Z}$ can be made into a positive definite quadratic lattice over $\boldsymbol{Z}$ with bilinear form B such that $B(x_1 \otimes y_1, x_2 \otimes y_2) = B_L(x_1, x_2)B_M(y_1, y_2)$ for any $x_i \in L, y_i \in M$. Hereafter the tensor product $L \otimes M$ means this positive definite quadratic lattice over $\boldsymbol{Z}$. Let x (resp. y) be a minimal vector of L (resp. M); then $x \otimes y \in L \otimes M$ implies $m(L \otimes M) \leq m(L)m(M)$. It is

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known by Steinberg (p. 47 in [3]) that there is an example of L, M such that $m(L \otimes M) < m(L)m(M)$.

DEFINITION. Let L be a positive definite quadratic lattice over $\boldsymbol{Z}$. We say that L is of E-type if every minimal vector of $L \otimes M$ is of the form $x \otimes y$ $(x \in L, y \in M)$ for any positive definite quadratic lattice M over $\boldsymbol{Z}$. Then x (resp. y) is a minimal vector of L (resp. M), and $m(L \otimes M)$ is equal to $m(L)m(M)$.

PROPOSITION 1. *If L_1, L_2 are of E-type,*[*] *then $L_1 \perp L_2, L_1 \otimes L_2$ are of E-type.*

Proof. Let M be a positive definite quadratic lattice over $\boldsymbol{Z}$. Let v be a minimal vector of $(L_1 \perp L_2) \otimes M$; then v is of the form $x + y$ $(x \in L_1 \otimes M, y \in L_2 \otimes M)$. Since x is orthogonal to y, we have $Q(v) = Q(x) + Q(y)$. The minimality of $Q(v)$ yields $x = 0$ or $y = 0$. Hence v is in $L_1 \otimes M$ or $L_2 \otimes M$, and v is of the form $u \otimes w$ $(u \in L_1$ or L_2, $w \in M)$. This means that $L_1 \perp L_2$ is of E-type. Every minimal vector of $L_1 \otimes L_2 \otimes M$ is of the form $x_1 \otimes y$ where x_1 (resp. y) is a minimal vector of L_1 (resp. $L_2 \otimes M$). As y is of the form $x_2 \otimes z$ $(x_2 \in L_2, z \in M)$, we have $x_1 \otimes y = x_1 \otimes x_2 \otimes z$, and $x_1 \otimes x_2$ is a minimal vector of $L_1 \otimes L_2$. Hence $L_1 \otimes L_2$ is of E-type.

PROPOSITION 2. *Let L be of E-type. If a submodule L_1 of L satisfies $m(L_1) = m(L)$, then L_1 is of E-type.*

Proof. Let M be a positive definite quadratic lattice over $\boldsymbol{Z}$. Since we have $m(L)m(M) = m(L \otimes M) \leq m(L_1 \otimes M) \leq m(L_1)m(M) = m(L)m(M)$, a minimal vector v of $L_1 \otimes M$ is one of $L \otimes M$. Hence v is of the form $x \otimes y$ $(x \in L, y \in M)$. As y is primitive in M, x is in L_1. Therefore L_1 is of E-type.

DEFINITION. Let n be a natural number. We put $\mu_n = \max \dfrac{m(A)}{\sqrt[n]{|A|}}$, where A runs over positive definite real symmetric matrices with degree n, and $m(A) = \min\limits_{x \in \boldsymbol{Z}^n - \{0\}} {}^t x A x$.

LEMMA 1. *If $n \geq 40$, then $\mu_n < \dfrac{n}{6}$.*

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[*] When we say that L is of E-type, L is assumed to be a positive definite quadratic lattice over $\boldsymbol{Z}$.

