# ON $\mathfrak{P}$-ADIC INTEGRAL REPRESENTATIONS OF FINITE GROUPS

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**1. Introduction.** It has been shown by Diederichsen [2] that for integral representations of a finite group, the irreducible constituents in any complete reduction are not necessarily unique up to order and unimodular equivalence. In this same article, it is shown that for certain finite groups, such as the cyclic group of order 4, there are infinitely many classes of indecomposable representations under unimodular equivalence.

A natural method for studying these problems of arithmetical representation theory would be the $\mathfrak{P}$-adic approach, and as a first step in this direction, using the methods of Hensel and of Brauer and Nesbitt [**1**], we shall show that the theory of representations of finite groups, over a ring of $\mathfrak{P}$-adic integers can always be brought back to the modular case, in so far as it is concerned with questions of unimodular equivalence, reduction, and decomposition.

More particularly, we shall show that for any finite group, if $\mathfrak{P}$ is a generator of the maximal ideal in the ring of $\mathfrak{P}$-adic integers considered, and if $\mathfrak{P}^{k_0}$ is the highest power of $\mathfrak{P}$ dividing the order of the group, then unimodular equivalence may be considered modulo $\mathfrak{P}^k$, for any $k > k_0$, while unimodular reduction and decomposition may be considered modulo $\mathfrak{P}^k$, for any $k > 2k_0$, without any loss of generality.

As a corollary, we shall show that if $\mathfrak{P}$ does not divide the order of the group then all questions of unimodular equivalence, reduction and decomposition are completely equivalent to these same questions modulo $\mathfrak{P}$.

**2. Modular binding systems.**[1] Let $\mathfrak{O}$ be a commutative ring with a 1-element and let $\mathfrak{A}$ be an ideal of $\mathfrak{O}$ (possibly the null ideal). Let $\mathfrak{H}$ be a hypercomplex system over $\mathfrak{O}$ and let $\Gamma$ and $\Delta$ be two $\mathfrak{A}$-modular representations of $\mathfrak{H}$, by matrices with entries in $\mathfrak{O}$, of degrees $n_1$ and $n_2$ respectively.

We shall consider the $\mathfrak{A}$-modular representations of $\mathfrak{H}$, by matrices with entries in $\mathfrak{O}$, having $\Gamma$ as a top constituent and $\Delta$ as the corresponding bottom constituent, i.e.

$$x \to \mathfrak{D}(x) = \begin{pmatrix} \Gamma(x) & \Lambda(x) \\ \theta(x) & \Delta(x) \end{pmatrix} \tag{1}$$

where $\theta(x) \equiv 0 \pmod{\mathfrak{A}}$ for all $x \in \mathfrak{H}$. Since $\mathfrak{D}$ is an $\mathfrak{A}$-modular representation of $\mathfrak{H}$, the following laws must hold:

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[1]For an account of the theory of ordinary binding systems, see [**4**, pp. 276–279; **2**, pp. 364–374].



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$$\mathfrak{D}(x+y) \equiv \mathfrak{D}(x) + \mathfrak{D}(y) \pmod{\mathfrak{A}},$$
$$\mathfrak{D}(cx) \equiv c\mathfrak{D}(x) \pmod{\mathfrak{A}},$$
$$\mathfrak{D}(xy) \equiv \mathfrak{D}(x)\mathfrak{D}(y) \pmod{\mathfrak{A}},$$

for all $x, y \in \mathfrak{H}$ and all $c \in \mathfrak{O}$. From these laws one deduces the following:

$$\text{(2)} \qquad \begin{aligned} \Lambda(x+y) &\equiv \Lambda(x) + \Lambda(y) && \pmod{\mathfrak{A}},\\ \Lambda(cx) &\equiv c\Lambda(x) && \pmod{\mathfrak{A}},\\ \Lambda(xy) &\equiv \Gamma(x)\Lambda(y) + \Lambda(x)\Delta(y) && \pmod{\mathfrak{A}}. \end{aligned}$$

Any system $\Lambda = \{\Lambda(x)\}$ ($x$ running through all the elements of $\mathfrak{H}$) of $n_1 \times n_2$ $\mathfrak{O}$-matrices obeying the laws (2) will be called an $\mathfrak{A}$-modular binding system determined by the representations $\Gamma$ and $\Delta$. Evidently, any such system determines an $\mathfrak{A}$-modular representation of $\mathfrak{H}$ of the type (1) for any choice of $\theta$, provided $\theta(x) \equiv 0 \pmod{\mathfrak{A}}$ for all $x \in \mathfrak{H}$. Because of the linearity of the congruences (2), it is easily verified that the set $\mathfrak{B}(\Gamma, \Delta, \mathfrak{A})$ of all $\mathfrak{A}$-modular binding systems determined by $\Gamma$ and $\Delta$ is an $\mathfrak{O}$-module under the following operations:

$$\text{(3)} \qquad \begin{aligned} \{(\Lambda + \Lambda')(x)\} &= \{\Lambda(x) + \Lambda'(x)\},\\ \{(c\Lambda)(x)\} &= \{c\Lambda(x)\}, \end{aligned}$$

for all $x \in \mathfrak{H}$, all $c \in \mathfrak{O}$, and all $\Lambda, \Lambda' \in \mathfrak{B}(\Gamma, \Delta, \mathfrak{A})$.

