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SECOND-ORDER NONCOMMUTATIVE DIFFERENTIAL AND LIPSCHITZ STRUCTURES DEFINED BY A CLOSED SYMMETRIC OPERATOR

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Abstract

The Banach *-operator algebras, exhibiting the second-order noncommutative differential structure and the noncommutative Lipschitz structure, that are determined by the unbounded derivation and induced by a closed symmetric operator in a Hilbert space, are explored.

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The aim of the present paper is to understand the noncommutative second-order differential structure and the noncommutative Lipschitz structure defined by a closed symmetric operator in a Hilbert space. Let S be a closed symmetric operator with dense domain $D(S)$ in a Hilbert space $\mathcal{H}$. Let $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ be the C^*-algebras consisting of all bounded operators and all compact operators on $\mathcal{H}$, respectively. Let $\mathcal{A}_S^1$ consist of all operators A in $\mathcal{B}(\mathcal{H})$ such that $AD(S) \subset D(S), A^*D(S) \subset D(S)$ and $SA - AS$ extends by closure to a bounded operator on H. Let $A_S := (SA - AS)^-$, where the bar above denotes the closure of the respective operator. Then $\mathcal{A}_S^1$ is a Banach *-algebra with norm $\|A\|_1 := \|A\| + \|A_S\|$, with $\|\cdot\|$ denoting the operator norm. Let $\mathcal{U}_S$ be the C^*-algebra obtained by completing $\mathcal{A}_S^1$ in $\|\cdot\|$. Let δ_S be the $*$-derivation defined by S as $\delta_S(A) = iA_S$ with domain $D(\delta_S) = \mathcal{A}_S^1$ in $\mathcal{U}_S$. Let $\mathcal{K}_S^1 := \mathcal{A}_S^1 \cap \mathcal{K}(\mathcal{H})$, $\mathcal{J}_S^1 := \{A \in \mathcal{K}_S^1 : A_S \in \mathcal{K}(\mathcal{H})\}$ and $\mathcal{F}_S^1$ be the closure in the norm $\|\cdot\|_1$ of all finite rank operators in $\mathcal{A}_S^1$. The algebra $\mathcal{A}_S^1$ is a Banach (D_1^*)-algebra [KS2] in the sense that it is a Banach *-algebra that is a dense *-subalgebra of a C^*-algebra satisfying $\|TR\|_1 \le \|T\|_1\|R\| + \|T\|\,\|R\|_1$ for all T, R in $\mathcal{A}_S^1$. The algebras $\mathcal{K}_S^1, \mathcal{J}_S^1, \mathcal{F}_S^1$

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are closed subalgebras of $(\mathcal{A}_S^1, \|\cdot\|_1)$ and $\mathcal{F}_S^1 \subset \mathcal{J}_S^1 \subset \mathcal{K}_S^1 \subset \mathcal{A}_S^1$. In [KS2, KS3, KS4], Kissin and Shulman have investigated the structure of these algebras, regarding them as noncommutative differential algebras defined by the derivation δ_S.

The classical Banach function algebra $C^1[a,b]$ (consisting of functions $f \in C[a,b]$ such that the derivative f' exists on $[a,b]$ and $f' \in C[a,b]$) as well as the Lipschitz algebra $\mathrm{Lip}[a,b]$ (consisting of functions $f \in C[a,b]$ such that the derivative f' exists almost everywhere on $[a,b]$ and $f' \in L^\infty[a,b]$) suggest that the algebras $\mathcal{A}_S^1, \mathcal{K}_S^1, \mathcal{J}_S^1$ and $\mathcal{F}_S^1$ represent the noncommutative Lipschitz structure defined by S (more precisely, defined by S relative to $\mathcal{B}(\mathcal{H})$). The noncommutative C^1-structure defined by S may be described more accurately by the following modified versions of these algebras. Let

$$\mathcal{A}_S^{(1)} := \{A \in \mathcal{U}_S : AD(S) \subset D(S), A^*D(S) \subset D(S), (SA - AS)^- \in \mathcal{U}_S\},$$

$\mathcal{K}_S^{(1)} := \mathcal{K}(H) \cap \mathcal{A}_S^{(1)}$, $\mathcal{J}_S^{(1)} := \{A \in \mathcal{K}_S^{(1)} : A_S \in \mathcal{K}(\mathcal{H})\}$ and $\mathcal{F}_S^{(1)}$ be the $\|\cdot\|_1$-closure of finite rank operators in $\mathcal{A}_S^{(1)}$. These Banach algebras, together with the Banach algebras considered in the previous paragraph, exhibit the first-order differential structure defined by S and described in terms of the derivation δ_S. We consider the second-order differential structure defined by S, which is exhibited by the algebras and is defined as follows.

Let $\mathcal{A}_S^2 := \{A \in \mathcal{A}_S^1 : \delta_S(A) \in \mathcal{A}_S^1\}$, which is a Banach *-algebra with norm $\|A\|_2 = \|A\| + \|\delta_S(A)\| + (1/2)\|\delta_S^2(A)\|$, $\mathcal{K}_S^2 = \mathcal{A}_S^2 \cap \mathcal{K}(H)$ and $\mathcal{J}_S^2 = \{A \in \mathcal{K}_S^1 : \delta_S(A) \in \mathcal{J}_S^1\}$, and let $\mathcal{F}_S^2$ be the closure in $\|\cdot\|_2$ of finite rank operators in $\mathcal{A}_S^2$. Notice that, for A in $\mathcal{A}_S^2$, $\delta_S^2(A) \in \mathcal{B}(\mathcal{H})$, and thus the algebra $\mathcal{A}_S^2$ corresponds to the algebra of C^1-functions whose derivatives are Lipschitzian. The analogues of the algebra of C^2-functions are given as follows. Let $\mathcal{A}_S^{(2)} = \{A \in \mathcal{A}_S^{(1)} : \delta_S(A) \in \mathcal{A}_S^{(1)}\}$, which is a closed subalgebra of $\mathcal{A}_S^2$, $\mathcal{K}_S^{(2)} = \mathcal{A}_S^{(2)} \cap \mathcal{K}(H)$, $\mathcal{J}_S^{(2)} = \{A \in \mathcal{K}_S^{(1)} : \delta_S(A) \in \mathcal{J}_S^{(1)}\}$, and let $\mathcal{F}_S^{(2)}$ be the closure in $\|\cdot\|_2$ of finite rank operators in $\mathcal{A}_S^{(2)}$. Thus the noncommutative second-order differential structure defined by S is manifested as the following complex of Banach algebras which are dense smooth subalgebras of C^*-algebras.

$$\begin{array}{ccccccccc}
\mathcal{A}_S^{(2)} & \subset & \mathcal{A}_S^2 & \subset & \mathcal{A}_S^{(1)} & \subset & \mathcal{A}_S^1 & \subset & \mathcal{U}_S \\
\cup & & \cup & & \cup & & \cup & & \\
\mathcal{K}_S^{(2)} & \subset & \mathcal{K}_S^2 & \subset & \mathcal{K}_S^{(1)} & \subset & \mathcal{K}_S^1 & & \\
\cup & & \cup & & \cup & & \cup & & \\
\mathcal{J}_S^{(2)} & \subset & \mathcal{J}_S^2 & \subset & \mathcal{J}_S^{(1)} & \subset & \mathcal{J}_S^1 & & \\
\cup & & \cup & & \cup & & \cup & & \\
\mathcal{F}_S^{(2)} & \subset & \mathcal{F}_S^2 & \subset & \mathcal{F}_S^{(1)} & \subset & \mathcal{F}_S^1 & &
\end{array}$$

An algebra of the form $\mathcal{B}_S^2$ (and, analogously, $\mathcal{B}_S^{(2)}$) should not be confused with $(\mathcal{B}_S)^2$ which is the linear span in $\mathcal{B}_S$ of the set $\{XY : X \in \mathcal{B}_S, Y \in \mathcal{B}_S\}$. Notice that, when S is a bounded operator, all the three norms $\|\cdot\|_2, \|\cdot\|_1$ and $\|\cdot\|$ are equivalent, $\mathcal{A}_S^{(2)} = \mathcal{A}_S^2 = \mathcal{A}_S^{(1)} = \mathcal{A}_S^1 = \mathcal{U}_S = \mathcal{B}(\mathcal{H})$, and the remaining Banach algebras coincide with the C^*-algebra $\mathcal{K}(\mathcal{H})$. A comparison with the classical C^1-algebra and the Lipschitz algebra in real analysis suggests that the noncommutative C^1-structure is likely to be more rigid than the noncommutative Lipschitz structure. The purpose of the present paper is to contribute to the understanding of the noncommutative second-order differential and Lipschitz structures defined by S using the method adopted in Kissin and Shulman [KS2] and in Weaver [W1, W2] for the investigation of the first-order structures. Throughout the paper, we assume that the closed symmetric operator S is such that the operator S^2 with domain $D(S^2) := \{x \in D(S) : Sx \in D(S)\}$ is a densely defined operator. This would ensure that S^2 is closable.

The paper is organized as follows. In Section 1, we develop basic properties of the Banach *-algebra $\mathcal{A}_S^2$, and compute the finite rank operators therein. The densely defined second-order derivation $\delta_S^2 : \mathcal{A}_S^1 \to \mathcal{B}(\mathcal{H})$, with domain $D(\delta_S^2) = \mathcal{A}_S^2$, turns out to be a closed operator in the C^1-norm $\|\cdot\|_1$ on $\mathcal{A}_S^1$ and the operator norm on $\mathcal{B}(\mathcal{H})$. We also discuss the regularity properties, such as spectral invariance and closure, under functional calculi. In Section 2, it is noticed that the derivations δ_S and δ_S^2 are W^*-derivations in the sense of Weaver [W1, W2] with the result that $\mathcal{A}_S^1$ and $\mathcal{A}_S^2$ are W^*-domain algebras [W1] which are duals of Banach spaces. This enables us to discuss Lipschitz functional calculus in these Banach algebras. In Section 3, we discuss approximation properties in $\mathcal{A}_S^2$; the approximation being by a $\|\cdot\|_1$-convergence of a $\|\cdot\|_2$-bounded sequence. In Section 4, closed essential left ideals in the algebra $\mathcal{F}_S^2$ are determined. As a whole, the paper seeks analogues for second-order derivation δ_S^2 of results pertaining to first-order operator δ_S in [KS2], and adds a new perspective to a noncommutative Lipschitz structure defined by S. The paper discusses only some basic properties. Many important issues such as duality [KS3], isomorphisms [KS3], second-order analogues of differential Schatten algebras [KS4], analogues of Calkin algebra, as well as higher-order differential structures defined by S remain to be investigated.

1. Noncommutative differential structure

Proposition 1.1.