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Proof. It is known by [1] that

$$\mu_n < \frac{2}{\pi}\Gamma\left(2+\frac{n}{2}\right)^{2/n}.$$

Since $\Gamma(x) = \sqrt{2\pi}x^{x-1/2}e^{-x+\mu(x)}$ $\left(x>0, \mu(x)=\frac{\theta}{12x}, 0<\theta<1\right)$, we have $\mu_n < \frac{2}{\pi}(2\pi)^{1/n}\left(2+\frac{n}{2}\right)^{1+3/n}e^{-4/n-1+1/3n(n+4)}$. Put $f(x) = \log\frac{x}{6} - \log\left\{\frac{2}{\pi}(2\pi)^{1/x}\left(2+\frac{x}{2}\right)^{1+3/x}e^{-4/x-1+1/3x(x+4)}\right\}$. If $f(x)>0$ for $x \geq 40$, then Lemma is true. Since $f(x) = \log x - \log 6 - \log\frac{2}{\pi} - \frac{1}{x}\log 2\pi - \left(1+\frac{3}{x}\right)\log\left(2+\frac{x}{2}\right) + \frac{4}{x} + 1 - \frac{1}{3x(x+4)}$, we get

$$x^2f'(x) = \log 2\pi + 3\log\left(2+\frac{x}{2}\right) - 3 - \frac{4}{x+4} + \frac{2x+4}{3(x+4)^2} > 3\log 22 - 3 - \frac{1}{11} > 0 \qquad \text{if } x \geq 40 .$$

Hence we have only to show $f(40) > 0$. This is easy to see.

We denote by κ the maximum of the number k which satisfies that $\mu_r \geqq \sqrt{r}$ and $r \leq k$ imply $r=1$.

LEMMA 2. *κ is not smaller than* 42.

Proof. It is known that μ_n $(1 \leq n \leq 8)$ is $1, \sqrt{4/3}, \sqrt[3]{2}, \sqrt[4]{4}, \sqrt[5]{8}, \sqrt[6]{64/3}, \sqrt[7]{64}$, and 2 respectively. Hence $\kappa \geq 8$. Put

$$g(x) = \log\frac{2}{\pi}(2\pi)^{1/x}\left(2+\frac{x}{2}\right)^{1+3/x}e^{-4/x-1+1/3x(x+4)} - \log\sqrt{x}.$$

Since $\log\mu_n - \log\sqrt{n} < g(n)$, we have only to show $g(x) \leq 0$ for $8 \leq x \leq 42$. Then $x^2g'(x) = \frac{x}{2} - \log 2\pi - 3\log\left(2+\frac{x}{2}\right) + 3 + \frac{4}{x+4} - \frac{2x+4}{3(x+4)^2}$. Putting $h(x) = x^2g'(x)$, we have

$$h'(x) = \frac{1}{2} - 3\frac{1}{x+4} - \frac{4}{(x+4)^2} - \frac{2}{3(x+4)^2} + \frac{4(x+2)}{3(x+4)^3} = \frac{1}{6(x+4)^3}(3x^3+18x^2-20x-176) .$$

Since $3x^3+18x^2-20x-176>0$ for $x\geq 8$, we get $h'(x)>0$. Moreover $h(8)$ is positive. Hence $g'(x)$ is positive for $x\geq 8$. $g(42)<0$ is easy to see.

Remark. Rogers' result [5] may improve the number 42.

LEMMA 3. *Let A, B be positive definite real symmetric matrices with degree n; then we have* $\operatorname{Tr}(AB)\geq n\sqrt[n]{|A|}\sqrt[n]{|B|}$.

Proof. Put $B=D[T]$ where D is diagonal and T is orthogonal. Let $a_1,\cdots,a_n$ and $d_1,\cdots,d_n$ be diagonals of TA^tT, D respectively. Then

$$\begin{aligned}\operatorname{Tr}(AB)&=\operatorname{Tr}(AD[T])=\operatorname{Tr}(TA^tTD)=\sum a_id_i\geq n\sqrt[n]{\prod(a_id_i)}\\&=n\sqrt[n]{|B|}\sqrt[n]{\prod a_i}\geq n\sqrt[n]{|B|}\sqrt[n]{|TA^tT|}=n\sqrt[n]{|A|}\sqrt[n]{|B|}.\end{aligned}$$