Two $\mathfrak{A}$-modular representations of the type (1)

$$\mathfrak{D}_i(x) = \begin{pmatrix} \Gamma(x) & \Lambda_i(x) \\ \theta_i(x) & \Delta(x) \end{pmatrix} \qquad (i = 1, 2),$$

where $\theta_i(x) \equiv 0 \pmod{\mathfrak{A}}$ for all $x \in \mathfrak{H}$, are said to be "strongly" equivalent if there is a matrix

$$P = \begin{pmatrix} I_{n_1} & T \\ & I_{n_2} \end{pmatrix},$$

where $I_{n_1}$ and $I_{n_2}$ are the unity of matrices of degrees $n_1$ and $n_2$ respectively, and where $T$ is any $n_1 \times n_2$ $\mathfrak{O}$-matrix, such that

$$P^{-1}\mathfrak{D}_1(x)P \equiv \mathfrak{D}_2(x) \pmod{\mathfrak{A}},$$

for all $x \in \mathfrak{H}$. Since

$$P^{-1} = \begin{pmatrix} I_{n_1} & -T \\ & I_{n_2} \end{pmatrix},$$

this implies that

$$\text{(4)} \qquad \Lambda_2(x) \equiv \Lambda_1(x) + \Big(\Gamma(x)T - T\Delta(x)\Big) \pmod{\mathfrak{A}},$$

for all $x \in \mathfrak{H}$.

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Conversely, if $\Lambda_1$ and $\Lambda_2$ are two $\mathfrak{A}$-modular binding systems in $\mathfrak{B}(\Gamma, \Delta, \mathfrak{A})$, for which there exists an $n_1 \times n_2$ $\mathfrak{O}$-matrix $T$ such that the condition (4) is satisfied, then evidently, the $\mathfrak{A}$-modular representations of the type (1), $\mathfrak{D}_1$ and $\mathfrak{D}_2$, determined by $\Lambda_1$ and $\Lambda_2$ respectively, are strongly equivalent and the transforming matrix is $P$ as given above.

If the condition (4) holds for the binding systems $\Lambda_1$ and $\Lambda_2$, they are said to be strongly equivalent.

It is easily verified that the set $\mathfrak{B}_0(\Gamma, \Delta, \mathfrak{A})$ of all $\mathfrak{A}$-modular binding systems $\Lambda \in \mathfrak{B}(\Gamma, \Delta, \mathfrak{A})$, which are strongly equivalent to zero, i.e., for which there exists an $n_1 \times n_2$ $\mathfrak{O}$-matrix $T$ such that

$$\Lambda(x) \equiv \Gamma(x)T - T\Delta(x) \pmod{\mathfrak{A}}, \tag{5}$$

for all $x \in \mathfrak{H}$, is an $\mathfrak{O}$-submodule of $\mathfrak{B}(\Gamma, \Delta, \mathfrak{A})$. Evidently, the statement $\Lambda \in \mathfrak{B}_0(\Gamma, \Delta, \mathfrak{A})$ means that the representation

$$x \to \mathfrak{D}(x) = \begin{pmatrix} \Gamma(x) & \Lambda(x) \\ \theta(x) & \Delta(x) \end{pmatrix},$$

where $\theta(x) \equiv 0 \pmod{\mathfrak{A}}$ for all $x \in \mathfrak{H}$, is fully reducible,[2] modulo $\mathfrak{A}$.

From now on, we shall always suppose that $\mathfrak{H}$ is the group algebra of some finite group $\mathfrak{G}$ of order $N$, and we shall confine outselves to the $\mathfrak{A}$-modular representations of $\mathfrak{H}$ which map the unity element of $\mathfrak{G}$ onto the unity matrix, modulo $\mathfrak{A}$. The fundamental theorem is the following:

THEOREM 1. *For any binding system* $\Lambda \in \mathfrak{B}(\Gamma, \Delta, \mathfrak{A})$,

$$N \cdot \Lambda \in \mathfrak{B}_0(\Gamma, \Delta, \mathfrak{A}).$$

*Proof.* Let $x$ and $y$ be any two elements of $\mathfrak{G}$ and let $\Lambda$ be any binding system in $\mathfrak{B}(\Gamma, \Delta, \mathfrak{A})$. Then

$$\Lambda(xy) \equiv \Gamma(x)\Lambda(y) + \Lambda(x)\Delta(y) \pmod{\mathfrak{A}},$$

$$\Lambda(xy)\Delta(y^{-1}) \equiv \Gamma(x)\Lambda(y)\Delta(y^{-1}) + \Lambda(x) \pmod{\mathfrak{A}},$$

and therefore

$$\Lambda(x) \equiv \Lambda(xy)\,\Delta((xy)^{-1}) \cdot \Delta(x) - \Gamma(x) \cdot \Lambda(y)\,\Delta(y^{-1}) \pmod{\mathfrak{A}}.$$

Now let $y$ run through all the elements of $\mathfrak{G}$ and sum:

$$N \cdot \Lambda(x) \equiv \sum_y \Lambda(xy)\,\Delta((xy)^{-1}) \cdot \Delta(x) - \Gamma(x) \cdot \sum_y \Lambda(y)\,\Delta(y^{-1}).$$

[2]In this article we will use the expressions *complete reduction* and *complete decomposition* of a representation to denote a reduction of this representation into its irreducible constituents and a decomposition into its indecomposable components respectively. The term *full reduction* will be used to denote a reduction which can be transformed into a decomposition where the components are equivalent to the constituents of the given reduction.

