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A NON-COPRIME HALL-HIGMAN REDUCTION THEOREM

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Abstract

In a well-known paper, Hall and Higman proved the reduction theorem on a coprime order operator group acting on a finite group. This theorem plays an important role in local analysis of finite group theory. In this paper, we generalize the Hall-Higman reduction theorem by dropping the restrictive hypothesis $(|G|, |H|) = 1$ and determine the detailed structure of G completely.

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1. Introduction

It is useful to consider the following critical case when we consider a group acting on another group. Let G be a finite group, and H an operator group of G.

HYPOTHESIS (*). *H acts nontrivially on G but acts trivially on every proper H-invariant subgroup of G.*

Our purpose is to determine the structure of a group G which satisfies hypothesis (*). In [2], Hall and Higman have considered this question when $(|G|, |H|) = 1$. They proved the following famous reduction theorem.

REDUCTION THEOREM. *Suppose that (G, H) satisfies hypothesis (*) with $(|G|, |H|) = 1$. Then G is a special p-group.*

The hypothesis $(|G|, |H|) = 1$ here is very restrictive. For example, the very important case of a subgroup H of G acting on G by conjugation cannot satisfy

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this coprime order hypothesis. In this paper, we drop this hypothesis and obtain the detailed structure of G. We also give some applications and examples.

The main results are as follows:

MAIN THEOREM. *Suppose that* (G, H) *satisfies hypothesis* (*). *Then, there is a unique maximal* H*-invariant subgroup* C *of* G. *Moreover,* $C = C_G(H) = N_G(H)$ *is a normal abelian subgroup of* G *and* $[G, H]C \leq C_G(C)$. *Furthermore,* G *must satisfy one of the following conditions:*

(I) $[G, H] \neq G$. *Then* $|G/C| = p$, $H/C_H(G)$ *is isomorphic to a subgroup of* C *and* H' *acts trivially on* G. *Furthermore,* $F(G) \neq C$ *if and only if* G *is a* p*-group.*

(II) $[G, H] = G$. *Then* $C \leq Z(G)$. *Furthermore,*

(1) *Assume* $R_S(G) \neq G$. *Then* G *is a* p*-group in* $\mathscr{A}\mathscr{A}_e \cap \mathscr{A}_e\mathscr{A}$ *with class* ≤ 2. H *acts trivially on* $\Phi(G)$ *and irreducibly on* $G/Z(G)$. G' *is an elementary* p*-subgroup. If* $p \neq 2$, *the* $x^p = 1$ *for every element* x *of* G.

(2) *Assume* $R_S(G) = G$. *Then* $C = Z(G) = F(G) = \Phi(G)$. *Moreover* $G/Z(G) = G_1 \times \ldots \times G_k$ *is a direct product of isomorphic nonabelian simple groups. For every* $i \in \{1, \ldots, k\}$, *there exists* $H_i \leq H$ *such that* $|H : H_i| = k$, $G_i \cong \mathrm{Inn}(G_i) \leq H_i/C_{H_i}(G_i) \lesssim \mathrm{Aut}(G_i)$. (*Hence* H *is nonsolvable*).

Our notation follows that of [4]. All the groups in this paper are finite. p always denotes a prime. $\mathscr{A}$ denotes the class of abelian groups while $\mathscr{A}_e$ denotes the class of elementary abelian groups. $\mathscr{C}_p$ denotes the class of $\{1, C_p\}$, where C_p is the group of order p. $R_S(G) = \cap\{N \mid N \trianglelefteq G$ such that G/N is solvable$\}$.

$[G]H$ expresses the semidirect product of the group G and H where $G \trianglelefteq [G]H$.

2. Preliminaries

LEMMA 2.1. *Let* M *be a subgroup of* G. *Suppose that* $M \neq 1$ *and* $N_G(P) \leq M$ *for every nontrivial* p*-subgroup of* M *and* $p \in \pi(M)$. *Then* M *is a Hall subgroup of* G *and* G *is a Frobenius group with Frobenius complement* M.