THEOREM 1. *If L is a positive definite quadratic lattice over $\boldsymbol{Z}$ with rank $L\leq\kappa$, then L is of E-type.*

Proof. Taking a positive definite quadratic lattice M over $\boldsymbol{Z}$, we put a minimal vector v of $L\otimes M=\sum_{i=1}^r x_i\otimes y_i\ (x_i\in L, y_i\in M)$. In these representations of v we take one with minimal r. Then $x_1,\cdots,x_r$ and $y_1,\cdots,y_r$ is linearly independent in L, M respectively. Noting $Q(v)=Q(\sum x_i\otimes y_i)=\sum_{i,j}B(x_i,x_j)B(y_i,y_j)=\operatorname{Tr}((B(x_i,x_j)(B(y_i,y_j))$, we get $Q(v)\geq r(|(B(x_i,x_j))||(B(y_i,y_j))|)^{1/r}$ by Lemma 3. On the other hand $Q(v)=m(L\otimes M)\leq m(L)m(M)\leq m(\boldsymbol{Z}[x_1,\cdots,x_r])m(\boldsymbol{Z}[y_1,\cdots,y_r])$. Therefore

$$r\leq\frac{m(\boldsymbol{Z}[x_1,\cdots,x_r])}{|(B(x_i,x_j))|^{1/r}}\,\frac{m(\boldsymbol{Z}[y_1,\cdots,y_r])}{|(B(y_i,y_j))|^{1/r}}\leq\mu_r^2\,.$$

By the definition of κ we have $r=1$. This completes the proof.

Remark. In the Steinberg's example for $m(L\otimes M)<m(L)m(M)$, rank $L\geq 292$.

THEOREM 2. *Let L be a positive definite quadratic lattice over $\boldsymbol{Z}$. If $m(L)\leqq 6$, and the discriminant dL_0 of any non-zero submodule L_0 of L is not smaller than* 1, *then L is of E-type.*

Proof. Let M be a positive definite quadratic lattice over $\boldsymbol{Z}$, and let a minimal vector v of $L\otimes M$ be $\sum_{i=1}^r x_i\otimes y_i$. As in the proof of Theorem 1 we may assume that $x_1,\cdots,x_r$, and $y_1,\cdots,y_r$ are linearly independent in L, M respectively. Put $L_0=\boldsymbol{Z}[x_1,\cdots,x_r]$ and $M_0=\boldsymbol{Z}[y_1,\cdots,y_r]$. Then $m(L\otimes M)=Q(v)\geq r\sqrt[r]{dL_0}\sqrt[r]{dM_0}\geq r\sqrt[r]{dM_0}$. On the other

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hand $m(L \otimes M) \leq m(L)m(M) \leq 6m(M_0)$. Hence we get $r/6 \leq m(M_0)/\sqrt[r]{dM_0} \leq \mu_r$. Lemma 1 implies $r \leq 40$, and Lemma 2 implies that L_0 is of E-type. Since $v \in L_0 \otimes M_0$ and $m(L \otimes M) \leq m(L_0 \otimes M_0)$, v is a minimal vector of $L_0 \otimes M_0$. Therefore v is of the form $x \otimes y$ $(x \in L_0, y \in M_0)$, and this completes the proof.

§ **2.** We apply the results of § 1 to our problem. Some other applications will appear in the forthcoming paper.

In this section E denotes a totally real algebraic number field with degree n, and $\mathfrak{O}$ is the maximal order of E. From Theorem III of p. 2 in [6] follows that $\mathrm{tr}_{E/Q}\, a^2 \geq n$ for any non-zero element a of $\mathfrak{O}$, and moreover the equality yields $a = \pm 1$. Let L be a positive definite quadratic lattice over $\boldsymbol{Z}$; then we denote by $\mathfrak{O}L$ the tensor product of $\mathfrak{O}$ and L as an extension of coefficient ring $\boldsymbol{Z}$ of L to $\mathfrak{O}$. By definition an element v of $\mathfrak{O}L$ gives the rational minimum of $\mathfrak{O}L$ if and only if $Q(v) = \min Q(u)$ where u runs over a non-zero element of $\mathfrak{O}L$ with $Q(u) \in \boldsymbol{Q}$. When we regard $\mathfrak{O}$ as a positive definite quadratic lattice over $\boldsymbol{Z}$ with the bilinear form $B(x, y) = \mathrm{tr}_{E/Q}\, xy$, we write $\tilde{\mathfrak{O}}$ instead of $\mathfrak{O}$.