(1) *The class $\mathcal{A}_S^2$ is a Banach *-algebra with norm $\|A\|_2 = \|A\| + \|\delta_S(A)\| + (1/2)\|\delta_S^2(A)\|$. Also, for any A in $\mathcal{A}_S^2$, $AD(S^{*2}) \subset D(S^{*2})$, and $\delta_S^2(A)|_{D(S^{*2})} = -[S^{*2}A - 2S^*AS^* + AS^{*2}]$.*

(2) *For each $i = 1, 2$, the algebra $\mathcal{A}_S^{(i)}$ is a closed *-subalgebra of $\mathcal{A}_S^i$.*

(3) *If δ_S is a generator and, in particular, if S is self-adjoint, the algebra $\mathcal{A}_S^2$ is dense in $\mathcal{U}_S$. Also, the algebra $\mathcal{A}_S^{(2)}$ is dense in $\mathcal{U}_S$.*

Proof. (1) First, we note that, for $A \in \mathcal{A}_S^2$, $AD(S^2) \subset D(S^2)$ and $A^*D(S^2) \subset D(S^2)$. Indeed let $A \in \mathcal{A}_S^2$. Then $A \in \mathcal{A}_S^1$ and, as $A^* \in \mathcal{A}_S^1$, $A^*D(S) \subset D(S)$. Let $y \in D(S^2)$.

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Then SA^*y is defined. Let $x \in D(S^*)$. Then

$$\begin{aligned} i(SA^*y, S^*x) &= i(SA^*y, S^*x) - i(A^*Sy, S^*x) + i(A^*Sy, S^*x) \\ &= (\delta_S(A^*)y, S^*x) + i(A^*Sy, S^*x) \\ &= (\delta_S(A)^*y, S^*x) + i(A^*Sy, S^*x) \\ &= (S\delta_S(A)^*y, x) + i(A^*Sy, S^*x) \end{aligned}$$

because $A \in \mathcal{A}_S^2$, with the result $\delta_S(A)^*D(S) \subset D(S)$. Also, since $y \in D(S^2)$, $A^*Sy \in D(S)$ and SA^*Sy is defined. Hence, in the above expression, $i(SA^*y, S^*x) = (S\delta_S(A)^*y, x) + i(SA^*Sy, x)$. It follows, from the definition of the domain of the adjoint of an unbounded operator, that $SA^*y \in D(S^{**}) = D(S)$, with S being closed. Thus $A^*y \in D(S^2)$. This proves $A^*D(S^2) \subset D(S^2)$. Similarly, it follows that $AD(S^2) \subset D(S^2)$.

Clearly, $\mathcal{A}_S^2$ is a complex vector space. We assume $A \in \mathcal{A}_S^2, B \in \mathcal{A}_S^2$ and verify that $AB \in \mathcal{A}_S^2$. As $\mathcal{A}_S^1$ is an algebra and $A, B \in \mathcal{A}_S^1$, we have $AB \in \mathcal{A}_S^1$. As $\delta_S(AB) = \delta_S(A)B + A\delta_S(B)$ and $\delta_S(A), \delta_S(B) \in \mathcal{A}_S^1$, we have $\delta_S(AB) \in \mathcal{A}_S^1$. Thus $AB \in \mathcal{A}_S^2$. To show that $\mathcal{A}_S^2$ is a *-algebra, we show that $A^* \in \mathcal{A}_S^2$ for $A \in \mathcal{A}_S^2$. We have $A \in \mathcal{A}_S^1, \delta_S(A) \in \mathcal{A}_S^1$. Since $\mathcal{A}_S^1$ is a *-algebra and δ_S is a *-derivation, $A^* \in \mathcal{A}_S^1, \delta_S(A^*) = \delta_S(A)^* \in \mathcal{A}_S^1$. Thus $A^* \in \mathcal{A}_S^2$.

We show that $(\mathcal{A}_S^2, \|\cdot\|_2)$ is complete. Let (A_n) be a Cauchy sequence in $\mathcal{A}_S^2$. Then (A_n) is $\|\cdot\|_1$-Cauchy in the Banach algebra $(\mathcal{A}_S^1, \|\cdot\|_1)$. Hence there exists A in $\mathcal{A}_S^1$ such that in the operator norm, both $\|A_n - A\| \to 0$ and $\|\delta_S(A_n) - \delta_S(A)\| \to 0$. Also, since $A_n \in \mathcal{A}_S^2, \delta_S(A_n) \in \mathcal{A}_S^1$ and since (A_n) is $\|\cdot\|_2$-Cauchy, $(\delta_S(A_n))$ is $\|\cdot\|_1$-Cauchy. Hence, for some $T \in \mathcal{A}_S^1$, $\|\delta_S(A_n) - T\| \to 0$, $\|\delta_S^2(A_n) - \delta_S(T)\| \to 0$. It follows that $T = \delta_S(A)$. Thus $A \in \mathcal{A}_S^2$ and $\|A_n - A\|_2 \to 0$, showing that $(\mathcal{A}_S^2, \|\cdot\|_2)$ is complete. The norm inequality $\|AB\|_2 \leq \|A\|_2\|B\|_2$ (A, B in $\mathcal{A}_S^2$) follows easily from the derivation property of δ_S. Thus $(\mathcal{A}_S^2, \|\cdot\|_2)$ is a Banach *-algebra.

Let $A \in \mathcal{A}_S^2$. We show that $AD(S^{*2}) \subset D(S^{*2})$ and $\delta_S^2(A)|_{D(S^{*2})} = -[S^{*2}A - 2S^*AS^* + AS^{*2}]$. By [R, Theorem 13.2, page 330], $S^{*2} \subset (S^2)^*$. Now let $y \in D(S^2), x \in D(S^{*2})$. Since $A \in \mathcal{A}_S^1$, $AD(S^*) \subset D(S^*)$, by [KS2, Lemma 3.1, page 16], and $\delta_S(A)|_{D(S^*)} = i(S^*A - AS^*)$. Also, $\delta_S(A) \in \mathcal{A}_S^1$. Hence $\delta_S(A)D(S^*) \subset D(S^*)$. Now $x \in D(S^{*2}), S^*x \in D(S^*), AS^*x \in D(S^*)$ and

$$\begin{aligned} &-(\delta_S^2(A)x, y) - (AS^{*2}x, y) + 2(S^*AS^*x, y) \\ &\quad = -(x, \delta_S^2(A^*)y) - (x, S^2A^*y) + 2(x, SA^*Sy) \\ &\quad = (x, A^*S^2y) = (Ax, S^2y). \end{aligned}$$

Hence $y \to (S^2y, Ax)$ is a bounded linear functional on $D(S^2)$ and so $Ax \in D(S^{2*})$. As $x \in D(S^{*2}), x \in D(S^*)$. Since $AD(S^*) \subset D(S^*), Ax \in D(S^*)$ and $y \to (S^*Ax, Sy) = (Ax, S^2y)$ is $\|\cdot\|$ bounded. Thus $S^*Ax \in D(S^*)$ and so $Ax \in D(S^{*2})$. This gives $AD(S^{*2}) \subset D(S^{*2})$ and $\delta_S^2|_{D(S^{*2})} = -[S^{*2}A - 2S^*AS^* + AS^{*2}]$. This completes the proof of (1).

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(2) Is obvious.

(3) If δ_S is a generator, then the set $C^\infty(\delta_S)$ of smooth vectors (in $\mathcal{B}(\mathcal{H})$) of δ_S is dense in $\mathcal{U}_S$ [S1]. Since $C^\infty(\delta_S) \subset \mathcal{A}_S^{(2)} \subset \mathcal{A}_S^2$, it follows that each of $\mathcal{A}_S^{(2)}$ and $\mathcal{A}_S^2$ is dense in $\mathcal{U}_S$. □

Given norms $|\cdot|$ and $\|\cdot\|$ on a vector space X, $|\cdot|$ is called *closable* with respect to $\|\cdot\|$ if, for any sequence (x_n) in X, the assumptions (x_n) is $|\cdot|$-Cauchy and $\|x_n\| \to 0$ imply that $|x_n| \to 0$. The following lemma captures, in the present framework, an important property of the C^2-norm on the commutative Banach algebra $C^2[a, b]$ of C^2-functions.

Lemma 1.2. *On the Banach algebra $\mathcal{A}_S^2$, each of the norms $\|\cdot\|_2$ and $\|\cdot\|_1$ is closable with respect to the operator norm $\|\cdot\|$, and $\|\cdot\|_2$ is closable with respect to $\|\cdot\|_1$.*

Proof. First, we show that $\|\cdot\|_1$ is closable with respect to $\|\cdot\|$ on $\mathcal{A}_S^1$ (and hence also on $\mathcal{A}_S^2$). As S is closed, δ_S is a closed operator. If $A_n \to 0$ in $\|\cdot\|$ and if A_n is a Cauchy sequence in $\|\cdot\|_1$, then $\delta_S(A_n)$ is Cauchy in $\|\cdot\|$. As δ_S is a closed operator, $\|\delta_S(A_n) \to 0$. Hence $\|A_n\|_1 \to 0$. If A_n is Cauchy in $\|\cdot\|_2$, then $\delta_S(A_n)$ and $\delta_S^2(A_n)$ are Cauchy in $\|\cdot\|$. As above, $\|\delta_S(A_n)\| \to 0$. Applying this again, $\|\delta_S^2(A_n) \to 0$. Hence $\|A_n\|_2 \to 0$. Since $\|\cdot\| \le \|\cdot\|_1 \le \|\cdot\|_2$, $\|\cdot\|_2$ is closable with respect to $\|\cdot\|_1$. □

The following follows immediately as in the previous lemma.

Proposition 1.3. *The operator δ_S^2 with domain $D(\delta_S^2) = \mathcal{A}_S^2$ is a closed operator from $(\mathcal{A}_S^1, \|\cdot\|_1)$ to $(\mathcal{B}(\mathcal{H}), \|\cdot\|)$.*

For x, y in $\mathcal{H}$, let $x \otimes y$ be the rank one operator defined as $z \to (z, x)y$. For a densely defined operator T, if $y \in D(T)$, $x \in D(T^*)$, then $\|x \otimes y\| = \|x\|\,\|y\|$, $(x \otimes y)^* = y \otimes x$, $(x \otimes y)(u \otimes v) = (v, x)(u \otimes y)$, $T(x \otimes y) = x \otimes Ty$, and $(x \otimes y)T$ extends to $(T^*x) \otimes y$. It is shown in [KS2, Lemma 3.1] that $x \otimes y \in \mathcal{A}_S^1$ if and only if $x, y \in D(S)$, and that any finite rank operator $F \in \mathcal{A}_S^1$ is of the form $F = \sum x_i \otimes y_i$, a finite sum, where $x_i, y_i \in D(S)$. We use this to prove the following analogue in the present framework.