PROOF. Let $p \in \pi(M)$ and $P \in \mathrm{Syl}_p(M)$. Since $N_G(P) \leq M$, we have $P \in \mathrm{Syl}_p(G)$. Hence M is a Hall subgroup of G. For every $x \in G$ with $M \cap M^x \neq 1$, there exists $p \mid |M \cap M^x|$. Let $1 \neq P \in \mathrm{Syl}_p(M \cap M^x)$. We have $N_G(P) \leq M$ and M^x and so $P \in \mathrm{Syl}_p(G) \cap \mathrm{Syl}_p(M) \cap \mathrm{Syl}_p(M^x)$. Since P and $P^x \in \mathrm{Syl}_p(M)$, there exists $m \in M$ such that $P^m = P^{x^{-1}}$ by Sylow's theorem. It follows that $x \in MN_G(P) \leq M$. Thus $M \cap M^x = 1$, for every $x \in G - M$. This yields that G is a Frobenius group with Frobenius complement M.

LEMMA 2.2. *Let* $N \leq Z(G)$, $N \cap \Phi(G) = 1$. *Then* $G = G_1 \times N$ *for a subgroup* G_1 *of* G.

PROOF. Suppose $N \neq 1$. Let $G_1 = \min\{M \mid NM = G\}$. We assert that $N \cap G_1 = 1$. In fact, if $N \cap G_1 \neq 1$, then there is a maximal subgroup M of G such that $M \ngeq N \cap G_1$ and $(N \cap G_1)M = G$. Thus $M \cap G_1 < G_1$ since $G_1 < G$, where $G = NG_1 = N(G_1 \cap ((N \cap G_1)M) = N(N \cap G_1)(M \cap G_1) = N(M \cap G_1)$, contrary to the choice of G. Hence $G = G_1 \times N$.

LEMMA 2.3. *Let $G = G_1 \times S = G_2 \times S$, where S is a solvable group and G_1 is a direct product of some nonabelian simple groups. Then $G_1 = G_2$.*

PROOF. $G_1 \cong G/S \cong G_2$. Let $G = G_{11} \times \ldots \times G_{1k}$, where G_{1i} is a nonabelian simple group for every $i \in \{1, 2, \ldots, k\}$. Since $G_{11} \leq G_2 \times S$, for each $g_{11} \in G_{11}$, we can uniquely express g_{11} as $g_{11} = g_2 s$ where $g_2 \in G_2$, $s \in S$. Set $\sigma : g_{11} \to s$. Then σ is a homomorphism from G_{11} into S, since $g_{11}^1 s^1 g_{11}^2 s^2 = g_{11}^1 g_{11}^2 s^1 s^2$. Now G_{11} is nonabelian simple and S is solvable. Hence $\operatorname{Ker}\sigma \neq 1$, $\operatorname{Ker}\sigma = G$ and so $G_{11} \leq G_2$. Similarly, $G_{1i} \leq G_2$ for every $i \in \{1, 2, \ldots, k\}$ and finally $G_1 = G_2$.

We say that a group H acts *irreducibly* on a group G provided that G has no nontrivial proper H-invariant subgroup.

LEMMA 2.4. *Suppose that a solvable group H acts faithfully and irreducibly on a finite group G. Then G is an elementary abelian p-group.*

PROOF. The result is trivial if $H = 1$. Consider $H \neq 1$. Since H is solvable, there exists a minimal normal q-subgroup Q of H such that $C_G(Q)$ is H-invariant. Irreducibility implies that $C_G(Q) = 1$. By the orbit formula, $|G| = |C_G(Q)| + kq \equiv 1 (\mathrm{mod}\, q)$, so G is a $q'$$\cdot$ group. By Glauberman's theorem [4, Theorem 7.5] there exists $P \in \mathrm{Syl}_p(G)$ such that P is Q-invariant for every $p \in \pi(G)$. The same theorem yields that the Q-invariant Sylow p-subgroup is unique since $C_G(Q) = 1$. Since $Q \trianglelefteq H$, we have $(P^h)^Q = (P^Q)^h = P^h$. Hence $P^h = P$ by uniqueness and so P is H-invariant. Again, irreducibility yields that $G = P$ and $\Phi(P) = 1$, that is , G is an elementary abelian p-group.

