

LIE AND JORDAN STRUCTURE IN OPERATOR ALGEBRAS

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Abstract

Let \mathfrak{A} be a C^* -algebra, α a $*$ -anti-automorphism of order 2, and $\mathfrak{A}^\alpha(\pm 1) = \{A \in \mathfrak{A}, \alpha(A) = \pm A\}$ the spectral subspaces of α . It follows that $\mathfrak{A}^\alpha(+1)$ is a Jordan algebra and $\mathfrak{A}^\alpha(-1)$ is a Lie algebra. We begin the classification of pairs of Jordan and Lie algebras which can occur in this manner by examining $\mathfrak{A} = \mathcal{L}(\mathcal{H})$, the algebra of bounded operators on a Hilbert space \mathcal{H} .

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0. Introduction

Recently Effros and the second-named author (Effros and Størmer (1979)) began the analysis of Jordan structure associated with positive identity-preserving projections P acting on a C^* -algebra \mathfrak{A} with identity $\mathbf{1}$. Thus $P; A \in \mathfrak{A} \rightarrow P(A) \in \mathfrak{A}$ satisfies $P \geq 0$, $P^2 = P$, and $P(\mathbf{1}) = \mathbf{1}$. The starting point of the analysis, and a subsequent paper (Størmer (1979)), is a proof that the range $P(\mathfrak{A}_h)$ of P on the hermitian elements \mathfrak{A}_h of \mathfrak{A} is a Jordan algebra when equipped with a suitable new product. One consequence of this result is that if \mathfrak{A} is a von Neumann algebra and α a normal positive identity preserving map of \mathfrak{A} into itself then the hermitian fixed points of α have a natural Jordan structure. An easy example of this latter result occurs if α is a $*$ -automorphism, or a $*$ -anti-automorphism, of \mathfrak{A} .

The starting point of the present note is the observation that if α is an anti-automorphism then the elements of \mathfrak{A} satisfying $\alpha(A) = -A$ form a Lie-algebra with respect to the natural Lie product $[A, B] = AB - BA$. This follows from the

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simple calculation

$$\begin{aligned} \alpha([A, B]) &= \alpha(AB) - \alpha(BA) \\ &= \alpha(B)\alpha(A) - \alpha(A)\alpha(B) = -[A, B]. \end{aligned}$$

Our purpose is to explore relations between this Lie structure, the Jordan structure of the fixed points, and positive projections associated with anti-automorphisms.

Suppose, more generally, that \mathfrak{A} is an algebra over a field \mathcal{k} and α is a \mathcal{k} -linear anti-automorphism of \mathfrak{A} then the eigenspaces $\mathfrak{A}^\alpha(\pm 1) = \{A; A \in \mathfrak{A}, \alpha(A) = \pm A\}$ are Jordan and Lie subalgebras of \mathfrak{A} under the products

$$A \circ B = (AB + BA)/2, \quad [A, B] = AB - BA,$$

respectively. Moreover the linear span of $\mathfrak{A}^\alpha(+1)$ and $\mathfrak{A}^\alpha(-1)$ is a subalgebra \mathcal{L} of \mathfrak{A} on which α acts as an anti-automorphism of order 2. This follows from the easily verified relations

$$\begin{aligned} [\mathfrak{A}^\alpha(-1), \mathfrak{A}^\alpha(+1)] &\subseteq \mathfrak{A}^\alpha(+1), \quad [\mathfrak{A}^\alpha(+1), \mathfrak{A}^\alpha(+1)] \subseteq \mathfrak{A}^\alpha(-1), \\ \mathfrak{A}^\alpha(+1) \circ \mathfrak{A}^\alpha(-1) &\subseteq \mathfrak{A}^\alpha(-1) \end{aligned}$$

which imply that \mathcal{L} is both a Lie algebra and a Jordan algebra. Thus if one considers different powers of α and the corresponding eigenspaces one can reduce a substantial part of spectral theory for α to the study of anti-automorphisms of order 2, and the associated Jordan and Lie structure.

In this note we describe parts of the latter theory when \mathfrak{A} is a C^* -algebra and α preserves the $*$ -operation. Our principal result concerns the case that $\mathfrak{A} = \mathcal{L}(\mathcal{H})$. We show that there is a pairing between the classical Lie algebras and their infinite dimensional analogues, and the irreducible reversible Jordan algebras of self-adjoint operators or the scalars.