Proposition 1.4. *Given x, y in $\mathcal{H}$, the rank one operator $x \otimes y \in \mathcal{A}_S^2$ if and only if both x and y are in $D(S^2)$. Further, any finite rank operator F in $\mathcal{A}_S^2$ is of the form $F = \sum x_i \otimes y_i$, a finite sum, with all $x_i \in D(S^2)$, $y_i \in D(S^2)$.*

Proof. Let $x \in D(S^2)$, $y \in D(S^2)$. Then for all $z \in \mathcal{H}$, $(x \otimes y)z = (z, x)y \in D(S)$. Also, $\delta_S(x \otimes y) = i\{S(x \otimes y) - (x \otimes y)S\} = i\{x \otimes Sy - S^*x \otimes y\}$. Further,

$$\begin{aligned}\delta_S(\delta_S(x \otimes y)) &= i\{\delta_S(x \otimes Sy) - \delta_S(S^*x \otimes y)\}\\ &= -\{S(x \otimes Sy) - (x \otimes Sy)S - S(S^*x \otimes y) + (S^*x \otimes y)S\}\\ &= -\{x \otimes S^2y - S^*x \otimes Sy - S^*x \otimes Sy + S^{*2}x \otimes y\}.\end{aligned}$$

In fact, $S \subset S^*$ and $Sx \in D(S), Sy \in D(S)$. Hence $\delta_S^2(x \otimes y) = -(x \otimes S^2y - 2Sx \otimes Sy + S^2x \otimes y)$. As $y \in D(S)$, $(x \otimes y)D(S) \subset D(S)$, and as $x \in D(S)$, $(x \otimes y)^*D(S) = (y \otimes x)D(S) \subset D(S)$. Also, $\delta_S(x \otimes y) = i\{x \otimes Sy - S^*x \otimes y\} = i\{x \otimes Sy - Sx \otimes y\} \in \mathcal{B}(\mathcal{H})$. Thus $x \otimes y \in \mathcal{A}_S^1$. Moreover, $\delta_S(x \otimes y)D(S) \subset D(S)$, $\{\delta_S(x \otimes y)\}^*D(S) =$

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$\delta_S(y \otimes x)D(S) \subset D(S)$, and $\delta_S(\delta_S(x \otimes y)) = -\{x \otimes S^2 y - 2Sx \otimes Sy + S^2 x \otimes y\}$ is a bounded linear operator on $\mathcal{H}$. Hence $x \otimes y \in \mathcal{A}_S^2$.

Conversely, let x, y in $\mathcal{H}$ be such that $x \otimes y \in \mathcal{A}_S^2$.We show that $x \in D(S^2), y \in D(S^2)$. Note that $x \otimes y \in \mathcal{A}_S^1$ and $\delta_S(x \otimes y) \in \mathcal{A}_S^1$. By [KS2, Lemma 3.1(ii)], $x \in D(S)$ and $y \in D(S)$. Also $\delta_S(x \otimes y)D(S) \subset D(S)$ and $\{\delta_S(x \otimes y)\}^* D(S) \subset D(S)$. Now, for any $z \in \mathcal{H}$,

$$\delta_S(x \otimes y)z = i\{x \otimes Sy - S^* x \otimes y\}z = i\{(z, x)Sy - (z, S^* x)y\}.$$

Since $\delta_S(x \otimes y)D(S) \subset D(S)$ and $(S^* x \otimes y)D(S) \subset D(S)$ as $y \in D(S)$, it follows that $(x \otimes Sy)D(S) \subset D(S)$, so that, for all $z \in D(S)$, we have $(x \otimes Sy)z = (z, x)Sy \in D(S)$. Choosing z such that (z, x) is nonzero, we get $Sy \in D(S)$, so that $y \in D(S^2)$. Now $x \otimes Sy \in \mathcal{A}_S^1$. Since $\delta_S(x \otimes y) \in \mathcal{A}_S^1$, we get $S^* x \otimes y \in \mathcal{A}_S^1$. Then, by above result stated in [KS2], $Sx = S^* x \in D(S)$. Thus $x \in D(S^2)$.

Now let $F \in \mathcal{A}_S^2$ be a finite rank operator, say $F = \sum x_i \otimes y_i$, a finite sum. We can assume all x_i to be linearly independent, and also all y_i to be linearly independent. For any $z \in \mathcal{H}$, $Fz = \sum (x_i \otimes y_i)z = \sum (z, x_i)y_i$. Since $F \in \mathcal{A}_S^2$, we have $F \in \mathcal{A}_S^1$ and $\delta_S(F) \in \mathcal{A}_S^1$. By [KS2, Lemma 3.1], all $x_i \in D(S)$ and all $y_i \in D(S)$. Also,

$$\begin{aligned}\delta_S(F) &= \sum \delta_S(x_i \otimes y_i) = i \sum \{S(x_i \otimes y_i) - (x_i \otimes y_i)S\} \\ &= i \sum \{x_i \otimes Sy_i - S^* x_i \otimes y_i\} = i \sum \{x_i \otimes Sy_i - Sx_i \otimes y_i\}.\end{aligned}$$

As $\delta_S(F) \in \mathcal{A}_S^1$, again [KS2, Lemma 3.1(ii)] implies that all $Sy_i \in D(S)$ and all $Sx_i \in D(S)$. Thus all $x_i \in D(S^2)$, and all $y_i \in D(S^2)$. This completes the proof. □

By [KS2, Lemma 3.1(iii)], $\mathcal{K}_S^1$ and $\mathcal{J}_S^1$ are closed *-ideals of $(\mathcal{A}_S^1, \|\cdot\|_1)$ and $(\mathcal{K}_S^1)^2 \subset \mathcal{J}_S^1$. The following contains an analogue of this in the present case.

Proposition 1.5. *$\mathcal{K}_S^2$ and $\mathcal{J}_S^2$ are closed *-ideals of $(\mathcal{A}_S^2, \|\cdot\|_2)$, and $(\mathcal{K}_S^2 \cap \mathcal{J}_S^1)^2 \subset \mathcal{J}_S^2$.*

Proof. Clearly, $\mathcal{K}_S^2$ is a closed *-ideal of $\mathcal{A}_S^2$. Let $A \in \mathcal{J}_S^2$. Then $A \in \mathcal{K}_S^2$, $\delta_S(A) \in \mathcal{K}(\mathcal{H})$, $\delta_S^2(A) \in \mathcal{K}(\mathcal{H})$. Let $B \in \mathcal{A}_S^2$. Then $\delta_S(B) \in \mathcal{A}_S^1, \delta_S^2(B) \in \mathcal{B}(\mathcal{H})$. Then $\delta_S(AB) = \delta_S(A)B + A\delta_S(B) \in \mathcal{K}(\mathcal{H})$ and $\delta_S^2(AB) = A\delta_S^2(B) + 2\delta_S(A)\delta_S(B) + \delta_S^2(A)B \in \mathcal{K}(\mathcal{H})$. Similarly, $BA \in \mathcal{J}_S^2, A^* \in \mathcal{J}_S^2$ showing that $\mathcal{J}_S^2$ is a *-ideal of $\mathcal{A}_S^2$. Clearly, $\mathcal{J}_S^2$ is closed in $(\mathcal{A}_S^2, \|\cdot\|_2)$. Now let $A, B \in \mathcal{K}_S^2 \cap \mathcal{J}_S^1$. Since $B \in \mathcal{K}_S^2$, we have $B \in \mathcal{A}_S^2$ and so $\delta_S(B) \in \mathcal{A}_S^1$. Since $B \in \mathcal{J}_S^1$, B is compact and $\delta_S(B) \in \mathcal{K}(\mathcal{H})$. Thus $\delta_S(B) \in \mathcal{K}_S^1$. Also, $A \in \mathcal{J}_S^1 \subset \mathcal{K}_S^1$. Therefore $A\delta_S(B) \in \mathcal{J}_S^1$. Similarly, $\delta_S(A)B \in \mathcal{J}_S^1$. Thus $\delta_S(AB) = A\delta_S(B) + \delta_S(A)B \in \mathcal{J}_S^1$. We already have $AB \in \mathcal{K}_S^1$. It follows that $AB \in \mathcal{J}_S^2$. □

Proposition 1.6. *The Banach algebras $(\mathcal{A}_S^2), \|\cdot\|_2)$, $(\mathcal{K}_S^2, \|\cdot\|_2)$, $(\mathcal{J}_S^2, \|\cdot\|_2)$ and $(\mathcal{F}_S^2, \|\cdot\|_2)$ are semisimple, $\mathcal{F}_S^2$ has no closed two sided ideals, and $\mathcal{F}_S^2 \subset I$ for any closed *-ideal $\mathcal{I}$ of $(\mathcal{A}_S^2, \|\cdot\|_2)$.*

Proof. Let $\mathcal{I}$ be a closed *-ideal of $\mathcal{F}_S^2$. Let $A \in \mathcal{I}$. Let $x \in D(S)$ such that $A^* x$ is nonzero. Now $A^* x \otimes y = (x \otimes y)A \in \mathcal{I}$ for all $x, y \in D(S^2)$. Then, for all z in $D(S^2)$,

$$(A^* x \otimes y)A^*(z \otimes x) = (A^* x \otimes y)(z \otimes A^* x) = \|A^* x\|^2 (z \otimes y).$$

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Hence $z \otimes y \in \mathcal{I}$. Since $\mathcal{I}$ contains all finite rank operators in $\mathcal{A}_S^2$, we get $\mathcal{F}_S^2 \subset \mathcal{I}$ and, since $\mathcal{I} \subset \mathcal{F}_S^2$, $\mathcal{F}_S^2 = \mathcal{I}$. If $\mathcal{I}$ is a closed *-ideal of $\mathcal{A}_S^2$, this argument implies that $\mathcal{F}_S^2 \subset \mathcal{I}$. The Banach algebra $\mathcal{A}_S^2$ is an A^*-algebra (that is, a Banach *-algebra with a C^*-norm). Hence it is *-semisimple, and so is semisimple. □

We consider the regularity properties of these Banach algebras. Following [KS1, KS2], *a Banach* (D_1^*)*-subalgebra of a* C^**-algebra* $(\mathcal{U}, \|\cdot\|)$ is a dense *-subalgebra $\mathcal{A}$ of $\mathcal{U}$ such that $\mathcal{A}$ is a Banach *-algebra with some norm $\|\cdot\|_1$ satisfying $\|xy\|_1 \le \|x\|\,\|y\|_1 + \|x\|_1\|y\|$, for all $x, y \in \mathcal{A}$. This models a noncommutative differential structure of order one, and the algebra $\mathcal{A}_S^1$ is a Banach $(D_1)^*$-subalgebra of the C^*-algebra $\mathcal{U}_S$. A Banach (D_2^*)-subalgebra of $\mathcal{U}$ [KS1] is a dense $*$-subalgebra $\mathcal{A}$ with seminorms $\|\cdot\|_1, \|\cdot\|_2$ such that:

(1) for each $i = 1, 2$ and for each $x, y \in \mathcal{A}$, there exist $D_i > 0$ satisfying $\|x\|_i = \|x^*\|_i$, $\|xy\|_i \le \|x\|_i\|y\|_i$, $\|xy\|_i \le D_i(\|x\|_i\|y\|_{i-1} + \|x\|_{i-1}\|y\|_i)$; and
(2) $\|\cdot\|_2$ is a norm and $(\mathcal{A}, \|\cdot\|_2)$ is a Banach *-algebra.

This is a noncommutative analogue of the Banach algebra of C^2-functions. The following theorem, which exhibits regularity properties of noncommutative C^2-structures defined by S, contains analogues in the present set-up of several well-known results on the Banach algebra of C^2-functions. For terminology, we refer to [BC, BIO, KS1]. A *Q-normed algebra* is a normed algebra in which the set of quasiregular elements is an open set.