3. Proof of the main theorem

LEMMA 3.1. *Suppose that (G, H) satisfies hypothesis (*). Then:*

(1) *$G/[G, H]$ is a cyclic p-group or the identity group.*

(2) *The unique maximal H-invariant subgroup of G is $C = C_G(H) = N_G(H)$ and C is a normal subgroup of G.*

(3) *$[G, H] \leq C_G(N)$ for every proper H-invariant subgroup N of G.*

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PROOF. (1) The conclusion is obvious if $G = [G, H]$. Suppose $[G, H] < G$. Since H acts trivially on $G/[G, H]$ but nontrivially on G, there exists $x \in G$ such that $[x, H] \neq 1$ and $\langle x\rangle[G, H]$ is an H-invariant subgroup of G. Hypothesis (*) yields that $G = \langle x\rangle[G, H]$, that is, $G/[G, H]$ is cyclic. Let $G/[G, H] = P_1/[G, H] \times \ldots \times P_k/[G, H]$ be the direct product of Sylow subgroups, where each P_i is H-invariant. We have $k = 1$ and $G/[G, H]$ is a cyclic p-group by hypothesis (*).

(2) It is clear that every proper H-invariant subgroup of G is contained in $C = C_G(H)$. Therefore C is the unique maximal H-invariant subgroup of G. We claim that $N_G(H) = C$. In fact, if $N_G(H) > C$, we have $N_G(H) = G$ since $N_G(H)$ is H-invariant. It is easy to see that $[G, H] = 1$ in this case, contrary to the hypothesis. Our next goal is to show that $C \trianglelefteq G$ by induction on $|G|$. If $C = 1$, there is nothing to prove. We consider $C \neq 1$. Set $\mathrm{Core}(C) = \bigcap_{x\in G} C^x$. Suppose Core $(C) = 1$. Then there is no nontrivial normal subgroup of G contained in C. For each $p \in \pi(C)$, let $1 \neq P$ be a p-subgroup of C. Since $N_G(P)$ is H-invariant and $N_G(P) \neq G$, we have $N_G(P) \leq C$. Now $G = K[C]$ by Lemma 2.1, where K is the Frobenius core and so is H-invariant. Since $C \neq 1$, it follows that $K \neq G$ and hence $K \leq C$. Thus $G \leq C \neq G$, a contradiction. Thus we have $\mathrm{Core}(C) \neq 1$. Consider H acting on $\bar{G} = G/\mathrm{Core}(C)$. Suppose H acts nontrivially on $\bar{G}$. Then $(\bar{G}, H)$ satisfies hypothesis (*) and the unique maximal H-invariant subgroup of $\bar{G}$, $C_{\bar{G}}(H) = L/\mathrm{Core}(C)$ is normal in $\bar{G}$ by induction. It is clear that $L \geq C$ and $L \neq G$. We conclude that $C = L \trianglelefteq G$. If H acts trivially on G, then $[G, H] \leq \mathrm{Core}(C) \leq C$ and $G/\mathrm{Core}(C)$ is cyclic, hence $C \trianglelefteq G$. This completes the proof of (2).

(3) Since $[C, H] = 1$ and $C \trianglelefteq G$, we have $[C, H, G] = 1 = [G, C, H]$. Thus we have $[G, H] \leq C_G(C)$. Hence $[G, H] \leq G_G(N)$ for every proper H-invariant subgroup N of G by (2).

In the following, C always denotes $C_G(H)$.

THEOREM 3.1. *Suppose that (G, H) satisfies hypothesis (*) with $[G, H] \neq G$. Then $|G/C| = p$ and C is an abelian group. Moreover $H/C_H(G)$ is isomorphic to a subgroup of C, and H' acts trivially on G.*

PROOF. Since $[G, H] < G$, $[G, H]$ is normal and H-invariant in G, and $[G, H] \leq C$ by Lemma 3.1 (3). Since H acts trivially on G/C and C is a maximal H-invariant subgroup of G, we find G/C has no nontrivial proper subgroup, and so $|G/C| = p$ for some prime p. Since H acts trivially on both G/C and C but nontrivially on G, there is some $a \in G$ such that $[a, H] \neq 1$ and $\langle a\rangle C = G$. For each $h \in H$, there is an unique $c_h \in C$ such that $a^h = c_h a$. Set $\sigma : h \to c_h$. Then σ is a homomorphism from H to C. In fact, if $a^{h_1} = c_{h_1}a$, $a^{h_2} = c_{h_2}a$, then $a^{h_1h_2} = (c_{h_1}a)^{h_2} = c_{h_1}c_{h_2}a$. Since $\mathrm{Ker}\,\sigma = C_H(a) = C_H\langle a, C\rangle = C_H(G)$, we have that $H/C_H(G)$ is isomorphic to a subgroup of C. It remains to prove that C is abelian.