Since, as $y$ runs through all the elements of $\mathfrak{G}$, $xy$ also runs through all the elements of $\mathfrak{G}$, we may write

$$N\cdot \Lambda(x) \equiv \sum_z \Lambda(z)\,\Delta(z^{-1})\cdot\Delta(x) - \Gamma(x)\cdot\sum_z \Lambda(z)\,\Delta(z^{-1}).$$

Setting $T = -\sum \Lambda(z)\Delta(z^{-1})$, we obtain

$$N\cdot \Lambda(x) \equiv \Gamma(x)T - T\Delta(x) \pmod{\mathfrak{A}}$$

for all $x \in \mathfrak{G}$. Since this last condition is a linear congruence, it will also hold for all the elements of $\mathfrak{H}$, so that

$$N\cdot \Lambda(x) \equiv \Gamma(x)T - T\Delta(x) \pmod{\mathfrak{A}}$$

for all $x \in \mathfrak{H}$, i.e. $N\cdot\Lambda \in \mathfrak{B}_0(\Gamma, \Delta, \mathfrak{A})$.

**3. On the connections between $\mathfrak{P}$-adic integral and modular representations of a finite group.** From now on, we will suppose that the ring $\mathfrak{O}$ considered in the preceding section is the ring of $\mathfrak{P}$-adic integers of some $\mathfrak{P}$-adic field[3] $K$.

Let $\mathfrak{P}$ be a generator of the maximal ideal of $\mathfrak{O}$ and let $\mathfrak{P}^{k_0}$ be the highest power of $\mathfrak{P}$ dividing $N$. $N$ is therefore a unit times $\mathfrak{P}^{k_0}$.

THEOREM 2. *If $\Gamma$ and $\Delta$ are two ordinary $\mathfrak{P}$-adic integral representations of $\mathfrak{H}$, then $\Gamma$ and $\Delta$ are unimodularly equivalent if and only if they are unimodularly equivalent, modulo $\mathfrak{P}^k$, for any $k > k_0$.*

*Proof.* If $\Gamma$ and $\Delta$ are unimodularly equivalent, modulo $\mathfrak{P}^k$, then they must be of the same degree ($n_1 = n_2 = n$), and there must exist an $n \times n$ $\mathfrak{O}$-matrix $T$ such that

$$\Gamma(x)T - T\Delta(x) \equiv 0 \pmod{\mathfrak{P}^k} \tag{6}$$

for all $x \in \mathfrak{H}$ and $|T| \not\equiv 0 \pmod{\mathfrak{P}}$. We will now apply the theory of the preceding section in the case that $\mathfrak{A}$ is the null ideal (0), since $\Gamma$ and $\Delta$ are ordinary $\mathfrak{P}$-adic integral representations. Then congruence modulo $\mathfrak{A}$ just means equality.

From (6), the matrices

$$\Gamma(x)\left(\frac{T}{\mathfrak{P}^k}\right) - \left(\frac{T}{\mathfrak{P}^k}\right)\Delta(x)$$

are integral for all $x \in \mathfrak{H}$, and one can easily verify that the set of all these matrices is a binding system in $\mathfrak{B}(\Gamma, \Delta, (0))$. (This is not necessarily true in the case that $\Gamma$ and $\Delta$ are modular representations.) Then, by Theorem 1,

$$N\left\{\Gamma(x)\left(\frac{T}{\mathfrak{P}^k}\right) - \left(\frac{T}{\mathfrak{P}^k}\right)\Delta(x)\right\} \in \mathfrak{B}_0(\Gamma, \Delta, (0)),$$

i.e., there exists an $n \times n$ $\mathfrak{O}$-matrix $T'$ such that

$$N\left(\Gamma(x)\left(\frac{T}{\mathfrak{P}^k}\right) - \left(\frac{T}{\mathfrak{P}^k}\right)\Delta(x)\right) = \Gamma(x)T' - T'\Delta(x)$$

[3]For an exposition of the theory of fields with valuations, see [**3**, chap. X].