1. Pairing of Lie and Jordan algebras

We begin by examining some pairing properties of Lie and Jordan algebras in finite dimensions.

Let M_n denote the $n \times n$ matrices and α a $*$ -anti-automorphism of M_n . Since there is a unique normalized trace on M_n it follows that

$$\text{Tr}(\alpha(A)) = \text{Tr}(A)$$

for all $A \in M_n$. Consequently if $A \in M_n^\alpha(-1)$ and $B \in M_n^\alpha(+1)$

$$\begin{aligned} \text{Tr}(AB^*) &= -\text{Tr}(\alpha(A)\alpha(B^*)) \\ &= -\text{Tr}(\alpha(B^*A)) = -\text{Tr}(AB^*) \end{aligned}$$

and hence

$$\text{Tr}(AB^*) = 0.$$

But M_n is the linear span of $M_n^\alpha(+1)$ and $M_n^\alpha(-1)$, because each $A \in M_n$ has the decomposition

$$A = A_+ + A_-,$$

where

$$A_\pm = (A \pm \alpha(A))/2 \in M_n^\alpha(\pm 1).$$

Thus one deduces that the Jordan algebra $M_n^\alpha(+1)$ is the orthogonal complement $M_n^\alpha(-1)^\perp$ of the Lie algebra $M_n^\alpha(-1)$ when M_n is equipped with the Hilbert space scalar product

$$(A, B) = \text{Tr}(AB^*).$$

This motivates the examination of Lie subalgebras of M_n which have Jordan subalgebras as their orthogonal complements, and vice versa.

The Cartan–Killing classification (Z-X Wan (1975)) of Lie subalgebras of M_n is given in terms of four families of ‘classical’ algebras A_n, B_n, C_n, D_n , together with five ‘exceptional’ algebras G_2, F_4, E_6, E_7, E_8 . These latter algebras will play no role in the sequel.

The algebra A_{n-1} consists of the $n \times n$ matrices with trace zero. The algebras B_n only occur if n is odd and C_n and D_n occur if n is even. The B_n, C_n, D_n are identifiable as eigenspaces $M_n^\alpha(-1)$ of $*$ -anti-automorphisms of the form

$$A \rightarrow \alpha(A) = UA^tU^{-1},$$

where A^t denotes the transpose of A and U is a unitary matrix whose form varies with the class. The three possibilities $U_{B_n}, U_{C_n}, U_{D_n}$, are usually expressed in the forms

$$U_{B_n} = \begin{bmatrix} 1 & & 0 \\ & 0 & \mathbf{1}_{(n-1)/2} \\ 0 & \mathbf{1}_{(n-1)/2} & 0 \end{bmatrix}, \quad U_{C_n} = \begin{bmatrix} 0 & \mathbf{1}_{n/2} \\ -\mathbf{1}_{n/2} & 0 \end{bmatrix},$$

$$U_{D_n} = \begin{bmatrix} 0 & \mathbf{1}_{n/2} \\ \mathbf{1}_{n/2} & 0 \end{bmatrix}$$

where $\mathbf{1}_m$ denotes the identity matrix in M_m . By repeated transposition of rows and columns one easily sees that these matrices are similar to the unitary matrices

PROOF 1. Let $A, B \in \mathfrak{g}$ and $C \in \mathfrak{g}^\perp$ then

$$\text{Tr}([C, A]B) = \text{Tr}(C[A, B]) = 0$$

because $[A, B] \in \mathfrak{g}$. Thus $[C, A] \in \mathfrak{g}^\perp$ which establishes 1.

2. Since \mathfrak{g}^\perp is a finite-dimensional self-adjoint Jordan algebra its simplicity will follow if we can establish that its only central projections are 0 and 1.

Let E be a central projection of \mathfrak{g}^\perp . Then if $A \in \mathfrak{g}$ we have

$$EAE - AE = [E, A]E = E[E, A] = EA - EAE,$$

where the second step uses part 1, that is $[E, A] \in \mathfrak{g}^\perp$ because $A \in \mathfrak{g}$ and $E \in \mathfrak{g}^\perp$. Thus

$$EA + AE = 2EAE.$$

Successive multiplication on the left and right by E yields

$$EA = EAE = AE$$

and consequently $E \in \mathfrak{g}'$. Thus

$$E \in (\mathfrak{g} + \mathfrak{g}^\perp)' = M'_n = \mathbf{C}\mathbf{1}.$$

3. It follows from the discussion preceding the proposition, and the observation that if $\mathfrak{g} = 0$, or M_n , then $\mathfrak{g}^\perp = M_n$, or 0, that the cases listed in part 3 are all possible. It remains to prove that this list is exhaustive.