For every $x \in C$, $C \trianglelefteq G$ yields that $x^a \in C$. Since H acts trivially on C, we get $x^a = (x^a)^h = (a^{-1}xa)^h = (a^h)^{-1}x^h a^h = a^{-1}c_h^{-1}xc_h a$ for every $h \in H$. Thus $xc_h = c_h x$, $c_h \in Z(C)$, $a^h = c_h a \in Z(C)\langle a\rangle$. Since $Z(C)$ char $C \trianglelefteq G$, $Z(C)\langle a\rangle$ is a subgroup of G. Hence $G_1 = \langle a^h \mid h \in H\rangle \leq Z(C)\langle a\rangle$. It is clear that $a \in G_1$, G_1 is H-invariant and H acts nontrivially on G_1. So $G = G_1 \leq Z(C)\langle a\rangle$ by hypothesis (*). It follows that $C/Z(C)$ is cyclic and C is abelian.

REMARK 1. By Lemma 3.1(3) and Theorem 3.1, we have that C is abelian in all cases (independently of whether $[G, H] = G$ or not).

THEOREM 3.2. *Suppose that (G, H) satisfies hypothesis (*) with $[G, H] = G$. Then*

(1) *every proper H-invariant subgroup of G is contained in $Z(G)$;*

(2) *furthermore, if $R_S(G) \neq G$, then we have the following:*

(a) *G is a p-group in $\mathscr{A}\mathscr{A}_e \cap \mathscr{A}_e\mathscr{A}$ with class at most two.*

(b) *$G' \leq \Phi(G) \leq Z(G)$, G' is an elementary abelian p-group. H acts trivially on $\Phi(G)$ and irreducibly on $G/Z(G)$.*

(c) *If $p \neq 2$, then $x^p = 1$ for every $x \in G$.*

PROOF. (1) For each proper H-invariant subgroup N of G, we have $G = [G, H] \leq C_G(N)$ by Lemma 3.1 (3). Thus $N \leq Z(G)$.

(2) If $R_S(G) \neq G$, then $R_S(G) \leq Z(G)$ by (1). So $G/Z(G)$ is solvable and $G' < G$, and $G' \leq Z(G)$ by (1). Therefore $G/Z(G)$ is abelian and G is nilpotent. It is easy to show that G is a p-group by hypothesis (*).

Proof of (b). It is clear that $G' \leq \Phi(G) \leq Z(G)$. H acts trivially on $\Phi(G)$ since $\Phi(G) \neq G$. By (1) and $\Phi(G) \leq Z(G)$ we conclude that $G/Z(G)$ is elementary abelian and H acts irreducibly on $G/Z(G)$. For $x, y \in G$, we have $y^p \in \Phi(G) \leq Z(G)$ and $[x, y] \in Z(G)$. Thus $1 = [x, y^p] = [x, y]^p$ and so G' is an elementary abelian p-group.

Proof of (a). $G/G' \in \mathscr{A}$, $G' \in \mathscr{A}_e$ by (b), and $G/Z(G) \in \mathscr{A}_e$. Hence $G \in \mathscr{A}\mathscr{A}_e \cap \mathscr{A}_e\mathscr{A}$ and the class of G is at most 2.

Proof of (c). If $p \neq 2$, then $p \mid p(p-1)/2$. Since H acts trivially on G^pG', $(G')^p = 1$ and $G' \leq Z(G)$, and we have $[x, h]^p = (x^{-1}x^h)^p = (x^p)^{-1}(x^p)^h[x^{-1}, x^h]^{p(p-1)/2} = 1$. The conclusion follows from that $[G, H] = G$ and $(xy)^p = x^p y^p [x, y]^{p(p-1)/2}$.

LEMMA 3.2.

(1) *Suppose that group H acts faithfully and irreducibly on a nonabelian simple group G. Then $G \cong \mathrm{Inn}(G) \leq H \leq \mathrm{Aut}(G)$.*

(2) *Suppose that a group H acts faithfully and irreducibly on a nonsolvable group G. Then $G = G_1 \times \ldots \times G_k$ is a direct product of isomorphic nonabelian simple*

groups, and for each $i \in \{1, \ldots, k\}$, *there is* $H_i \leq H$ *with* $|H : H_i| = k$, *such that* $G_i \cong \mathrm{Inn}(G_i) \leq H_i/C_{Hi}(G_i) \lesssim \mathrm{Aut}(G_i)$. *(Hence H is nonsolvable).*

PROOF. (1) Consider $L = [G]H$. Then $C_L(G) \trianglelefteq L$. It is obvious that $\mathrm{Inn}(G) \cong G$ and $H = N_H(G)/C_H(G) \leq \mathrm{Aut}(G)$. We only need to prove $\mathrm{Inn}(G) \leq H$.