Theorem 1.7. *For $\mathcal{B} = \mathcal{A}_S^2, \mathcal{K}_S^2, \mathcal{J}_S^2, \mathcal{F}_S^2$, let $\mathcal{A}$ stand for their respective C^*-algebra completions. The following hold.*

(1) *$\mathcal{B}$ is a differential Banach algebra of order two and total order less than or equal to two, $\mathcal{B}$ is a Banach (D_2^*)-algebra and $\mathcal{B}$ is a smooth subalgebra of a C^*-algebra.*
(2) *$\mathcal{B}$ is a Q-normed algebra in the C^*-norm on $\mathcal{A}$, and the algebras $\mathcal{B}$ and $\mathcal{A}$ have the same K-theory.*
(3) *$\mathcal{B}$ is closed under the holomorphic functional calculus of $\mathcal{A}$, and is also closed under the C^3-functional calculus of self-adjoint elements of $\mathcal{A}$.*
(4) *The algebra $\mathcal{B}$ is hermitian and spectrally invariant in $\mathcal{A}$.*
(5) *The map $\mathcal{I} \to \mathcal{I} \cap \mathcal{B}$ is a one-to-one correspondence between the closed ideals of $\mathcal{A}$ and the C^*-norm closed ideals of $\mathcal{B}$. The inverse of this correspondence is given by $\mathcal{I} \to \mathcal{I}^-$, the closure of the ideal $\mathcal{I}$ of $\mathcal{B}$ in the C^*-norm $\|\cdot\|$ on $\mathcal{A}$. Not every ideal in $\mathcal{B}$ closed in $\|\cdot\|_2$ is of this form.*
(6) *Let $\pi : \mathcal{B} \to \mathcal{B}(\mathcal{K})$ be a *-representation of $\mathcal{B}$ into bounded operators on a Hilbert space $\mathcal{K}$. Then π is continuous in the C^*-norm on $\mathcal{B}$, and it extends uniquely to a representation of $\mathcal{A}$ into $\mathcal{B}(\mathcal{K})$.*
(7) *Let $\mathcal{U}_S$ be unital. Every completely positive map $\phi : \mathcal{A}_S^2 \to \mathcal{B}(\mathcal{K})$ extends uniquely as a completely positive map $\phi : \mathcal{U}_S \to \mathcal{B}(\mathcal{K})$.*

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Proof. (1) Consider $\mathcal{A}_S^2$. Let $T = (T_0, T_1, T_2)$ on $\mathcal{A}_S^2$ be $T_0(A) = \|A\|, T_1(A) = \|\delta_S(A)\|, T_2(A) = (1/2)\|\delta_S^2(A)\|$. Clearly, T is a differential norm of order two. Further, $T_1(AB) \leq T_0(A)T_1(B) + T_1(A)T_0(B)$ and $T_2(AB) \leq \|A\|T_2(B) + T_1(A)T_1(B) + T_2(A)\|B\|$ showing that T is of logarithmic order $p = \log_2 1 + 1 = 1$ [BC]. By [BC, Proposition 3.10], T is of total order less than or equal to two. Also, the total norm of T is $T_{\text{tot}} := T_0 + T_1 + T_2 = \|\cdot\|_2$. The same arguments apply to other algebras. Thus $\mathcal{B}$ is a differential Banach algebra of order two and total order less than or equal to two. By [BC], $\mathcal{A}_S^2$ is a smooth subalgebra of its C^*-completion in operator norm.

(2) That $\mathcal{B}$ is a Q-normed algebra with the C^*-norm from $\mathcal{A}$ follows from [BC, Proposition 3.12] or [KS1, Theorem 5], and hence closure under holomorphic functional calculus and K-theory isomorphism follows by [C].

(3) The closure under C^3-functional calculus follows from [BC, Proposition 6.4] or [KS1, Theorem 12].

(4) The fact that $\mathcal{B}$ is a Q-subalgebra of $\mathcal{A}$ gives hermiticity and spectral invariance (see also [KS1, Theorem 5]). Notice that the C^*-norm from $\mathcal{A}$ is the greatest C^*-norm on $\mathcal{B}$.

(5) As $\mathcal{B}$ is a Banach D_2^*-subalgebra of $\mathcal{U}$, the assertion follows from [KS1, Theorem 13]. Let $\mathcal{I}$ be a closed ideal of the C^*-algebra $\mathcal{U}_S$. Then the set $\mathcal{I}_S^2 := \{A \in \mathcal{A}_S^2 \cap \mathcal{I} : \delta_S(A) \in \mathcal{I}, \delta_S^2(A) \in \mathcal{I}\}$ is a $\|\cdot\|_2$-closed ideal of $\mathcal{A}_S^{(2)}$.

(6) This follows from the fact that every *-representation of a Q-normed algebra into a C^*-algebra is norm continuous.

(7) The completely positive map ϕ on the unital Banach *-algebra $\mathcal{A}_S^2$ is Stinespring representable [B1] in the sense that it is of form $\phi(T) = V^*\pi(T)V$ where $\pi : \mathcal{A}_S^2 \to \mathcal{B}(\mathcal{K})$ ($\mathcal{K}$ a Hilbert space) is a *-homomorphism and $V : \mathcal{K} \to \mathcal{H}$ is a projection. Now π, and hence ϕ, extends to the C^*-completion of $\mathcal{A}_S^2$, and Arveson's famous completely positive extension theorem applies. □

2. Noncommutative Lipschitz structure

We consider the Lipschitz structure defined by S following the ideas in [W1, W2]. Let $\mathcal{M} \subset \mathcal{N}$ be von Neumann algebras with same unit. A *W^*-derivation* $\delta : \mathcal{M} \to \mathcal{N}$ is an unbounded linear map whose domain $\text{dom}(\delta)$ is a unital *-subalgebra of $\mathcal{M}$ such that (i) $\text{dom}(\delta)$ is ultra weakly dense in $\mathcal{M}$ (ii) the graph of δ is ultra weakly closed in $\mathcal{M} \oplus \mathcal{N}$ and (iii) δ is a *-derivation. Then $\text{dom}(\delta)$ is called *a W^*-domain algebra*. It is a Banach *-algebra with norm $\|x\|_1 := \|x\| + \|\delta(x)\|$. A W^*-domain algebra is envisaged as a noncommutative Lipschitz algebra; equivalently, as a noncommutative metric space. The following brings out an essential difference between the Banach *-algebras $\mathcal{A}_S^{(1)}$ and $\mathcal{A}_S^1$, illuminating the difference between a noncommutative C^1-structure and a noncommutative Lipschitz structure. Let $\mathcal{M}_S := W^*(\mathcal{U}_S)$ be the von Neumann algebra generated by the C^*-algebra $\mathcal{U}_S$. Notice that $\mathcal{M}_S = W^*(\mathcal{A}_S^1)$ and $\mathcal{U}_S = C^*(\mathcal{A}_S^1)$.

Proposition 2.1. *Let S be as above.*

(1) *The derivation $\delta_S : \mathcal{M}_S \to \mathcal{B}(\mathcal{H})$ with domain* $\text{dom}(\delta_S) = \mathcal{A}_S^1$ *is a W^*-derivation.*

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(2) *The Banach* **-algebra* $\mathcal{A}_S^1$ *is dual of a Banach space, and the weak***-topology* σ^1 *on* $\mathcal{A}_S^1$ *is described as* $A_\alpha \to A$ *in* σ^1 *if and only if* $A_\alpha \to A$ *ultra weakly in* $\mathcal{M}_S$ *and* $\delta_S(A_\alpha) \to \delta_S(A)$ *ultra weakly in* $\mathcal{B}(\mathcal{H})$.

Proof. (1) Let $\mathcal{M}_*$ be the predual of $\mathcal{M}_S$, consisting of all ultra weakly continuous linear functionals on $\mathcal{M}_S$, so that the ultra weak topology on $\mathcal{M}_S$ is the weak *-topology $\sigma(\mathcal{M}_S, \mathcal{M}_*)$. Clearly, $\mathcal{A}_S^1$ is ultra weakly dense in $\mathcal{M}_S$. The graph of δ_S is $G(\delta_S) = \{(A, \delta_S(A)) : A \in \mathcal{A}_S^1)\}$, a subspace of $\mathcal{M}_S \oplus \mathcal{B}(\mathcal{H})$. We prove that $G(\delta_S)$ is closed in the ultra weak topology on the direct sum von Neumann algebra $\mathcal{M}_S \oplus \mathcal{B}(\mathcal{H})$. Let (A, B) be in the closure of the graph $G(\delta_S)$ in the ultra weak topology on $\mathcal{M}_S \oplus \mathcal{B}(\mathcal{H})$. Let (A_α) be a net in $\mathcal{A}_S^1$ such that $A_\alpha \to A$ ultra weakly in $\mathcal{M}_S$ and $\delta_S(A_\alpha) \to B$ ultra weakly in $\mathcal{B}(\mathcal{H})$. We show that $AD(S) \subset D(S), A^*D(S) \subset D(S)$, $\delta_S(A)$ is bounded and $B = \delta_S(A)$.

Notice that, since $A_\alpha \in \mathcal{A}_S^1$, $A_\alpha D(S) \subset D(S), A_\alpha^* D(S) \subset D(S)$ and $\delta_S(A_\alpha)$ are bounded operators. Now, since $\mathcal{M}_S$ is ultra weakly closed, the operator $A \in \mathcal{M}_S$ is bounded; similarly B is bounded, and for all ψ, η in $\mathcal{H}$, $((A_\alpha - A)\psi, \eta) \to 0, ((\delta_S(A_\alpha) - B)\psi, \eta) \to 0$. Now let $\psi \in D(S), \eta \in D(S^*)$. Then $(B\psi, \eta) = \lim_\alpha(\delta_S(A_\alpha)\psi, \eta) = i \lim_\alpha((SA_\alpha - A_\alpha S)\psi, \eta) = i \lim_\alpha(A_\alpha\psi, S^*\eta) - i(AS\psi, \eta) = i(A\psi, S^*\eta) - i(AS\psi, \eta)$. As $S^{**} = S$ since S is closed, we have $A\psi \in D(S^{**}) = D(S)$ and $B\psi = i(SA - AS)\psi$. Thus $AD(S) \subset D(S)$ and $(SA - AS)$ extends to a bounded operator. Next we show that $A^*D(S) \subset D(S)$. Let ψ and η be as above. Then $(\eta, (A_\alpha - A)^*\psi) \to 0, (\eta, (\delta_S(A_\alpha) - B)^*) \to 0$. Now $(\eta, B^*\psi) = \lim_\alpha(\eta, \delta_S(A_\alpha)^*\psi) = \lim_\alpha(\eta, \delta_S(A_\alpha^*)\psi) = \lim_\alpha i(\eta, (SA_\alpha^* - A_\alpha^* S)\psi) = \lim_\alpha i(\eta, SA_\alpha^*\psi) - \lim_\alpha i(\eta, A_\alpha^* S\psi) = \lim_\alpha i(\eta, SA_\alpha^*\psi) - i(\eta, A^*S\psi)$. Thus $(\eta, SA_\alpha^*\psi)$ converges and $\lim_\alpha(\eta, SA_\alpha^*\psi) = -i(\eta, B^*\psi) + (\eta, A^*S\psi)$. Also, $\lim_\alpha(\eta, SA_\alpha^*\psi) = \lim_\alpha(S^*\eta, A_\alpha^*\psi) = (S^*\eta, A^*\psi)$. Thus $(A^*\psi, S^*\eta) = i(B^*\psi, \eta) + (A^*S\psi, \eta)$. Hence $A^*\psi \in D(S^{**}) = D(S)$. Thus $A^*D(S) \subset D(S)$. It follows that $G(\delta_S)$ is ultra weakly closed in $\mathcal{M}_S \oplus \mathcal{B}(\mathcal{H})$.