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for all $x \in \mathfrak{H}$. But since $N = u\,\mathfrak{P}^k$, where $u$ is a unit of $\mathfrak{O}$, we may write

$$\Gamma(x)\left(\frac{T}{\mathfrak{P}^{k-k_0}}\right) - \left(\frac{T}{\mathfrak{P}^{k-k_0}}\right)\Delta(x) = \Gamma(x)T'' - T''\,\Delta(x)$$

for all $x \in \mathfrak{H}$, where $T'' = u^{-1}\,T'$ is an $n \times n$ $\mathfrak{O}$-matrix. Then

$$\Gamma(x)(T - \mathfrak{P}^{k-k_0}T'') - (T - \mathfrak{P}^{k-k_0}T'')\,\Delta(x) = 0$$

for all $x \in \mathfrak{H}$ and setting $T^* = T - \mathfrak{P}^{k-k_0}\,T''$, we obtain

$$\Gamma(x)\,T^* - T^*\Delta(x) = 0$$

for all $x \in \mathfrak{H}$, and $T^* \equiv T \pmod{\mathfrak{P}^{k-k_0}}$. Since $k > 0$, this implies that $T^* \equiv T \pmod{\mathfrak{P}}$, and therefore $|T^*| \equiv |T| \not\equiv 0 \pmod{\mathfrak{P}}$, so that $T^*$ is unimodular. The converse is immediate.

THEOREM 3. *If $\mathfrak{D}$ is an ordinary $\mathfrak{P}$-adic integral representation of $\mathfrak{H}$ of degree $n$ and if there is an $n \times n$ $\mathfrak{O}$-matrix $Q$ such that*

$$Q^{-1}\mathfrak{D}(x)Q = \begin{pmatrix} \Gamma(x) & \Lambda(x) \\ \mathfrak{P}^k\theta(x) & \Delta(x) \end{pmatrix},$$

*for all $x \in \mathfrak{H}$, and $|Q| \not\equiv 0 \pmod{\mathfrak{P}}$, so that $\Gamma$ and $\Delta$ are modular constituents of $\mathfrak{D}$, modulo $\mathfrak{P}^k$, of degrees $n_1$ and $n_2$ respectively, then there is an $n \times n$ $\mathfrak{O}$-matrix $M$ such that*

$$M^{-1}\mathfrak{D}(x)M = \begin{pmatrix} \Gamma^*(x) & \Lambda(x) \\ & \Delta^*(x) \end{pmatrix},$$

*for all $x \in \mathfrak{H}$, and $|M| \not\equiv 0 \pmod{\mathfrak{P}}$, where the ordinary constituents $\Gamma^*$ and $\Delta^*$, of degrees $n_1$ and $n_2$ respectively, are such that*

$$\begin{aligned} \Gamma^*(x) &\equiv \Gamma(x) \\ \Delta^*(x) &\equiv \Delta(x) \end{aligned} \qquad \pmod{\mathfrak{P}^{k-k_0}}$$

*for all $x \in \mathfrak{H}$.*

*Proof.* Let $\mathfrak{D}_0 = Q^{-1}\mathfrak{D}Q$ and assume that we have determined a finite sequence of $n \times n$ $\mathfrak{O}$-matrices

$$Q_1 = \begin{pmatrix} I_{n_1} & \\ & I_{n_2} \end{pmatrix}, \quad Q_2 = \begin{pmatrix} I_{n_1} & \\ \mathfrak{P}^{k-k_0}T_2 & I_{n_2} \end{pmatrix}, \ldots, \quad Q_m = \begin{pmatrix} I_{n_1} & \\ \mathfrak{P}^{k-k_0}T_m & I_{n_2} \end{pmatrix},$$

and a finite sequence of natural numbers $k = k_1 < k_2 < \ldots < k_m$, such that for all $i = 1, 2, \ldots, m$,

(8) $$Q_i \equiv Q_{i-1} \pmod{\mathfrak{P}^{k_{i-1}-k_0}},$$

(9) $$\mathfrak{D}_i = Q_i^{-1}\mathfrak{D}_0 Q_i = \begin{pmatrix} \Gamma_i & \Lambda \\ \mathfrak{P}^{k_i}\theta_i & \Delta_i \end{pmatrix},$$

where the degrees of the $\Gamma_i$ and the $\Delta_i$ are $n_1$ and $n_2$ respectively. We will show that one can always extend both these sequences by another term.

Since $\mathfrak{D}_m$ is an ordinary $\mathfrak{P}$-adic integral representation of $\mathfrak{H}$,

$$\mathfrak{D}_m(x+y) = \mathfrak{D}_m(x) + \mathfrak{D}_m(y),$$

$$\mathfrak{D}_m(cx) = c\mathfrak{D}_m(x),$$

$$\mathfrak{D}_m(xy) = \mathfrak{D}_m(x)\mathfrak{D}_m(y),$$

for all $x, y \in \mathfrak{H}$, and all $c \in \mathfrak{O}$, and from this one can deduce that

$$\theta_m(x+y) = \theta_m(x) + \theta_m(y),$$

$$\theta_m(cx) = c\theta_m(x),$$

$$\theta_m(xy) = \Delta_m(x)\theta_m(y) + \theta_m(x)\Gamma_m(y),$$

for all $x, y \in \mathfrak{H}$ and all $c \in \mathfrak{O}$. Therefore $\theta_m$ is surely a binding system in $\mathfrak{B}(\Delta_m, \Gamma_m, (\mathfrak{P}^{k_m}))$. Then by Theorem 1, there exists an $n_2 \times n_1$ $\mathfrak{O}$-matrix $T$ such that