LEMMA. *Let L be a positive definite quadratic lattice over $\boldsymbol{Z}$. If $\tilde{\mathfrak{O}}$ or L is of E-type, then a vector of $\mathfrak{O}L$ which gives the rational minimum of $\mathfrak{O}L$ is already in L.*

Proof. As indicated in the introduction B denotes the bilinear form of L. We define a new bilinear form $\tilde{B}$ on $\mathfrak{O}L$ which is defined by $\tilde{B}(x, y) = \mathrm{tr}_{E/Q}\, B(x, y)$ $(x, y \in \mathfrak{O}L)$. This quadratic lattice is denoted by $(\mathfrak{O}L, \tilde{B})$. As $\tilde{B}(a_1x_1, a_2x_2) = \mathrm{tr}_{E/Q}\, a_1a_2 \cdot B(x_1, x_2)$ for $a_i \in \mathfrak{O}, x_i \in L$, a quadratic lattice $(\mathfrak{O}L, \tilde{B})$ is isometric to $\tilde{\mathfrak{O}} \otimes L$. Take a vector v of $\mathfrak{O}L$ which gives the rational minimum of $\mathfrak{O}L$; then we have

$$0 \neq \tilde{B}(v, v) = nQ(v) \leq nm(L) = m(\tilde{\mathfrak{O}})m(L) = m(\tilde{\mathfrak{O}} \otimes L) = m((\mathfrak{O}L, \tilde{B})) .$$

Hence v is a minimal vector of $(\mathfrak{O}L, \tilde{B})$. Regarding v as an element of $\tilde{\mathfrak{O}} \otimes L$, we get $v = a \otimes x (a \in \mathfrak{O}, x \in L)$, where a is a minimal vector of $\tilde{\mathfrak{O}}$, and so $a = \pm 1$. This implies $v \in L$.

THEOREM. *Let L, M be positive definite quadratic lattices over $\boldsymbol{Z}$. Assume that* $\mathrm{rank}\, L \leq \kappa$ *or $\tilde{\mathfrak{O}}$ is of E-type. Then, for any isometry σ from $\mathfrak{O}L$ on $\mathfrak{O}M$ over the ring $\mathfrak{O}$, we get $\sigma(L) = M$.*

Proof. Lemma implies that a vector v of L which gives the rational minimum of $\mathfrak{O}L$ is in L, and $\sigma(v)$ is also in M since $\sigma(v)$ gives the rational minimum of $\mathfrak{O}M$. Therefore σ induces an isometry from $\mathfrak{O}v^{\perp}$ on $\mathfrak{O}\sigma(v)^{\perp}$. Inductively we get $\sigma(\boldsymbol{Q}L) = \boldsymbol{Q}M$. $\sigma(\mathfrak{O}L) = \mathfrak{O}M$ yields $\sigma(L) = M$.

Remark 1. If $n \leq \kappa$ or $n/m \leq 6$ where $m\boldsymbol{Z} = \{\mathrm{tr}_{E/\boldsymbol{Q}}\, a\,;\, a \in \mathfrak{O}\}\, (m > 0)$, then Theorem 1, 2 in § 1 imply that $\tilde{\mathfrak{O}}$ is of E-type.

Remark 2. Assume that $E = E_1E_2$ where E_i is a totally real algebraic number field with maximal order $\mathfrak{O}_i$. Moreover we assume that $(dE_1, dE_2) = 1$ and $\tilde{\mathfrak{O}}_i$ is of E-type $(i = 1, 2)$. Then $\tilde{\mathfrak{O}} \cong \tilde{\mathfrak{O}}_1 \otimes \tilde{\mathfrak{O}}_2$ is of E-type.

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