(2) This follows from (1) above as in [W1, Proposition 2]. Indeed, the Banach space $\mathcal{A}_S^1$ is isometrically isomorphic to the graph of δ_S by the map $A \to \{A, \delta_S(A)\}$, and the graph of δ_S is an ultra weakly closed (and hence norm closed) subspace of $\mathcal{M}_S \oplus \mathcal{B}(\mathcal{H})$. Now $\mathcal{M} \oplus \mathcal{B}(\mathcal{K})$ is the dual of the direct sum Banach space $\mathcal{M}_* \oplus C^1(\mathcal{H})$, where $\mathcal{M}_*$ is the predual of $\mathcal{M}_S$ and $C^1(\mathcal{H})$ is the Banach space of trace class operators on $\mathcal{H}$ whose dual is $\mathcal{B}(\mathcal{H})$. Hence it follows that $\mathcal{A}_S^1$ is a dual space. In fact, it is the dual of $(\mathcal{M}_* \oplus C^1(\mathcal{H}))/\mathcal{L}$, where $\mathcal{L}$ is the annihilator of graph of δ_S in $\mathcal{M}_* \oplus C^1(\mathcal{H})$. □

The following continues from the above in view of [W1, Corollaries 4 and 5]. For a metric space (X, d), the *Lipschitz algebra* Lip(X) consists of all bounded complex valued Lipschitz functions f on X, where the Lipshitz number $L(f)$ of f is $L(f) = \sup\{|f(x) - f(y)|/d(x, y) : x, y \in X, x \neq y\} < \infty$. It is a Banach *-algebra with norm $\|f\|_\infty + L(f)$, and is a subalgebra of the abelian von Neumann algebra $L^\infty(X)$ of essentially bounded Borel measurable functions on X. For an operator T, $Sp(T)$ denotes the spectrum of T.

Corollary 2.2. *Let* S *be as above.*

(1) *Let* $X = X^* \in \mathcal{A}_S^1$. *Let* $f \in Lip(Sp(X))$. *Let* $\delta_S(X)$ *commute with* X. *Then* $f(X) \in \mathcal{A}_S^1$ *and* $\|\delta_S(f(X))\| \leq L(f)\|\delta_S(X)\|$.

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(2) *Let $\mathcal{J}$ be a σ^1-closed *-ideal of $\mathcal{A}_S^1$. Then $\mathcal{J}$ is the σ^1-closure of $(\mathcal{J})^2$, where $(\mathcal{J})^2$ is the linear span of $\{AB : A \in \mathcal{J}, B \in \mathcal{J}\}$.*

(3) *Let $\mathcal{J}$ be a *-ideal of $\mathcal{A}_S^1$. Then $\delta_S(\mathcal{J})$ is contained in the ultra weak closure of $\mathcal{J}\mathcal{B}(\mathcal{H}) + \mathcal{B}(\mathcal{H})\mathcal{J}$.*

(4) *Let $\mathcal{I}$ and $\mathcal{J}$ be *-ideals of $\mathcal{A}_S^1$. Then $\mathcal{I} \cap \mathcal{J}$ is contained in the σ^1-closure of $\mathcal{I}\mathcal{J}$ and, if $\mathcal{I}$ and $\mathcal{J}$ are σ^1-closed, then $\mathcal{I} \cap \mathcal{J}$ is the σ^1-closure of $\mathcal{I}\mathcal{J}$.*

We consider the second-order Lipschitz structure. Let $\mathrm{Lip}^2[a,b] := \{f \in \mathrm{Lip}[a,b] : f' \in \mathrm{Lip}[a,b]\} = \{f \in C^1[a,b] : f' \in \mathrm{Lip}[a,b]\}$ a Banach *-algebra with norm $\|f\|_{\mathrm{Lip}^2} = \|f\|_\infty + \|f'\|_\infty + (1/2)\max\{\|f''\|_\infty, L(f')\}$. Let ϕ be the linear operator $\phi : \mathcal{A}_S^2 \to \mathcal{M}_S \oplus \mathcal{M}_S \oplus \mathcal{B}(\mathcal{H})$, $\phi(A) = (A, \delta_S(A), \delta_S^2(A))$. The operator $\delta_S^2 : \mathcal{A}_S^1 \to \mathcal{B}(\mathcal{H})$ is $\delta_S^2(A) = \delta_S(\delta_S(A))$ with domain $\mathrm{dom}(\delta_S^2) = \mathcal{A}_S^2$. The following theorem gives the Lip^2-functional calculus in $\mathcal{A}_S^2$.

THEOREM 2.3.

(1) *The graph of the operator $\delta_S^2 : \mathcal{A}_S^1 \to \mathcal{B}(\mathcal{H})$, dom$\delta_S^2 = \mathcal{A}_S^2$, given by $G(\delta_S^2) = \{(A, \delta_S^2(A)) : A \in \mathcal{A}_S^2\}$ is closed in $\mathcal{A}_S^1 \oplus \mathcal{B}(\mathcal{H})$, where $\mathcal{A}_S^1$ carries the σ^1-topology and $\mathcal{B}(\mathcal{H})$ carries the ultra weak topology. The range of the map ϕ is an ultra weakly closed subspace of $\mathcal{M}_S \oplus \mathcal{M}_S \oplus \mathcal{B}(\mathcal{H})$ with the product ultra weak topology.*

(2) *The algebra $\mathcal{A}_S^2$ is dual of a Banach space. The weak *-topology on $\mathcal{A}_S^2$, denoted by σ^2, is given as $A_\alpha \to A$ in σ^2 if and only if $A_\alpha \to A$ ultra weakly, $\delta_S(A_\alpha) \to \delta_S(A)$ ultra weakly and $\delta_S^2(A_\alpha) \to \delta_S^2(A)$ ultra weakly.*

(3) *Let $X = X^* \in \mathcal{A}_S^2$. Let $f \in \mathrm{Lip}^2(sp(X))$. Let X commute with $\delta_S(X)$. Then $f(X) \in \mathcal{A}_S^2$ and*

$$\|\delta_S^2 f((X))\| \leq L(f)\|\delta_S^2(X)\| + L(f')\|(\delta_S(X))^2\|.$$

PROOF. (1) Follows by application of Proposition 2.1(1) from which (2) follows as in Proposition 2.1(2). Indeed, $\mathcal{A}_S^2$ is isometrically isomorphic to a closed subspace of $\mathcal{A}_S^1 \oplus \mathcal{B}(\mathcal{H})$, and the latter is a dual space.

(3) The proof is a second-order analogue of that of [W1, Theorem 1]. The function f can be extended as a Lipschitz function without changing the Lipschitz constant $L(f)$ to the interval $[-\|X\|, \|X\|]$. Now let f be a polynomial $f(t) = \sum a_n t^n$. Then $f(X) \in \mathrm{dom}(\delta_S^2)$ and, since X and $\delta_S(X)$ commute, we get $\delta_S(f(X)) = \sum n a_n X^{n-1}\delta_S(X)$ as well as

$$\begin{aligned}
\delta_S^2(f(X)) &= \sum n a_n \delta_S(X^{n-1}\delta_S(X)) \\
&= \sum n a_n \{X^{n-1}\delta_S^2(X) + \delta_S(X^{n-1})\delta_S(X)\} \\
&= \sum n a_n \{X^{n-1}\delta_S^2(X) + (n-1)X^{n-2}(\delta_S(X))^2\} \\
&= \sum n a_n X^{n-1}\delta_S^2(X) + \sum n(n-1)a_n X^{n-2}(\delta_S(X))^2 \\
&= f'(X)\delta_S^2(X) + f''(X)(\delta_S(X))^2.
\end{aligned}$$

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Hence

$$\|\delta_S^2(f(X))\| \leq L(f)\|\delta_S^2(X)\| + L(f')\|(\delta_S(X))^2\|.$$

Now let $I = [-\|X\|, \|X\|]$. Let $f \in \mathrm{Lip}^2(I)$. Then $f'' \in L^\infty(I)$. Choose a sequence of polynomials g_n such that $g_n \to f''$ in $L^1(I)$ and $\|g_n|_I\|_\infty \leq \|f''\|_\infty = L(f')$. Let $f_n(t) = f'(0) + \int_0^t g_n(t)\, dt$. Then f_n are polynomials and $\|f_n - f'\|_\infty \to 0$. Hence the L^1-norm $\|f_n - f'\|_1 \to 0$ and $\|f_n'|_I\|_\infty \leq L(f')$. Let $h_n(t) = f(0) + \int_0^t f_n(t)\, dt$. Again h_n are polynomials, $\|h_n|_I - f\|_\infty \to 0$, $\|h_n'|_I\|_\infty \leq \|f'\|_\infty = L(f)$. Then, by the above estimates,

$$\begin{aligned}\|\delta_S^2(h_n(X))\| &\leq \|h_n'\|_\infty\|\delta_S^2(X)\| + \|h_n''\|_\infty\|(\delta_S(X))^2\| \\ &\leq L(f)\|\delta_S^2(X)\| + L(f')\|(\delta_S(X))^2\|.\end{aligned}$$

Therefore there is a subnet (h_α) of the sequence (h_n) such that $\delta_S^2(h_\alpha(X)) \to Y$ for some Y in the ultra weak topology. Now $h_n(X) \to f(X)$ uniformly, $h_n'(X) \to f'(X)$ uniformly and $\delta_S(h_n(X)) \to \delta_S(f(X))$ uniformly. Since the graph of δ_S^2 is σ^2-closed, $f(X) \in \mathrm{dom}(\delta_S^2)$ and $\|\delta_S^2(f(X))\| \leq L(f)\, \|\delta_S^2(X)\| + L(f')\|\, (\delta_S(X))^2\|$. □

3. (~)-convergence

Let $\mathcal{X}$ be a linear subspace of a normed linear space $(\mathcal{Y}, \|\cdot\|)$. Let $\|\cdot\|_1$ be a norm on $\mathcal{X}$ such that $\|x\| \leq \|x\|_1$ for all $x \in \mathcal{X}$. Following [KS2], we say that a sequence (x_n) in $\mathcal{X}$ *(~)-converges to* $y \in \mathcal{Y}$ if $\sup \|x_n\|_1 < \infty$ and $\|x_n - y\| \to 0$ as $n \to \infty$. For a subset M of $\mathcal{X}$, its (~)-closure in $\mathcal{Y}$ (respectively, in $\mathcal{X}$) is the set of all elements in $\mathcal{Y}$ (respectively, in $\mathcal{X}$) which are (~)-limits of elements from M. Then M is (~)-closed in $\mathcal{Y}$ (respectively, in $\mathcal{X}$) if it coincides with its (~)-closure in Y (respectively, in X). This auxiliary mode of convergence has been found useful in understanding the first-order structure in [KS2]. By [KS2, Theorem 3.3], the Banach algebra $\mathcal{A}_S^1$ is (~)-closed in $\mathcal{B}(\mathcal{H})$, and every closed subspace of $(\mathcal{J}_S^1, \|\cdot\|_1)$ is (~)-closed in $\mathcal{J}_S^1$. The following gives an analogue of this in the present set-up. We say that a sequence (A_n) in $\mathcal{A}_S^2$ *(~)-converges to* $A \in \mathcal{B}(\mathcal{H})$ if $\|A_n\|_2 < \infty$, $\|A_n - A\| \to 0$ in operator norm. Thus (~)-convergence in $\mathcal{A}_S^2$ is an analogue of a sequence of C^2-functions bounded in C^2-norm and converges uniformly to a continuous function. In the following, we shall use a technical result [KS2, Corollary 2.7, page 7] that states that if ϕ is a closed linear map from a Banach space $(X, \|\cdot\|_X)$ to a Banach space $(Z, \|\cdot\|_Z)$ with domain domϕ such that the set W_ϕ consisting of bounded linear functionals f on Z (with $fo\phi$ extendable as bounded linear functionals on X) is norm dense in the dual Z^* of Z, then any closed subspace of the Banach space $(\mathrm{dom}\phi, \|\cdot\|_1)$, $\|x\|_1 := \|x\|_X + \|\phi(x)\|_Z$, is (~)-closed in $\mathcal{X}$.