$$N\theta_m(x) \equiv \Delta_m(x)\, T - T\,\Gamma_m(x) \qquad (\text{mod } \mathfrak{P}^{k_m}),$$

for all $x \in \mathfrak{H}$. Then since $N = u\mathfrak{P}^{k_0}$, where $u$ is a unit, we may write

$$\mathfrak{P}^{k_0}\theta_m(x) \equiv \Delta_m(x)\, S - S\,\Gamma_m(x) \qquad (\text{mod } \mathfrak{P}^{k_m}), \tag{10}$$

where $S = u^{-1}T$ is an $n_2 \times n_1$ $\mathfrak{O}$-matrix. Let

$$P = \begin{pmatrix} I_{n_1} & \\ -\mathfrak{P}^{k_m-k_0}S & I_{n_2} \end{pmatrix}$$

and let

$$Q_{m+1} = Q_m P = \begin{pmatrix} I_{n_1} & \\ \mathfrak{P}^{k-k_0}\, T_{m+1} & I_{n_2} \end{pmatrix}$$

where $T_{m+1} = T_m - \mathfrak{P}^{k_m-k}S$. Then since $P \equiv I_n$ (mod $\mathfrak{P}^{k_m-k_0}$), we have

$$Q_{m+1} \equiv Q_m \qquad (\text{mod } \mathfrak{P}^{k_m-k_0}). \tag{11}$$

Also, setting

$$\mathfrak{D}_{m+1}(x) = Q_{m+1}^{-1}\mathfrak{D}_0(x)Q_{m+1} = P^{-1}(Q_m^{-1}\mathfrak{D}_0(x)Q_m)P = P^{-1}\mathfrak{D}_m(x)P =$$

$$\begin{pmatrix} \Gamma_m(x) - \mathfrak{P}^{k_m-k_0}\Lambda(x)S & \Lambda(x) \\ \mathfrak{P}^{k_m-k_0}(\mathfrak{P}^{k_0}\theta_m(x) - (\Delta_m(x)S - S\Gamma_m(x))) - \mathfrak{P}^{2(k_m-k_0)}S\Lambda(x)S & \Delta_m(x) + \mathfrak{P}^{k_m-k_0}S\Lambda(x) \end{pmatrix}$$

we see, from (10), that the lower left-hand entry of this matrix is divisible by $\mathfrak{P}^{2(k_m-k_0)}$, so that we may write

$$\mathfrak{D}_{m+1}(x) = \begin{pmatrix} \Gamma_{m+1}(x) & \Lambda(x) \\ \mathfrak{P}^{2(k_m-k_0)}\theta_{m+1}(x) & \Delta_{m+1}(x) \end{pmatrix} \tag{12}$$

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for all $x \in \mathfrak{H}$, where

$$\Gamma_{m+1}(x) = \Gamma_m(x) - \mathfrak{P}^{k_m-k_0}\Lambda(x)S$$

$$\Delta_{m+1}(x) = \Delta_m(x) + \mathfrak{P}^{k_m-k_0}S\Lambda(x)$$

$$\mathfrak{P}^{2(k_m-k_0)}\theta_{m+1}(x) = \mathfrak{P}^{k_m-k_0}(\mathfrak{P}^{k_0}\theta_m(x) - (\Delta_m(x)S - S\Gamma_m(x))) - \mathfrak{P}^{2(k_m-k_0)}S\Lambda(x)S.$$

Setting $k_{m+1} = 2(k_m - k_0) = 2k_m - 2k_0 > 2k_m - k_m = k_m$ we obtain the desired result.

Then, by induction, there is an infinite sequence of $n \times n$ $\mathfrak{O}$-matrices

$$Q_1 = \begin{pmatrix} I_{n_1} & \\ & I_{n_2} \end{pmatrix}, \quad Q_2 = \begin{pmatrix} I_{n_1} & \\ \mathfrak{P}^{k-k_0}T_2 & I_{n_2} \end{pmatrix}, \quad Q_3 = \begin{pmatrix} I_{n_1} & \\ \mathfrak{P}^{k-k_0}T_3 & I_{n_2} \end{pmatrix}, \ldots$$

and an infinite sequence of natural numbers $k = k_1 < k_2 < k_3 < \ldots$ such that for all $i = 1, 2, 3, \ldots,$

$$Q_i \equiv Q_{i-1} \pmod{\mathfrak{P}^{k_{i-1}-k_0}} \tag{13}$$

and

$$\mathfrak{D}_i(x) = Q_i^{-1}\mathfrak{D}_0(x)Q_i = \begin{pmatrix} \Gamma_i(x) & \Lambda(x) \\ \mathfrak{P}^{k_i}\theta_i(x) & \Delta_i(x) \end{pmatrix} \tag{14}$$

for all $x \in \mathfrak{H}$, where the $\Gamma_i$ and the $\Delta_i$ are of degree $n_1$ and $n_2$ respectively. From (13), we see that the sequence $\{Q_i\}$ converges, and that if we set