(a) We claim $C_L(H) \neq 1$.

If this is false, then $H \cong L/G \leq \mathrm{Aut}(G)/\mathrm{Inn}(G)$ since $L = N_L(G)/C_L(G) \leq \mathrm{Aut}(G)$. Now $\mathrm{Aut}(G)$ is solvable by [1, Theorem 4.239] and so H is solvable. By Lemma 2.4, G is a p-group, which contradicts the fact that G is nonabelian simple.

(b) $G = \{g \mid g \in G, \exists\, h \in H, \text{such that } gh \in C_L(G)\}$.

Set $G_1 = \{g \mid g \in G, \exists\, h \in H, \text{ such that } gh \in C_L(G)\}$. Since $G_H(G) = 1$ and $C_L(G) \neq 1$, we have $1 \in G_1$ and $G_1 \neq \{1\}$. Let $g_1, g_2 \in G_1$, $h_1, h_2 \in H$ be such that $g_i h_i \in C_L(G)$, $i = 1, 2$. Then $h_1 = g_1^{-1}(g_1 h_1) = g_1 h_1 g_1^{-1}$, $g_1 h_1 = h_1 g_1$, $g_1^{-1} h_1^{-1} = (g_1 h_1)^{-1} \in C_L(G)$, so $g_1^{-1} \in G_1$. Moreover $(g_2 g_1)(h_1 h_2) = g_2(g_1 h_1) h_2 = g_1 h_1 g_2 h_2 \in C_L(G)$. Thus G_1 is a subgroup of G. For each $h \in H$, $g_1^h h_1^h = (g_1 h_1)^h \in C_L(G)^h = C_L(G)$. Thus $g_1^h \in G_1$. It follows that $1 \neq G_1$ is an H-invariant subgroup of G and so $G = G_1$.

(c) There is an injective map from G to H. In fact, for each $g \in G$, there is $h_g \in H$ such that $gh_g \in C_L(G)$. If $gh_g = gh_{g'}$, then $h_g^{-1} h'_g = (gh_g)^{-1} g h_{g'_g} \in C_L(G) \cap H = C_H(G) = 1$, so $h_g = h_{g'}$. Thus h_g is uniquely determined by g. Let $\sigma : g \to h_g^{-1}$. We assert that σ is an injective homomorphism from G to H. In fact, suppose $g_1 h_{g_1}, g_2 h_{g_2} \in C_L(G)$. Then $g_1 g_2 h_{g_2} h_{g_1} = g_2 h_{g_2} g_1 h_{g_1} \in C_L(G)$. Thus $\sigma(g_1 g_2) = (h_{g_2} h_{g_1})^{-1} = h_{g_1}^{-1} h_{g_2}^{-1} = \sigma(g_1)\sigma(g_2)$, so σ is a homomorphism. Now $\mathrm{Ker}\,\sigma \trianglelefteq G$. If $\mathrm{Ker}\,\sigma = G$, then $h_g = 1$, for every $g \in G$ and so $G \leq C_L(G)$. Thus G is abelian, a contradiction. Therefore $\mathrm{Ker}\ \sigma = 1$ since G is simple.

For each $I_g \in \mathrm{Inn}(G)$, $x \in G$, $x^{gh_g} = x$, $x^g = x^{h_g^{-1}}$. Hence H contains every I_g. It follows that $G \cong \mathrm{Inn}(G) \leq H \leq \mathrm{Aut}(G)$.