Let us consider the various possibilities for the Jordan algebra \mathfrak{g}^\perp .

If \mathfrak{g}^\perp is irreducible and its self-adjoint part is Q_n then $\mathfrak{g} (= \mathfrak{g}^{\perp\perp})$ is C_n . But if \mathfrak{g}^\perp is irreducible and its self-adjoint part is S_n then \mathfrak{g} is B_n or D_n .

Next if \mathfrak{g}^\perp is reducible there are two possibilities. Either the self-adjoint algebra \mathfrak{A} generated by \mathfrak{g}^\perp is simple or it is not.

In the first of these cases we can identify M_n with $\mathfrak{A} \otimes M_k$, for some k , where \mathfrak{g}^\perp is identified with $\mathfrak{g}^\perp \otimes \mathbf{C}\mathbf{1}_k$. Now let $\{e_{ij}; i, j = 1, 2, \dots, k\}$ denote the usual matrix units in M_k , then it follows easily that the orthogonal complement of \mathfrak{g}^\perp within \mathfrak{A} is a Lie algebra \mathfrak{h} and

$$\mathfrak{g} = \left\{ A; A = \sum_{i,j=1}^k A_{ij} \otimes e_{ij}, \sum_{i=1}^k A_{ii} \in \mathfrak{h} \right\}.$$

Thus if $A, B \in \mathfrak{A}$ then $A \otimes e_{12}$ and $B \otimes e_{21}$ belong to \mathfrak{g} so that

$$AB \otimes e_{11} - BA \otimes e_{22} = [A \otimes e_{12}, B \otimes e_{21}] \in \mathfrak{g}.$$

Hence $[A, B] \in \mathfrak{h}$. Thus \mathfrak{h} contains all the commutators in \mathfrak{A} and \mathfrak{h} must be

$A_{m-1} \otimes M_k$ for some m . But then

$$A = \sum_{i,j=1}^k A_{ij} \otimes e_{ij} \in \mathfrak{g}$$

if, and only if,

$$\text{Tr} \left(\sum_{i=1}^k A_{ii} \right) = 0,$$

that is $\text{Tr}(A) = 0$ and $\mathfrak{g} = A_{n-1}$.

Next assume \mathfrak{A} is not simple. Then it has exactly two nontrivial central projections E and $F = \mathbf{1} - E$ (see, for example, Jordan and others (1934) or Størmer (1968)) and $\mathfrak{A}E = \mathfrak{g}^\perp E$. Now we argue that $EM_n E \cap \mathfrak{g}$ is the orthogonal complement of $E\mathfrak{g}^\perp$ in $EM_n E$. Indeed if $EAE \in \mathfrak{g}$ then

$$\text{Tr}((EAE)EB) = \text{Tr}(EAE B) = 0$$

for all $B \in \mathfrak{g}^\perp$. Thus $EAE \in (E\mathfrak{g}^\perp)^\perp$. Conversely if $EAE \in (E\mathfrak{g}^\perp)^\perp$ then for all $B \in \mathfrak{g}^\perp$ one has

$$0 = \text{Tr}((EAE)EB) = \text{Tr}(EAE B)$$

so $EAE \in \mathfrak{g}$.

Next remark that since $E\mathfrak{g}^\perp = E\mathfrak{A}$ is simple, and of the form $M_m \otimes \mathbf{C}\mathbf{1}_k$, the arguments in the earlier part of the proof establish that $(E\mathfrak{g}^\perp)^\perp = EM_n E \cap \mathfrak{g} = 0$ or A_{k-1} . In the latter case \mathfrak{g}^\perp is the scalars and \mathfrak{A} cannot have two central projections. Thus we are left with the possibility $EM_n E \cap \mathfrak{g} = 0$ or, equivalently, $E\mathfrak{g}^\perp = E\mathfrak{A} = EM_n E$. Now $E\mathfrak{A} \simeq F\mathfrak{A}$ (Størmer (1968)) and by symmetry

$$F\mathfrak{g}^\perp = F = FM_n F.$$

Thus $n = 2m$ is even and we can make the identifications $\mathfrak{A} = M_m \oplus M_m$ with $\mathfrak{g}^\perp = \{A \oplus A^t; A \in M_m\}$. Writing elements of \mathfrak{A} , \mathfrak{g} , etcetera as 2×2 matrices over M_m one readily computes that

$$\mathfrak{g} = \left\{ \begin{bmatrix} A & B \\ C & -A^t \end{bmatrix}; A, B, C \in M_m \right\}.$$