$$Q^* = \lim_{i\to\infty} Q_i.$$

then $Q^*$ is of the form

$$Q^* = \begin{pmatrix} I_{n_1} & \\ \mathfrak{P}^{k-k_0}T^* & I_{n_2} \end{pmatrix} \tag{15}$$

and therefore unimodular. Furthermore, if we set

$$\mathfrak{D}^*(x) = Q^{*-1}\mathfrak{D}_0(x)Q^* = \left(\lim_{i\to\infty} Q_i^{-1}\right)\mathfrak{D}_0(x)\left(\lim_{i\to\infty} Q_i\right)$$
$$= \lim_{i\to\infty} Q_i^{-1}\mathfrak{D}_0(x)\,Q_i = \lim_{i\to\infty}\mathfrak{D}_i(x),$$

we see, from (14), that $\mathfrak{D}^*$ is of the form

$$\mathfrak{D}^*(x) = \begin{pmatrix} \Gamma^*(x) & \Lambda(x) \\ & \Delta^*(x) \end{pmatrix} \tag{16}$$

for all $x \in \mathfrak{H}$, where

$$\Gamma^*(x) = \lim_{i\to\infty}\Gamma_i(x), \quad \Delta^*(x) = \lim_{i\to\infty}\Delta_i(x).$$

Also,

$$\mathfrak{D}^*(x) = Q^{*-1}\mathfrak{D}_0(x)Q^* = \begin{pmatrix} \Gamma(x) + \mathfrak{P}^{k-k_0}\Lambda(x)T^* & \Lambda(x) \\ & \Delta(x) - \mathfrak{P}^{k-k_0}T^*\Lambda(x) \end{pmatrix}$$

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so that

$$\Gamma^*(x) \equiv \Gamma(x), \quad \Delta^*(x) \equiv \Delta(x) \qquad (\text{mod } \mathfrak{P}^{k-k_0}).$$

for all $x \in \mathfrak{H}$. Setting $M = QQ^*$ we obtain the desired result. One can of course extend the preceding theorem, by induction, to the case of an arbitrary number of constituents.

It is to be noted that the conditions $k > k_0$ in Theorem 2 and $k > 2k_0$ in Theorem 3 are not necessarily the best possible, and it is possible that refinements of these conditions could be found.

COROLLARY 1. *If $\mathfrak{D}$ is an ordinary $\mathfrak{P}$-adic integral representation of $\mathfrak{H}$, then for all complete reductions of $\mathfrak{D}$, the irreducible constituents are unique up to order and unimodular equivalence if and only if for all complete reductions of $\mathfrak{D}$, modulo $\mathfrak{P}^k$, where $k > 2k_0$, the irreducible constituents are unique up to order and unimodular equivalence, modulo $\mathfrak{P}^{k-k_0}$.*

*Proof.* (a) Assume that for all complete reductions of $\mathfrak{D}$, the irreducible constituents are unique up to order and unimodular equivalence. Let

$$\bar{\mathfrak{D}} \sim \begin{pmatrix} \bar{\Gamma}_1 & & * \\ & \bar{\Gamma}_2 & \\ & & \cdots & \\ & & & \bar{\Gamma}_r \end{pmatrix}, \quad \bar{\mathfrak{D}} \sim \begin{pmatrix} \bar{\Gamma}'_1 & & * \\ & \bar{\Gamma}'_2 & \\ & & \cdots & \\ & & & \bar{\Gamma}'_s \end{pmatrix}$$

be two complete reductions of $\mathfrak{D}$, modulo $\mathfrak{P}^k$. By Theorem 3, there exist two complete reductions of $\mathfrak{D}$ into ordinary irreducible constituents

$$\mathfrak{D} \sim \begin{pmatrix} \Delta_1 & & * \\ & \Delta_2 & \\ & & \cdots & \\ & & & \Delta_r \end{pmatrix}, \quad \mathfrak{D} \sim \begin{pmatrix} \Delta'_1 & & * \\ & \Delta'_2 & \\ & & \cdots & \\ & & & \Delta'_s \end{pmatrix}$$

such that

$$\begin{aligned} \Gamma_i(x) &\equiv \Delta_i(x) \quad (i = 1, 2, \ldots, r) \\ \Gamma'_j(x) &\equiv \Delta'_j(x) \quad (j = 1, 2, \ldots, s) \end{aligned} \qquad (\text{mod } \mathfrak{P}^{k-k_0}),$$

for all $x \in \mathfrak{H}$. Since for ordinary $\mathfrak{P}$-adic integral representations, the number of irreducible constituents in any complete reduction is invariant [**2**, pp. 359-360], $r = s$. Then from the assumption, there is some arrangement $k_1, k_2, \ldots, k_r$ of the numbers $1, 2, \ldots, r$, such that

$$\Delta_{k_i} \sim \Delta'_i \qquad (i = 1, 2, \ldots, r).$$

Therefore for this arrangement of the indices,

$$\bar{\Gamma}_{k_i} \sim \bar{\Gamma}'_i \qquad (\text{mod } \mathfrak{P}^{k-k_0}) \ (i = 1, 2, \ldots, r).$$