Proof of (2). H acts irreducibly on G, so G is characteristically simple. Hence $G = G_1 \times \ldots \times G_k$ is a direct product of isomorphic nonabelian simple groups. Let $H_1 = N_H(G_1)$. Let $H = H_1 + H_1 a_2 + \ldots + H_1 a_n$, $\{1 = a_1, a_2, \ldots, a_n\}$ be the transversal of H_1 in H. Since G_1 is a minimal normal subgroup of G, $G_1^{a_i} = G_1^{a_j}$, if and only if $a_i = a_j$. Now $1 \neq \langle G_1^h : h \in H\rangle = G_1 \times G_1^{a_2} \times \ldots \times G_1^{a_n} = G$ since $\langle G_1^h : h \in H\rangle$ is an H-invariant subgroup of G. It is clear that $k = n$, hence $|H : H_1| = k$. If $1 \neq K_1 < G$ and K_1 is H_1-invariant, then $1 \neq K = K_1 \times \ldots \times K_1^{a_k} < G$ and K is H-invariant, contrary to irreducibility. It follows that H_1 acts irreducibly on G. Hence $G_i \cong \mathrm{Inn}(G_i) \leq H_i/C_{Hi}(G_i) \lesssim \mathrm{Aut}(G_i)$ by (1). The same argument for i implies the conclusion.

THEOREM 3.3. *Suppose that* (G, H) *satisfies the hypothesis* (*) *with* $R_S(G) = G$. *Then* $G = [G, H]$ *and* $C = C_G(H) = Z(G) = F(G) = \Phi(G)$. *G is a perfect*

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quasinilpotent group, and $\bar{G} = G/Z(G) = G_1 \times \ldots \times G_k$ *is a direct product of isomorphic nonabelian simple groups. For each* $i \in \{1, \ldots, k\}$, *there is* $H_i \leq H$ *with* $|H : H_i| = k$, *such that* $G_i \cong \mathrm{Inn}(G_i) \leq H_i/C_{H_i}(G_i) \lesssim \mathrm{Aut}(G_i)$. *(Hence H is nonsolvable).*

PROOF. $G = [G, H]$ follows from Theorem 3.1. Since $R_S(G) = G$, $G = G'$ and G is perfect. Thus $F(G) \neq G \neq Z(G)$, hence $F(G) = C = Z(G)$ by Theorem 3.2 (1) and Lemma 3.1. Hence $\bar{G} = G/Z(G)$ has no nontrivial proper H-invariant subgroup, so $\bar{G} = G_1 \times \ldots \times G_k$ has the properties mentioned in Lemma 3.2. G is perfect nilpotent by [3, X Section 13]. There is only $\Phi(G) = Z(G)$ left to prove. We prove this by induction on $|G|$.

Firstly, we consider $\Phi(G) = 1$. If $Z(G) \neq 1$, then $G = Z(G) \times G_1$ by Lemma 1.2. Since $G_1 \cong G/Z(G)$ is a direct product of isomorphic nonabelian simple groups, $Z(G) \times G_1 = G = G^h = Z(G)^h \times G_1^h$, $\forall h \in H$. Hence $G_1 = G_1^h$ by Lemma 1.3. thus $G_1 \neq G$ and G_1 is H-invariant. This yields that $G_1 \leq Z(G)$ and $G \leq Z(G) \neq G$, a contradiction. We conclude that $Z(G) = C = 1$ if $\Phi(G) = 1$.

Now, consider $\Phi(G) \neq 1$. Consider H acting on $G/\Phi(G)$. If H acts trivially on $G/\Phi(G)$, then $G = [G, H] \leq \Phi(G)$, a contradiction. Hence $(G/\Phi(G), H)$ satisfies hypothesis (*) and $R_S(G/\Phi(G)) = G/\Phi(G)$, $\Phi(G/\Phi(G)) = 1$. The proceeding argument yields that $1 \neq Z(G/\Phi(G)) \geq Z(G)/\Phi(G)$ and so $\Phi(G) = z(G) = C$.

The main Theorem follows from Theorem 3.1, Theorem 3.2 and Theorem 3.3.

4. Applications and examples

COROLLARY 4.1. *Let G be a solvable group. Let H be an operator group of G. Suppose H acts trivially on every H-invariant* $\mathscr{A}\mathscr{C}_p$*-subgroup of G and every H-invariant p-subgroup of G which lies in* $\mathscr{A}\mathscr{A}_e \cap \mathscr{A}_e\mathscr{A}$ *with class at most two. Then H acts trivially on G.*

PROOF. Assume that the conclusion is false and let G be a counterexample of minimal order. Since the hypotheses in Corollary 4.1 are inherited for H-invariant subgroups, by the choice of G, H acts trivially on every proper H-invariant subgroup of G. Thus (G, H) satisfies hypothesis (*). Since G is solvable, by Theorem 3.1 and Theorem 3.2, $G \in \mathscr{A}\mathscr{C}_p$ or G is a p-group in $\mathscr{A}\mathscr{A}_e \cap \mathscr{A}_e\mathscr{A}$ with class at most two. Thus H acts trivially on G, contrary to our choice. This shows that there is no counterexample and the corollary is proved.