THEOREM 3.1.

(1) *The Banach algebra $\mathcal{A}_S^2$ is (~)-closed in $\mathcal{B}(\mathcal{H})$.*
(2) *Every closed subspace of $(\mathcal{J}_S^2, \|\cdot\|_2)$ is (~)-closed in $(\mathcal{J}_S^2)$. In particular, the ideal $\mathcal{F}_S^2$ is (~)-closed in $(\mathcal{J}_S^2)$.*

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Proof. (1) Let $A_n \in \mathcal{A}_S^2, A \in \mathcal{B}(\mathcal{H})$ be such that $\|A_n - A\| \to 0, \sup \|A_n\|_2 < \infty$. Then $\sup \|\delta_S^2(A_n)\| = r < \infty$. Now the ball B_r of radius r in $\mathcal{B}(\mathcal{H})$ is weak *-compact. Hence there exists $R \in B_r$ such that each neighbourhood of R contains an infinite number of elements from $\{\delta_S^2(A_n)\}$. Let $x \in D(S^2), y \in D(S^{*2})$. Then there exists a sequence (A_{n_k}) from $\{A_n\}$ such that $(\delta_S^2(A_{n_k})x, y) \to (Rx, y)$. Then

$$\begin{aligned}(Rx, y) &= \lim(\delta_S^2(A_{n_k})x, y)\\ &= -\lim(\{S^2A_{n_k} - 2SA_{n_k}S + AS^2\}x, y)\\ &= -\{\lim(S^2A_{n_k}x, y) - 2(SASx, y) + (AS^2x, y)\}.\end{aligned}$$

Notice that the $(\sim)$-convergence with $\|\cdot\|_2$-implies $(\sim)$-convergence with norm $\|\cdot\|_1$. Since $\mathcal{A}_S^1$ is $(\sim)$-closed by [KS1], $A \in \mathcal{A}_S^1$ and $AD(S) \subset D(S)$. Thus

$$\begin{aligned}(Rx, y) &= -\lim(A_{n_k}x, S^{2*}y) + 2(SASx, y) - (AS^2x, y)\\ &= -(Ax, S^{2*}y) + 2(SASx, y) - (AS^2x, y).\end{aligned}$$

Thus $Ax \in D(S^{2**}) = D(S^2)$ and $(Ax, S^{2*}y) = (S^{2-}Ax, y)$. Then

$$(Rx, y) = -\{(S^{2-}Ax, y) + 2(SASx, y) - (AS^2x, y)\}$$

for all y in a dense subspace of $\mathcal{H}$. Thus $Rx = -(S^{2-}Ax + 2SASx - AS^2x) = \delta_S^2 Ax$, $A \in \mathcal{A}_S^2$ and $\mathcal{A}_S^2$ is $(\sim)$-closed in $\mathcal{B}(\mathcal{H})$.

(2) We shall apply [KS2, Corollary 2.4, page 7] stated above. Let $\phi := \delta_S^2|_{\mathcal{J}_S^2} : D(\phi) = \mathcal{J}_S^2 \subset \mathcal{A}_S^1 \to \mathcal{K}(\mathcal{H})$. By Proposition 1.3, it is a closed linear map in the $\|\cdot\|_1 - \|\cdot\|$ topologies. For x, y in $\mathcal{H}$, let $F_{x,y}(A) = (Ax, y)$, which is a bounded linear functional on $\mathcal{K}(\mathcal{H})$. Now take $x \in D(S^2), y \in D(S^2)$. Then, for any A in $\mathcal{J}_S^2$,

$$\begin{aligned}F_{x,y}(\phi(A)) &= (\delta_S^2(A)x, y) = -(\{S^{2-}A - 2SAS + AS^2\}x, y)\\ &= -\{(S^{2-}Ax, y) - 2(SASx, y) + (AS^2x, y)\}\\ &= -\{F_{x,S^2y}(A) - 2F_{Sx,Sy}(A) + F_{S^2x,y}(A)\}.\end{aligned}$$

Thus $F_{x,y} o \phi$ extends as a bounded linear functional on $\mathcal{K}(\mathcal{H})$. Since $D(S^2)$ is dense in $\mathcal{H}$, the set span $\{F_{x,y} : x \in D(S^2), y \in D(S^2)\}$ is dense in the dual of $\mathcal{K}(\mathcal{H})$, identified with trace class operators. By [KS2, Corollary 2.4], any closed subspace of $(\mathcal{J}_S^2, \|\cdot\|_2)$ is $\sim$-closed in $\mathcal{J}_S^2$. In particular, $\mathcal{F}_S^2$ is $(\sim)$-closed in $\mathcal{J}_S^2$. □

An estimate for the first-order functional calculus in $\mathcal{A}_S^1$ is given in[KS2, Lemma 2.6]. The following gives an estimate for the second-order functional calculus in $\mathcal{A}_S^2$. Our proof is different: it uses differential algebras as discussed in [BC, BIO].

Proposition 3.2. *Let $X = X^* \in \mathcal{A}_S^2$. Let $d = \|X\|$ the operator norm. Let h be a C^3-function on $[-d, d]$. Let $\|h\|_{(3)} := \|h\|_\infty + \|h'\|_\infty + \|h''\|_\infty + \|h'''\|_\infty$. Then $\|h(X)\|_2 \le C\|h\|_{(3)}$.*

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PROOF. Notice that $h(X) \in \mathcal{A}_S^2$ by Theorem 1.7(3). The Banach algebra norm $\|\cdot\|_2$ is the total norm of the differential norm T considered in the proof of Theorem 1.7(1) above. The differential norm T is of total order less than or equal to two. Thus, by the definition of the derived norm [BC], $\|\cdot\|_2$ is a derived norm of order less than or equal to two. By [BC, Proposition 6.4, page 270], $\|h(X)\|_2 \leq C\|h\|_{(3)}$, with the constant C depending only on X. □

It is shown in [KS2, Theorem 2.8] that, given $X = X^*$ in $\mathcal{A}_S^1$, there exists a sequence ϕ_n of functions in $C^\infty(R)$, each vanishing on a neighbourhood of zero, such that $\|X^2 - \phi_n(X)\|_1 \to 0$ as $n \to \infty$. The following theorem gives a partial analogue of this. The proof follows that given in [KS2] as much as possible.

THEOREM 3.3. *Let $\mathcal{X}$ consists of all smooth functions f on the real line R, vanishing on a neighbourhood of zero. Let $X = X^* \in \mathcal{A}_S^2$.*

(1) *There exists a sequence ϕ_n in $\mathcal{X}$ such that $\|X^4 - \phi_n(X)\|_2 \to 0$ as $n \to \infty$.*
(2) *X^4 lies in the $\|\cdot\|_2$-closed ideal of the $\|\cdot\|_2$-closed subalgebra $\mathcal{A}_S^2(X)$ of $\mathcal{A}_S^2$ generated by X.*

PROOF. The following constructions are as in [KS2, proof of Theorem 2.8]. Let $n \geq 3$. Let $u_n = u_n(t)$ be the segment of the straight line $u = nt/(n-2) - 2d/(n-2)$ on the plane joining the points $(2d/n, 0)$ and (d, d). Let T_n be the circle that touches the t-axis at $(d/n, 0)$ and also touches the graph of $u_n(t)$ at point $P_n = P(t_n, u_n)$. Let $v_n(t)$ be the arc of T_n between the points $(d/n, 0)$ and P_n. Now define the following functions $\alpha(\cdot), \beta(\cdot), \gamma(\cdot)$ and $\delta(\cdot)$ to be even as $\alpha_n(t) = 0$ if $0 \leq t \leq d/n$, $\alpha_n(t) = v_n(t)$ if $d/n \leq t \leq t_n$ and $\alpha_n(t) = u_n(t)$ if $t_n \leq t \leq d$.

$$\beta_n(t) = 2\int_0^t \alpha_n(s)\,ds \quad \text{if } 0 \leq t \leq d, \beta_n(-t) = \beta_n(t),$$
$$\gamma_n(t) = 3\int_0^t \beta_n(s)\,ds \quad \text{if } 0 \leq t \leq d, \gamma_n(-t) = \gamma_n(t),$$
$$\delta_n(t) = 4\int_0^t \gamma_n(s)\,ds \quad \text{if } 0 \leq t \leq d, \delta_n(-t) = \delta_n(t).$$

Then $\alpha_n(t) = \beta_n(t) = \gamma_n(t) = \delta_n(t) = 0$ in $[-d/n, d/n]$. Also, $\alpha_n \in C^1[-d, d], \beta_n \in C^2[-d, d], \gamma_n \in C^3[-d, d], \delta_n \in C^4[-d, d]$. These functions satisfy the following conditions.

(i) $\|t - \alpha_n(t)\| \leq 2d/n$, $\|\alpha_n'\| = \|u_n'\| = n/(n-2) \leq 3$.
(ii) $\lim_{n\to\infty} \|t^2 - \beta(t)\| = 0, \sup\{\|\beta_n\|, \|\beta_n'\|, \|\beta_n''\|\} < \infty$.
(iii) $\lim_{n\to\infty} \|t^3 - \gamma(t)\| = 0, \sup\{\|\gamma_n\|, \|\gamma_n'\|, \|\gamma_n''\|, \|\gamma_n'''\|\} < \infty$.
(iv) $\lim_{n\to\infty} \|t^4 - \delta_n(t)\| = 0$.