(b) Assume that for all complete reductions of $\mathfrak{D}$ (mod $\mathfrak{P}^k$) the irreducible constituents are unique up to order and unimodular equivalence, modulo $\mathfrak{P}^{k-k_0}$. Let

$$\mathfrak{D} \sim \begin{pmatrix} \Delta_1 & & * \\ & \Delta_2 & \\ & & \cdots & \\ & & & \Delta_r \end{pmatrix}, \quad \mathfrak{D} \sim \begin{pmatrix} \Delta'_1 & & * \\ & \Delta'_2 & \\ & & \cdots & \\ & & & \Delta'_r \end{pmatrix}$$

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be two complete reductions of $\mathfrak{D}$ into ordinary irreducible constituents. Then by Theorem 3,

$$\bar{\mathfrak{D}} \sim \begin{pmatrix} \bar{\Delta}_1 & & & * \\ & \bar{\Delta}_2 & & \\ & & \ldots & \\ & & & \bar{\Delta}_r \end{pmatrix}, \quad \bar{\mathfrak{D}} \sim \begin{pmatrix} \bar{\Delta}'_1 & & & * \\ & \bar{\Delta}'_2 & & \\ & & \ldots & \\ & & & \bar{\Delta}'_r \end{pmatrix}$$

must be complete reductions of $\mathfrak{D}$ into irreducible constituents, modulo $\mathfrak{P}^k$, and from the assumption, there is some arrangement $k_1, k_2, \ldots, k_r$ of the indices $1, 2, \ldots, r$, such that

$$\bar{\Delta}_{k_i} \sim \Delta'_i \qquad (\text{mod } \mathfrak{P}^{k-k_0}) \quad (i = 1, 2, \ldots, r).$$

But since $\Delta_{k_i}$ and $\Delta'_i$ are ordinary $\mathfrak{P}$-adic integral representations, and $k - k_0 > k_0$, by Theorem 2,

$$\Delta_{k_i} \sim \Delta'_i \qquad (i = 1, 2, \ldots, r).$$

COROLLARY 2. *If $\mathfrak{P}$ does not divide $N$, and if $\mathfrak{D}$ is an ordinary $\mathfrak{P}$-adic integral representation of $\mathfrak{H}$, then for all complete reductions of $\mathfrak{D}$, the irreducible constituents are unique up to order and unimodular equivalence.*

*Proof.* This is a direct consequence of Corollary 1, since in this case $k_0 = 0$ and we may take $k = k - k_0 = 1$, representations, modulo $\mathfrak{P}$, being representations over a field, the irreducible constituents in all complete reductions are unique up to order and unimodular equivalence, modulo $\mathfrak{P}$.

THEOREM 4. *If $\mathfrak{D}$ is an ordinary $\mathfrak{P}$-adic integral representation of $\mathfrak{H}$, and if*

$$\bar{\mathfrak{D}} \sim \bar{\mathfrak{D}}_0 = \begin{pmatrix} \bar{\Gamma} & \\ & \bar{\Delta} \end{pmatrix}$$

*is a unimodular decomposition of $\mathfrak{D}$, modulo $\mathfrak{P}^k$, where $k > 2k_0$, then there is a decomposition of $\mathfrak{D}$, into ordinary $\mathfrak{P}$-adic integral components*

$$\mathfrak{D} \sim \mathfrak{D}^{\#} = \begin{pmatrix} \Gamma^{\#} & \\ & \Delta^{\#} \end{pmatrix}$$

*such that*

$$\Gamma^{\#}(x) \equiv \Gamma(x), \quad \Delta^{\#}(x) \equiv \Delta(x) \qquad (\text{mod } \mathfrak{P}^{k-k_0}),$$

*for all $x \in \mathfrak{H}$.*

*Proof.* Let

$$\mathfrak{D}_0 = \begin{pmatrix} \Gamma & \mathfrak{P}^k \Lambda \\ \mathfrak{P}^k \theta & \Delta \end{pmatrix}.$$

Then, by Theorem 3, there is a reduction of $\mathfrak{D}$ into ordinary $\mathfrak{P}$-adic integral constituents

$$\mathfrak{D} \sim \mathfrak{D}^* = \begin{pmatrix} \Gamma^* & \mathfrak{P}^k \Lambda \\ & \Delta^* \end{pmatrix}$$

such that

$$\Gamma^*(x) \equiv \Gamma(x), \quad \Delta^*(x) \equiv \Delta(x) \qquad (\text{mod } \mathfrak{P}^{k-k_0}),$$

for all $x \in \mathfrak{H}$. By an interchange of rows and columns, which amounts to a unimodular transformation, one can always obtain

$$\mathfrak{D} \sim \begin{pmatrix} \Delta^* & \\ \mathfrak{P}^k \Lambda & \Gamma^* \end{pmatrix}.$$

By a second application of Theorem 3, one obtains the desired result.