COROLLARY 4.2. *Suppose that H is a solvable operator group of G and H acts nontrivially on G but acts trivially on every proper H-invariant* $\mathscr{A}\mathscr{C}_p$*-subgroup of*

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G and every proper H-invariant p-subgroup with class at most two which lies in $\mathscr{A}\mathscr{A}_e \cap \mathscr{A}_e\mathscr{A}$. Then G is solvable.

PROOF. Assume that the conclusion is false and let G be a counterexample of minimal order. We assert that (G, H) satisfies hypothesis (*).

In fact, H acts nontrivially on G. Suppose H acts nontrivially on a proper H-invariant subgroup G_1 of G. Then G_1 satisfies the assumption of Corollary 4.2; thus G_1 is solvable by the choice of G. Hence (G, H) satisfies hypothesis (*). By Theorem 3.2, $C = C_G(H) \trianglelefteq G$ and C is abelian. Consider H acting on G/C.

Assume $C \neq 1$. If H acts trivially on G/C, then G/C is cyclic by Lemma 3.1 and so G is solvable, a contradiction. If H acts nontrivially on G/C, then $(G/C, H)$ satisfies hypothesis (*) and so satisfies the hypothesis of Corollary 4.2. The choice of G yields that G/C is solvable and hence so is G, contrary to our choice.

Therefore $C = 1$. Since (G, H) satisfies hypothesis (*), Lemma 3.1 yields that every proper H-invariant subgroup of G is contained in $C = 1$. Hence H acts irreducibly on G. Since H is solvable, by Lemma 2.4, G is an elementary abelian p-group, contrary to our assumption. This shows that there is no counterexample and the corollary is proved.

We say that a group G is 3-*step solvable* if $G^{(3)} = 1$.

COROLLARY 4.3. *Suppose that A is an abelian subgroup of G and suppose A lies in the centre of every 3-step solvable subgroup of G which contains A. Then A lies in the centre of G.*

PROOF. Consider A acting on G by conjugation. Assume that the conclusion is false and let G be a counterexample. Now $A \not\leq Z(G)$ means that A acts nontrivially on G. For every $M \leq G$, where M is A-invariant and either $M \in \mathscr{A}\mathscr{C}_p$ or M is a p-group in $\mathscr{A}\mathscr{A}_e \cap \mathscr{A}_e\mathscr{A}$ with class at most two, we have $M^{(2)} = 1$ and $M \trianglelefteq MA$, $(MA)' \leq M$. Hence $(MA)^{(3)} = 1$. By assumption, $A \leq Z(MA)$, that is, A acts trivially on M. Corollary 4.2 yields that G is solvable. Now, Corollary 4.1 forces that A acts trivially on G, contrary to our assumption. This completes the proof of the theorem.

COROLLARY 4.4. *A p-element x of G lies in $Z(G)$ if and only if both of the following hold*

(1) *there exists $P \in Syl_p(G)$, such that $P \cap \{x^g \mid g \in G\} = x$;*

(2) *x centralizes every p'-characteristic subgroup of M, where M is a 3-step solvable subgroup of G which contains x.*

PROOF. Similar to the proof of Corollary 4.3

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REMARK 2.

(1) Corollaries 4.3 and 4.4 are generalisations of the main results in [5].

(2) Suppose (G, H) satisfies hypothesis (*) with $(|G|, |H|) = 1$. Then G is described by case II(1) of the Main Theorem, by Glauberman's theorem. We can easily prove that G is a special p-group and obtain the original Hall-Higman theorem.

(3) The tools used in the proof of the applications above are elementary; all of them can be found in [4].

We give an example for each case in the Main Theorem.

(I) $G = A_4 = [B_4]C_3$ where B_4 is Klein 4-group. Let $H = B_4$ with H acting on G by conjugation. Every proper H-invariant subgroup of G is contained in B_4.

(II) (1) Let G be any cyclic group of order P, p an odd prime, $1 \neq H \leq \mathrm{Aut}(G)$.

(II) (2) (i) Let G be a nonabelian characteristic simple group, $H = \mathrm{Aut}(G)$.

(ii) Let $G/Z(G)$ be a nonabelian simple group, $G = H$, H acting on G by conjugation.

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