Let $d = \|X\|$. By the functional calculus [KS1, Theorem 12], $\beta_n(X) \in \mathcal{A}_S^2, \gamma_n(X) \in \mathcal{A}_S^2$ and $\delta_n(X) \in \mathcal{A}_S^2$. It follows from above (i)–(iv) that $\|t^4 - \delta_n(t)\|_{(3)} \to 0$ as $n \to \infty$. Hence, by Proposition 3.2, $\|X^4 - \delta_n(X)\|_2 \to 0$. Choose functions $\phi_n \in \mathcal{X}$ such that

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$\|\delta_n - \phi_n\|_{(3)} \to 0$ in $C^3[-d, d]$. Then, by Proposition 3.2, $\|\delta_n(X) - \phi_n(X)\|_2 \to 0$, and so $\|X^4 - \phi_n(X)\|_2 \to 0$. Let $\mathcal{A}^2_S(X)$ be the closed subalgebra of $\mathcal{A}^2_S$ generated by X. Let $\mathcal{I}_0 = \{\phi(X) : \phi \in \mathcal{X}\}$, $\mathcal{I} = \|\cdot\|_2$-closure of $\mathcal{I}_0$ in $\mathcal{A}^2_S$. As $X\mathcal{I}_0 \subset \mathcal{I}_0$, $\mathcal{I}$ is a $\|\cdot\|_2$-closed ideal of $\mathcal{A}^2_S(X)$ and, by the above, $X^4 \in \mathcal{I}$. □

In the above, we do not know whether $X^3 \in \mathcal{I}$. Let $(\mathcal{A}, \|\cdot\|_2)$ be a dense Banach *-subalgebra of a C^*-algebra $(\mathcal{U}, \|\cdot\|)$. Let $\mathcal{A}_+$ be the set of all self-adjoint elements $h = h^*$ of $\mathcal{A}$ such that its spectrum $Sp(h)$ is nonnegative. Let $\mathcal{A}_+^{\text{square}} = \{x^2 : x \in \mathcal{A}_+\}$. It is shown in [KS2, Theorem 2.5] that if $\mathcal{A}$ is a Banach (D_1^*)-subalgebra of $\mathcal{U}$, then $\mathcal{A}_+ = (\sim)$-closure of $\mathcal{A}_+^{\text{square}}$. The following gives a D_2-analogue of this. It applies to the Banach algebra $\mathcal{A}^2_S$. Notice that, in view of [KS1], $\mathcal{A}^2_S$ is spectrally invariant in its C^*-completion in the operator norm, and hence in $\mathcal{U}_S$.

Theorem 3.4. *Let $\mathcal{A}$ be a unital Banach (D_2^*)-subalgebra of a C^*-algebra $\mathcal{U}$. Then $\mathcal{A}_+ = (\sim)$-closure of $\mathcal{A}_+^{\text{square}}$.*

Proof. We have $\mathcal{A}_+^{\text{square}} \subset \mathcal{A}_+$. Also $\mathcal{A}_+$ is $(\sim)$-closed. Hence $(\sim)$-closure $(\mathcal{A}_+^{\text{square}}) \subset \mathcal{A}_+$. To prove the reverse inclusion, let $a \in \mathcal{A}_+$. Then $0 \le a \le \|a\|1$, and $\text{sp}(a) \subset [0, \|a\|]$. For any $\epsilon > 0$, the function $k_\epsilon(t) = (t + \epsilon)^{1/2}$ is analytic on $\text{sp}(a)$. For sufficiently small ϵ, $k_\epsilon(t) \in C^\infty[0, \|a\| + 1]$. By the functional calculus in the C^*-algebra $\mathcal{U}$, $b_\epsilon := k_\epsilon(a) = (a + \epsilon 1)^{1/2} \in \mathcal{U}$, and by the C^∞-functional calculus in D_2-algebra (Theorem 1.7), $b_\epsilon \in \mathcal{A}$. Then $b_\epsilon^2 = a + \epsilon 1$ and $\|a - b_\epsilon^2\| = \epsilon \to 0$. Also, $\|b_\epsilon^2\|_2 \le \|a\|_2 + 1$ showing that b_ϵ^2 $(\sim)$-converges to a. □

4. One sided ideals in $(\mathcal{F}^2_S, \|\cdot\|_2)$

Since S is closed, its domain $D(S)$ is a Hilbert space with inner product $< x, y >_1 = (x, y) + (Sx, Sy)$. Also, S^2 is a densely defined closable operator, and its domain $D(S^2)$ is an inner product space with the inner product $\langle x, y\rangle_2 := (x, y) + (Sx, Sy) + (S^2x, S^2y)$. We show that $D(S^2)$ is a Hilbert space. Let $|x|_2 := \|x\| + \|Sx\| + \|S^2x\|$ be the norm on $D(S^2)$ defined by the inner product $\langle\, , \rangle_2$. First, notice that $|\cdot|_2$ is closable with respect to the Hilbert space norm $\|\cdot\|$ on $\mathcal{H}$. Indeed, let (x_n) be a Cauchy sequence in $|\cdot|_2$ and $\|x_n\| \to 0$. Then $\|Sx_n - Sx_m\| \to 0$. Since S is closed, $\|Sx_n\| \to 0$. Similarly, since S^2 is closable, $\|S^2x_n\| \to 0$. Thus $|x_n|_2 \to 0$, showing that $|\cdot|_2$ is closable. This implies that the completion $\mathcal{L}$ of $D(S^2)$ in $|\cdot|_2$ is contained in $\mathcal{H}$. Now let $x \in \mathcal{L}$. Choose a sequence x_n in $D(S^2)$ such that $|x_n - x|_2 \to 0$. Then $\|x_n - x\| \to 0$ and Sx_n is $\|\cdot\|$-Cauchy. By the closure of the operator S, $x \in D(S)$ and $Sx_n \to Sx$ in $\|\cdot\|$. Further, since $Sx_n \in D(S)$ and S^2x_n is $\|\cdot\|$-Cauchy, again, by the closure of S, it follows that $Sx \in D(S)$ and $S^2x_n \to S^2x$ in $\|\cdot\|$. Thus $x \in D(S^2)$. It follows that $\mathcal{L} = D(S^2)$ and thus $D(S^2), |\cdot|_2)$ is a Hilbert space.

For any $K \subset D(S^2)$, let $I_l(K)$ be the closure, in the norm $\|\cdot\|_2$ of the Banach *-algebra $\mathcal{A}^2_S$, of the linear span of $\{x \otimes y : x \in K, y \in D(S^2)\}$, and $I_r(K)$ be the closure in $\|\cdot\|_2$ of the linear span of $\{x \otimes y : x \in D(S^2), y \in K\}$. Since, for any operator T, $T(x \otimes y) = x \otimes Ty$, $(x \otimes y)T = T^*x \otimes y$, and since $(x \otimes y)^* = y \otimes x$, it follows that $I_l(K)$ is a closed left ideal of $\mathcal{F}^2_S$, $I_r(K)$ is a closed right ideal of $\mathcal{F}^2_S$ and $I_l(K) = I_r(K)^*$. Further,

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let I be a nontrivial left ideal of $\mathcal{F}_S^2$. Let $L(I) = \{x \in D(S^2) : x \otimes y \in I$ for all $y \in D(S^2)\}$. Since $D(S^2)$ is dense in $\mathcal{H}$, for any nonzero A in I, there exists $x \in D(S^2)$ such that A^*x is nonzero. Now $A^*x \in D(S^2)$ and, for all $y \in D(S^2)$, $(A^*x) \otimes y = (x \otimes y)A \in I$. Hence $A^*x \in L(I)$ and $L(I)$ is nonzero. Similarly, if I is a right ideal of $\mathcal{F}_S^2$, then $R(I) = \{x \in D(S^2) : y \otimes x \in I$ for all $y \in D(S^2)\}$ is nonzero. Recall that a closed ideal I in $(\mathcal{F}_S^2, \|\cdot\|_2)$ is *essential* if $I = (\mathcal{F}_S^2 I)^{-\|\cdot\|_2}$, which is the $\|\cdot\|_2$-closure of the linear span of the set $\mathcal{F}_S^2 I = \{T_1T_2 : T_1 \in \mathcal{F}_S^2, T_2 \in I\}$. The following provides a second-order analogue of [KS2, Theorem 4.1, page 24] that determines the essential left ideals of the algebra $\mathcal{F}_S^1$.

THEOREM 4.1.

(i) *Let K be a linear subspace of $D(S^2)$. The following hold:*

(1) *$I_l(K)$ is an essential left ideal of $(\mathcal{F}_S^2, \|\cdot\|_2)$; and*

(2) *$K \subset L(I_l(K))$; $L(I_l(K))$ equals the closure of K in $(D(S^2), |\cdot|_2)$, and $L(I_l(K)) = R(I_r(K))$.*

(ii) *Let I be a closed nontrivial left ideal of $(\mathcal{F}_S^2, \|\cdot\|_2)$. The following hold:*

(1) *$L(I)$ is a nontrivial closed subspace of $(D(S^2), |\cdot|_2)$; and*

(2) *$I_l(L(I))$ is the $\|\cdot\|_2$-closure of span $\mathcal{F}_S^2 I$, it is the largest essential ideal contained in I, and it contains all finite rank operators in I.*

PROOF. (i)(1) The $\|\cdot\|_2$-closure $(\mathcal{F}_S^2 I_l(K))^{-\|\cdot\|_2}$ is in $I_l(K)$, as $I_l(K)$ is a closed left ideal of $\mathcal{F}_S^2$. For any $x \in K, y \in D(S^2)$, $(y \otimes y)(x \otimes x) = \|y\|^2(x \otimes y) \in \mathcal{F}_S^2 I_l(K)$. Hence $x \otimes y \in \mathcal{F}_S^2 I_l(K)$ and, by the definition of $I_l(K)$, $I_l(K) \subset (\mathcal{F}_S^2 I_l(K))^{-\|\cdot\|_2}$. Thus $I_l(K) = (\mathcal{F}_S^2 I_l(K))^{-\|\cdot\|_2}$ and $I_l(K)$ is essential.