But this is a Lie algebra if, and only if, $m = 1$. Thus $\mathfrak{g} = A_1$ and $\mathfrak{g}^\perp = \mathbf{C}\mathbf{1}$.

It now follows from the structure theory of simple self-adjoint Jordan algebras (Jordan and others (1934); Størmer (1966)) that the only remaining possibility is that the self-adjoint part of \mathfrak{g}^\perp is a spin factor of real dimension greater than 6 and $n = 2^k$ for some $k \geq 3$ (see, for example, Effros and Størmer (1979) or Størmer (1966)). Now let E be a nonzero minimal projection in \mathfrak{g}^\perp and remark that if $A \in \mathfrak{g}^\perp$ and $B \in \mathfrak{g}$ then

$$\text{Tr}(A(EBE)) = \text{Tr}((EAE)B) = 0.$$

Hence $EgE \subseteq (Eg^\perp E)^\perp$. But conversely if $EAE \in (Eg^\perp E)^\perp$ and $B \in g^\perp$ then

$$\text{Tr}((EAE)B) = \text{Tr}((EAE)EB) = 0$$

and hence $(Eg^\perp E)^\perp \cap EM_n E \subseteq EgE$. Thus EgE is the orthogonal complement of $Eg^\perp E$ in $EM_n E$. But since E is minimal in g^\perp one has $Eg^\perp E = CE$ and consequently $EgE = (CE)^\perp \cap EM_n E$. But $F = 1 - E$ is also a minimal projection in g^\perp and thus one concludes that $EgE \simeq FgF \simeq A_{n/2-1}$. Now choose $A \in EgE$ to be rank one and let $S \in g^\perp$ be a self-adjoint unitary such that

$$SES = F.$$

It follows from part 1 of the proposition that $[S, A] \in g^\perp$. But $[S, A]$ has rank less than or equal to two and the spin algebra g^\perp cannot contain a nonzero matrix of rank less than four. This is a contradiction and consequently g^\perp cannot be a spin factor of dimension greater than 6.

The final statement of the proposition is immediate because the condition $A \circ B \in g^\perp$ for all $A, B \in g$ excludes the possibilities A_{n-1} with $n \geq 3$ and M_n .

COROLLARY 2. *If g is a self-adjoint exceptional Lie subalgebra of M_n then g^\perp cannot be a Jordan algebra.*

This is a partial reiteration of statement 3 of Proposition 1.

COROLLARY 3. *If g is a nonzero self-adjoint Lie subalgebra of M_n then there is a *-anti-automorphism α of order 2 of M_n with $g = M_n^\alpha(-1)$ and $g^\perp = M_n^\alpha(+1)$ if, and only if, g^\perp is a Jordan algebra and*

$$g \circ g \subseteq g^\perp.$$

PROOF. If $g = M_n^\alpha(-1)$ for some α and $A, B \in g$ then

$$\alpha(A \circ B) = (\alpha(AB) + \alpha(BA))/2 = A \circ B.$$

Thus $g \circ g \subseteq M_n^\alpha(+1) = g^\perp$. Conversely if this latter condition is satisfied then Proposition 1 implies that g is one of the algebras A_1, B_n, C_n, D_n . The existence of an α in the cases B_n, C_n, D_n , follows from the definitions of these algebras whilst the case $n = 2$ and $g = A_1$, is dealt with by defining α such that

$$\alpha\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

2. Projections on $\mathcal{L}(\mathcal{H})$

Next we examine the pairing of infinite-dimensional Jordan and Lie algebras but with a slight change of emphasis. If α is a *-anti-automorphism of order 2 of

a C^* -algebra \mathfrak{A} with identity and ι is the identity map then $P = (\iota + \alpha)/2$ is a positive projection which preserves the identity. The eigenspaces $\mathfrak{A}^\alpha(+1)$, and $\mathfrak{A}^\alpha(-1)$, now correspond to the fixed points of P , and the null space of P , respectively. Thus we analyze the corresponding pairing problem for projections. This approach is related to the work of Arazy and Friedman (1978) who discussed the characterization of all contractions of the compact operators $\mathcal{L}C(\mathcal{H})$ on a separable Hilbert space and it is also a natural extension of parts of Effros and Størmer (1979) and Størmer (1979). First it is necessary to introduce Lie algebras which are infinite-dimensional analogues of the B_n, C_n, D_n .

