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Sums of finite-dimensional spaces

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Analogues are developed to the sum theorems in the dimension theory of metric spaces. It is shown that, within the class of metric spaces, any locally countable, σ-locally finite, or closure-preserving sum of finite-dimensional sets is countable-dimensional. Similar results are obtained under the more general hypothesis of countable-dimensional rather than finite-dimensional sets.

The Sum Theorem in dimension theory can be stated as follows (all spaces in this paper are metric, and by dimension we mean Lebesque covering dimension):

SUM THEOREM. *If* X *is the union of a locally countable collection of* n*-dimensional closed subsets, then* X *is* n*-dimensional* [cf. *2*, Theorem II.1].

One might ask what could be concluded about X if the words "n-dimensional closed subsets" in the hypothesis were replaced by the words "finite-dimensional subsets". A similar question might also be raised: what would we know about X if the same words were replaced by "countable-dimensional subsets" (a countable-dimensional space is defined to be a union of countably many finite-dimensional spaces [*1*]). It will be shown below in Theorem 2 that the answer to both questions is the same; namely, X must be countable-dimensional.

The following Lemma is the key to the succeeding work.

LEMMA. X *is countable-dimensional iff* X *is the union of a locally finite collection of countable-dimensional subsets.*

Proof. If X is the union of a locally finite collection

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$A = \{A_\lambda : \lambda \in \Lambda\}$ of countable-dimensional subsets, then for each $\lambda \in \Lambda$ we can write

$$A_\lambda = \bigcup_{i=1}^{\infty} A_{\lambda,i} ,$$

where $A_{\lambda,i}$ is finite-dimensional for all $i = 1,2,\ldots$. Now for each $i = 1,2,\ldots$ we define

$$P_i = \{A_{\lambda,i} : \lambda \in \Lambda\} ,$$

and

$$P_i = \bigcup P_i .$$

Since A is locally finite and for all $\lambda \in \Lambda$, $A_{\lambda,i} \subset A_\lambda$, it is clear that P_i is a locally finite cover of P_i consisting of finite-dimensional sets. For every point $p \in P_i$ we define $N(p)$ to be a P_i-open set which contains p and meets at most finitely many members of P_i ; then $N = \{N(p) : p \in P_i\}$ is a relatively open cover of P_i . By the paracompactness of P_i there exists a locally finite P_i-closed refinement Q of N which covers P_i . For any $Q \in Q$, there exists $N(p) \in N$ such that $Q \subset N(p) \subset P_i$; $N(p)$ is covered by a finite union of members of P_i , each of which is finite-dimensional, so $N(p)$ is finite-dimensional [2, Corollary to Theorem II.4], and Q is therefore itself finite-dimensional by the Monotone Theorem [2, Theorem II.3].

Now for all $n = 1,2,\ldots$ we define

$$R_{i,n} = \bigcup \{Q \in Q : \dim Q \leqq n\} .$$

By the Sum Theorem each $R_{i,n}$ has dimension $\leqq n$ (as Q is a locally finite closed collection in P_i), and

$$P_i = \bigcup_{n=1}^{\infty} R_{i,n}$$

since Q is a cover of P_i consisting of finite-dimensional subsets, so

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$$X = \bigcup_{\lambda\in\Lambda} A_\lambda = \bigcup_{\lambda\in\Lambda} \left[\bigcup_{i=1}^{\infty} A_{\lambda,i}\right] = \bigcup_{i=1}^{\infty} \left[\bigcup_{\lambda\in\Lambda} A_{\lambda,i}\right] = \bigcup_{i=1}^{\infty} P_i = \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} R_{i,n} ,$$

which proves that X is countable-dimensional.

THEOREM 1. *X is countable-dimensional iff each point of X has a countable-dimensional neighborhood.*

Proof. For each $p \in X$ let $N(p)$ be a countable-dimensional neighborhood of p. Then $N = \{N(p) : p \in X\}$ has a locally finite refinement P which covers X, and each member of P is countable-dimensional, hence X is itself countable-dimensional by the Lemma.

We are now in a position to prove the main result of this communication.