(2) Notice that, for $x, y \in D(S^2)$,

$$\begin{aligned}|x|_2 &= \{\|x\|^2 + \|Sx\|^2 + \|S^2x\|^2\}^{1/2}\\ &\le \|x\| + \|Sx\| + \|S^2x\|.\end{aligned}$$

Therefore

$$\begin{aligned}|x|_2^2 &\le \{\|x\| + \|Sx\| + \|S^2x\|\}^2\\ &\le \{\|x\|^2 + \|Sx\|^2 + \|S^2x\|^2 + 2\|x\|\,\|Sx\| + 2\|Sx\|\,\|S^2x\| + 2\|S^2x\|\,\|x\|\}\\ &\le 3\{\|x\|^2 + \|Sx\|^2 + \|S^2x\|^2\}\\ &= 3|x|_2^2,\\ |x|_2\|y\| &\le (\|x\| + \|Sx\| + \|S^2x\|)\|y\|\\ &= \|x \otimes y\| + \|Sx \otimes y\| + \|S^2x \otimes y\|\\ &\le \|x \otimes y\| + \|Sx \otimes y - x \otimes Sy\| + \|x \otimes Sy\|\\ &\quad + \|S^2x \otimes y - 2Sx \otimes Sy + x \otimes S^2y\| + 2\|Sx \otimes Sy\| + \|x \otimes S^2y\|\\ &\le \|x \otimes y\|_2 + \|x\|\,\|Sy\| + 2\|Sx\|\,\|Sy\| + \|x\|\,\|S^2y\|.\end{aligned}$$

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Also

$$\begin{aligned}\|x\otimes y\|_2 &\le \|x\otimes y\| + \|\delta_S(x\otimes y)\| + \|\delta_S^2(x\otimes y)\|\\ &= \|x\otimes y\| + \|x\otimes Sy - Sx\otimes y\| + \|\delta_S(x\otimes Sy - Sx\otimes y)\|\\ &= \|x\otimes y\| + \|x\otimes Sy - Sx\otimes y\| + \|x\otimes S^2y - 2Sx\otimes Sy + S^2x\otimes y\|\\ &\le \|x\|\,\|y\| + \|x\|\,\|Sy\| + \|Sx\|\,\|y\| + \|x\|\,\|S^2y\| + 2\|Sx\|\,\|Sy\| + \|S^2x\|\,\|y\|\\ &\le \|x\|(\|y\| + \|Sy\| + \|S^2y\|) + \|y\|(\|x\| + \|Sx\| + \|S^2x\|) + 2\|Sx\|\,\|Sy\|\\ &\le 3^{1/2}(\|x\|\,|y|_2 + |x|_2\|y\|) + 2\|Sx\|\,\|Sy\|.\end{aligned}$$

Clearly, $K \subset L(I_l(K))$. We show that $L(I_l(K)) \subset K^{-|\cdot|_2}$. Let $y \in D(S^2), \|y\| = 1$, and let $z \in L(I_l(K)), z \notin K$. Then $z\otimes y \in I_l(K)$, and there exists a sequence $A_n = \sum_{i=1}^{m_n} x_n^i \otimes y_n^i \in I_l(K), x_n^i \in K, y_n^i \in D(S^2)$ such that $\|z\otimes y - A_n\|_2 \to 0$. Then

$$\|(y\otimes y)(z\otimes y) - (y\otimes y)A_n\|_2 = \|z\otimes y - A_n^* y\otimes y\|_2 = \|(z - z_n)\otimes y\|_2 \to 0,$$

where $z_n := A_n^* y$. Thus $\|(z - z_n)\otimes y\| \to 0$ and $\|(z - z_n)\otimes y\|_1 \to 0$. Hence $\|z - z_n\| \to 0$. This also implies that $\|S(z - z_n)\| \to 0$. Indeed,

$$\begin{aligned}\|(z - z_n)\otimes y\|_1 &= \|(z - z_n)\otimes y\| + \|S\{(z - z_n)\otimes y\} - \{(z - z_n)\otimes y\}S\|\\ &= \|(z - z_n)\otimes y\| + \|(z - z_n)\otimes Sy - \{S(z - z_n)\}\otimes y\|\\ &\ge \|z - z_n\|\,\|y\| + |\|(z - z_n)\otimes Sy\| - \|\{S(z - z_n)\}\otimes y\||\\ &= \|z - z_n\| + |\|z - z_n\| - \|S(z - z_n)\||.\end{aligned}$$

Thus $\|S(z - z_n)\| \to 0$. Then, by the above norm relations,

$$\begin{aligned}|z - z_n|_2 &= |z - z_n|_2\|y\|\\ &\le \|(z - z_n)\otimes y\|_2 + \|z - z_n\|\,\|Sy\| + \|z - z_n\|\,\|S^2y\| + 2\|S(z - z_n)\|\,\|Sy\|\\ &\to 0\end{aligned}$$

as $n \to \infty$. Therefore $z \in K^{-|\cdot|_2}$ and $L(I_l(K)) \subset K^{-|\cdot|_2}$. On the other hand, let $z \in K^{-|\cdot|_2}$. Then there exists a sequence $(z_n) \subset K$ such that $|z - z_n|_2 \to 0$, so that $\|S(z - z_n)\| \to 0$. Then, again by the norm relations discussed above,

$$\|z\otimes y - z_n\otimes y\|_2 \le 3^{1/2}(\|z - z_n\||y|_2 + |z - z_n|_2\|y\|) + 2\|S(z - z_n)\|\,\|Sy\| \to 0.$$

Hence $z\otimes y \in I_l(K)$ and $z \in L(I_l(K))$. Thus $L(I_l(K)) = K^{-|\cdot|_2}$. Similarly, we can prove that $R(I_r(K)) = K^{-|\cdot|_2}$.

(ii) (1) Let I be a closed nontrivial left ideal of $(\mathcal{F}_S^2, \|\cdot\|_2)$. Then $L(I)$ is nonzero, where $L(I) = \{x \in D(S^2) : x\otimes y \in I \text{ for all } y \in D(S^2)\}$. If $L(I) = D(S^2)$ then, by Proposition 1.4, $I = \mathcal{F}_S^2$, which contradicts the nontriviality of I. Thus $L(I)$ is a nontrivial subspace of $D(S^2)$. We show that $L(I)$ is closed. Let $x \in D(S^2)$, and let (x_n) in $L(I)$ be such that $|x_n - x|_2 \to 0$. Then $x_n \to x$ in $\mathcal{H}$. By the norm relations discussed above,

$$\|x\otimes y - x_n\otimes y\|_2 \le 3^{1/2}(\|x_n - x\||y|_2 + |x_n - x|_2\|x\|) + 2\|S(x_n - x)\|\,\|Sy\| \to 0$$

as $n \to \infty$ for all $y \in D(S^2)$. Since I is closed and $x_n\otimes y \in I$, we get $x\otimes y \in I$. Then $x \in L(I)$ and $L(I)$ is closed.

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(2) We show that $I_l(L(I))$ contains all finite rank operators from I. Let $F \in I$ be a finite rank operator. By Proposition 1.4, $F = \sum x_i \otimes y_i$, which is a finite sum, where $x_i \in D(S^2)$ and $y_i \in D(S^2)$ with (y_i) assumed to be linearly independent. For u, v in $D(S^2)$, $(u \otimes v)F = F^*u \otimes v \in I$. Therefore $F^*u = \sum_{i=1}^n (y_i, u)x_i \in L(I)$ for all $u \in D(S^2)$. For a fixed i, choose $(u_i)_{i=1}^n$ in $D(S^2)$ such that $(y_i, u_i) = 1, (y_i, u_j) = 0$ for all $j \neq i$. It follows that all $x_i \in L(I)$ and $F \in I_l(L(I))$.

We show that $(\mathcal{F}_S^2 I)^{-\|\cdot\|_2} \subset I_l(L(I))$. Let $A \in \mathcal{F}_S^2$ and $B \in I$. Then there exist finite rank operators A_n in $\mathcal{F}_S^2$ such that $\|A - A_n\|_2 \to 0$. Now all A_nB are finite rank operators in I and, by above arguments, $A_nB \in I_l(L(I))$. As $\|AB - A_nB\| \to 0$ and as $I_l(L(I))$ is closed in $\|\cdot\|_2$, we get $AB \in I_l(L(I))$. Hence $(\mathcal{F}_S^2 I)^{-\|\cdot\|_2} \subset I_l(L(I))$.

Next, we show that $I_l(L(I)) \subset (\mathcal{F}_S^2 I)^{-\|\cdot\|_2}$. Let $x \in L(I)$ and $y \in D(S^2)$. Then $x \otimes y \in I_l(L(I))$ and $x \otimes y \in I$. Now $(y \otimes y)(x \otimes y) = \|y\|^2(x \otimes y) \in \mathcal{F}_S^2 I$. Thus $x \otimes y \in \mathcal{F}_S^2 I$. Since $I_l(L(I))$ is the closed linear span of all $x \otimes y, x \in L(I), y \in D(S^2)$, $I_l(L(I)) \subset (\mathcal{F}_S^2 I)^{-\|\cdot\|_2}$. It follows that $I_l(L(I)) = (\mathcal{F}_S^2 I)^{-\|\cdot\|_2}$.

Further, $I_l(L(I))$ is essential and, by construction, $I_l(L(I)) \subset I$. Let J be an essential left ideal in I. Then $L(J) \subset L(I)$ and $I_l(L(J)) \subset I_l(L(I))$. As J is essential, $J = (\mathcal{F}_S^2 J)^{-\|\cdot\|_2} = I_l(L(J)) \subset I_l(L(I))$, showing that $I_l(L(I))$ is the largest essential ideal in I. □

Theorem 4.2. *The map ψ defined as $\psi(I) = L(I)$ gives a one-to-one correspondence between the set of nontrivial closed essential left ideals of $(\mathcal{F}_S^2, \|\cdot\|_2)$ and the set of nontrivial closed subspaces of $(D(S^2), |\cdot|_2)$.*

Proof. Given a closed nontrivial essential left ideal I of $\mathcal{F}_S^2$, $L(I)$ is a nontrivial closed subspace of $D(S^2)$, and then $I_l(L(I)) = (\mathcal{F}_S^2 I)^{-\|\cdot\|_2} = I$. Thus ψ is one-to-one. If $I \subset J$, then $L(I) \subset L(J)$, and, by the injectivity of ψ, $L(I) \neq Ł(J)$ if $I \neq J$. Let $K \subset D(S^2)$ be a nontrivial $|\cdot|_2$-closed subspace. Then $I_l(K)$ is essential and $\psi(I_l(K)) = L(I_l(K))$. Hence $I_l(K) \neq \mathcal{F}_S^2$ and ψ is surjective. Also, if $K \subset K_1$, then $I_l(K) \subset I_l(K_1)$. If $I_l(K) = I_l(K_1)$, then $L(I_l(K)) = K^- = K = L(I_l(K_1)) = K_1^- = K_1$. Since $K \neq K_1$, $I_l(K) \neq I_l(K_1)$. Thus ψ is a one-to-one partial order-preserving map and $\psi(I) \subset \psi(J)$ if and only if $I \subset J$. □

Thus a closed left ideal I of $(\mathcal{F}_S^2, \|\cdot\|_2)$ is essential if and only if $I = I_l(K)$ for a $K \subset D(S^2)$. In this case, $I = I_l(L(I))$. Further, it is maximal essential if and only if the closure of K in $(D(S^2), |\cdot|_2)$ is of codimension one in $D(S^2)$. The following can be proved exactly, as in [KS2, Theorems 4.2(iv) and 4.3]. An operator A is *essential* for $\mathcal{F}_S^2$ if $A \in (\mathcal{F}_S^2 A)^{-\|\cdot\|_2}$, the closure in $(\mathcal{F}_S^2, \|\cdot\|_2)$.

Theorem 4.3.

(i) *Let I be a closed left ideal of $(\mathcal{F}_S^2, \|\cdot\|_2)$ and let J be the intersection of all maximal essential left ideals containing I. Then $I_l(L(I)) = I_l(L(J))$. If I is essential, then $I = I_l(L(J))$. If all closed left ideals of $(\mathcal{F}_S^2, \|\cdot\|_2)$ are essential, then every closed left ideal is the intersection of all maximal closed left ideals containing I.*

(ii) *All left ideals $(\mathcal{F}_S^2 A, \|\cdot\|_2)$ are essential.*

(iii) *All finite rank operators in $\mathcal{F}_S^2$ are essential.*

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