COROLLARY. *If $\mathfrak{D}$ is an ordinary $\mathfrak{P}$-adic integral representation of $\mathfrak{H}$, then for all complete decompositions of $\mathfrak{D}$, the indecomposable components are unique up to order and unimodular equivalence if and only if for all complete decompositions of $\mathfrak{D}$, modulo $\mathfrak{P}^k$, where $k > 2k_0$, the indecomposable components are unique up to order and unimodular equivalence, modulo $\mathfrak{P}^{k-k_0}$.*

The method of proof for this corollary is essentially the same as for Corollary 1 of Theorem 3.

THEOREM 5. *If $\mathfrak{P}$ does not divide $N$, then all $\mathfrak{P}^k$-modular representations of $\mathfrak{H}$ are fully reducible, for any $k > 0$.*

*Proof.* Let $\mathfrak{D}$ be a $\mathfrak{P}^k$-modular representation of $\mathfrak{H}$, and let

$$\bar{\mathfrak{D}} \sim \bar{\mathfrak{D}}_0 = \begin{pmatrix} \bar{\Gamma} & \bar{\Lambda} \\ & \bar{\Delta} \end{pmatrix}$$

be a reduction of $\mathfrak{D}$, modulo $\mathfrak{P}^k$. $\Lambda$ is a modular binding system in $\mathfrak{B}(\Gamma, \Delta, (\mathfrak{P}^k))$. By Theorem 1, there is an $n_1 \times n_2$ $\mathfrak{O}$-matrix $T$ such that

$$N\Lambda(x) \equiv \Gamma(x)\, T - T\, \Delta(x) \qquad (\text{mod } \mathfrak{P}^k)$$

for all $x \in \mathfrak{H}$. Since $\mathfrak{P}$ does not divide $N$, $N$ is a $\mathfrak{P}$-adic unit, so that

$$\Lambda(x) \equiv \Gamma(x)\, S - S\, \Delta(x) \qquad (\text{mod } \mathfrak{P}^k)$$

for all $x \in \mathfrak{H}$, where $S = N^{-1}T$ is an $n_1 \times n_2$ $\mathfrak{O}$-matrix, i.e.,

$$\overline{\Lambda(x)} = \overline{\Gamma(x)}\, \bar{S} - \bar{S}\, \overline{\Delta(x)}$$

Transforming $\mathfrak{D}_0$ by

$$\bar{P} = \begin{pmatrix} \bar{I}_{n_1} & -\bar{S} \\ & \bar{I}_{n_2} \end{pmatrix},$$

one obtains

$$\bar{\mathfrak{D}} \sim \begin{pmatrix} \bar{\Gamma} & \\ & \bar{\Delta} \end{pmatrix}.$$

This last theorem is evidently also true in the case of $\mathfrak{P}$-adic integral representations of $\mathfrak{H}$.

In conclusion, we shall give a counter-example to show that in the case that $\mathfrak{P}$ divides $N$, the irreducible constituents in all complete reductions of a $\mathfrak{P}$-adic

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integral representation of $\mathfrak{H}$, are not necessarily unique up to order and unimodular equivalence. This counter-example is the same as that considered by Diederichsen [**2**, pp. 373-374] in the case of an arbitrary principal ideal ring.

Consider the group of all symmetries of the square, whose generators obeys the following conditions:

$$a^4 = b^2 = (ab)^2 = 1.$$

The two representations

$$\left\{a \to \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \; b \to \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right\}$$

and

$$\left\{a \to \begin{pmatrix} 1 & -1 \\ & \end{pmatrix}, \; b \to \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}\right\}$$

are irreducible and rationally equivalent. The module of all integral matrices which commute with these two representations, has only one generator, namely

$$T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad (\,|T| = 2)$$

and it is easily seen that any matrix commuting with these representations, modulo any power of 2, say $2^k$, must be congruent to a multiple of this generator, modulo $2^k$, and consequently must have a determinant which is divisible by 2. Therefore, modulo any power of 2, these two representations are not unimodularly equivalent.

Now consider the two following representations:

$$\mathfrak{D}_1 = \left\{a \to \left(\begin{array}{cc|cc} & -1 & & \\ 1 & & & \\ \hline & & & -1 \\ & & 1 & \end{array}\right), \; b \to \left(\begin{array}{cc|cc} & 1 & 1 & \\ 1 & & & -1 \\ \hline & & & 1 \\ & & 1 & \end{array}\right)\right\}$$

and

$$\mathfrak{D}_2 = \left\{a \to \left(\begin{array}{cc|cc} & -1 & & \\ 1 & & & \\ \hline & & & -1 \\ & & 1 & \end{array}\right), \; b \to \left(\begin{array}{cc|cc} -1 & & & 1 \\ & 1 & 1 & \\ \hline & & -1 & \\ & & & 1 \end{array}\right)\right\}$$

These two representations are unimodularly equivalent, for $U\mathfrak{D}_1 = \mathfrak{D}_2 U$, where

$$U = \left(\begin{array}{cc|cc} 2 & 1 & 1 & -1 \\ -1 & 2 & 1 & 1 \\ \hline 2 & -3 & -1 & -2 \\ 3 & 3 & 2 & -1 \end{array}\right),$$

and none of their irreducible constituents are unimodularly equivalent, modulo any power of 2, by the above discussion.

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