THEOREM 2. *X is countable-dimensional iff X is the union of a locally countable collection of countable-dimensional subsets.*

Proof. For any $p \in X$ there exists a neighborhood $N(p)$ which is covered by a countable collection $\{A_i : i = 1,2,\ldots\}$ of countable-dimensional sets. For each $i = 1,2,\ldots$, A_i is the union of a countable collection $\{A_{i,j} : j = 1,2,\ldots\}$ of finite-dimensional sets. Hence

$$N(p) \subset \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} A_{i,j} ,$$

so $N(p)$ is countable-dimensional. Thus every point of X has a countable-dimensional neighborhood, and X is countable-dimensional by Theorem 1.

Theorem 2 answers the second question from the introductory paragraph, and the first question now can be answered in the following, since every finite-dimensional space is countable-dimensional.

COROLLARY 3. *X is countable-dimensional iff X is the union of a locally countable collection of finite-dimensional subsets.*

A similar question can be asked about another sum theorem [2, Corollary to Theorem II.1], and the answer appears below in Theorem 5. The following result will be needed.

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THEOREM 4. *X is countable-dimensional iff X is the union of a σ-locally finite collection of countable-dimensional subsets.*

Proof. Let $\mathcal{A} = \bigcup_{i=1}^{\infty} \mathcal{A}_i$ be a cover of X, where $\mathcal{A}_i$ is a locally finite collection of countable-dimensional sets of all $i = 1,2,\ldots$. By the Lemma, $\bigcup \mathcal{A}_i$ is the union of a countable collection $\{B_{i,j} : j = 1,2,\ldots\}$ of finite-dimensional sets, so

$$X = \bigcup \mathcal{A} = \bigcup_{i=1}^{\infty} \bigcup \mathcal{A}_i = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} B_{i,j} ,$$

which implies that X is countable-dimensional.

THEOREM 5. *X is countable-dimensional iff $X = \bigcup_{\lambda<\tau} A_\lambda$, where for all $\lambda < \tau$, A_λ is countable-dimensional and $\bigcup_{\mu<\lambda} A_\mu$ is closed.*

Proof. For all $\lambda < \tau$ and all $i = 1,2,\ldots$ we define

$$B_{\lambda,i} = A_\lambda - S_{1/i}\left(\bigcup_{\mu<\lambda} A_\mu\right) .$$

Let

$$\mathcal{B}_i = \{B_{\lambda,i} : \lambda < \tau\}$$

for all $i = 1,2,\ldots$: we now show that each $\mathcal{B}_i$ is locally finite. For any $p \in X$, there exists a first $\lambda < \tau$ such that $p \in A_\lambda$. By hypothesis $\bigcup_{\mu<\lambda} A_\mu$ is closed, so

$$N = S_{1/i}(p) - \bigcup_{\mu<\lambda} A_\mu$$

is a neighborhood of p . If $\nu < \lambda$, then

$$N \cap B_{\nu,i} \subset N \cap A_\nu \subset N \cap \left(\bigcup_{\mu<\nu} A_\mu\right) = \emptyset ;$$

on the other hand, if $\nu > \lambda$, then

$$N \cap B_{\nu,i} \subset S_{1/i}(p) \cap \left(X - S_{1/i}\left(\bigcup_{\mu<\nu} A_\mu\right)\right) \subset S_{1/i}(p) \cap \left(X - S_{1/i}(A_\lambda)\right) = \emptyset$$

as $p \in A_\lambda$. Thus N is a neighborhood of p which meets at most the one element $B_{\lambda,i}$ of $\mathcal{B}_i$, so $\mathcal{B}_i$ is locally finite.

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$$B = \bigcup_{i=1}^{\infty} B_i$$

is a σ-locally finite collection of countable-dimensional sets, the Theorem will follow from Theorem 4 if B is a cover. But for any $p \in X$ there exists a first $\lambda < \tau$ such that $p \in A_\lambda$, and an integer i such that

$$d(p, \bigcup_{\mu<\lambda} A_\mu) \geqq 1/i ,$$

in which case $p \in B_{\lambda,i}$ and the proof is complete.

References

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