The weakly closed Jordan subalgebras of $\mathcal{L}(\mathcal{H})$ have been classified (Størmer (1966); Størmer (1972)) in terms of four JW-factors, real symmetric factors, quaternionian factors, spin factors and hermitian parts of von Neumann algebras. In the finite-dimensional discussion of the previous section the first two occurred in association with classical Lie algebras and we next introduce infinite-dimensional analogues of the Lie algebras by generalizing this association. One natural method of introducing these infinite-dimensional Lie algebras is by orthogonality with respect to the trace on $\mathcal{L}(\mathcal{H})$ but this leads to annoying details of approximation of trace-class operators. A second method is through spectral theory of $*$ -anti-automorphisms. The real symmetric irreducible JW-factor \mathfrak{A}_r is the hermitian (self-adjoint) elements of $\mathcal{L}(\mathcal{H})$ which are invariant under a $*$ -anti-automorphism of order 2, the transpose map $A \rightarrow A^t$. Thus $\mathfrak{A}_r = \mathcal{L}(\mathcal{H})^t(+1)_h$. The eigenspace $\mathfrak{A}_r = \mathcal{L}(\mathcal{H})^t(-1)$ is a Lie algebra \mathfrak{g} with the bracket operation $[A, B] = AB - BA$. If \mathcal{H} has finite dimension n then $\mathfrak{g} = B_n$ for n odd and $\mathfrak{g} = D_n$ for n even. If \mathcal{H} is infinite dimensional we denote the algebra \mathfrak{g} by B_∞ (or D_∞). Similarly, the quaternion algebra \mathfrak{A}_q is formed by the elements of $\mathcal{L}(\mathcal{H})$ which are invariant under a $*$ -anti-automorphism $A \rightarrow A^q$. If \mathcal{H} has finite dimension n the associated Lie algebra $\mathfrak{g} = \mathcal{L}(\mathcal{H})^q(-1)$ is C_n and hence we denote the infinite-dimensional analogue by C_∞ .

The connection between B_∞, C_∞ and B_n, C_n can be made more precise by considering an n -dimensional projector $E \in \mathfrak{A}_r$, or $E \in \mathfrak{A}_q$. It follows immediately from the calculations of the previous section that $EgE = (E\mathfrak{A}E)^\perp$ in $E\mathcal{L}(\mathcal{H})E$ and EgE is isomorphic to B_n, C_n or D_n . The first case occurs if n is odd and $E \in \mathfrak{A}_r$, the second if $E \in \mathfrak{A}_q$, and the third if n is even and $E \in \mathfrak{A}_r$.

THEOREM 4. *Let \mathcal{H} be a complex Hilbert space and P a positive normal identity preserving projection of $\mathcal{L}(\mathcal{H})$ into itself such that $\mathfrak{A} = P(\mathcal{L}(\mathcal{H})_h)$ is a JC-algebra. Let \mathfrak{g} denote the null space of P .*

1. *If $n = \dim \mathcal{H} < +\infty$ then \mathfrak{g} is a Lie algebra if and only if either $\mathfrak{A} = \mathbf{R1}$, in which case $\mathfrak{g} = A_{n-1}$ or \mathfrak{A} is reversible and irreducible, in which case \mathfrak{g} is one of $0, B_n, C_n, D_n$.*

2. If \mathcal{H} is infinite-dimensional then \mathfrak{g} is a Lie algebra if and only if \mathfrak{A} is reversible and irreducible, in which case \mathfrak{g} is one of $0, B_\infty, C_\infty$.

PROOF. If $n = \dim \mathcal{H} < +\infty, \mathfrak{A} = \mathbf{R}\mathbf{1}$ and $\mathfrak{g} = A_{n-1}$ then P is a state with null space A_{n-1} and hence P is the trace. Next assume \mathfrak{A} is irreducible and reversible. One can have $\mathfrak{A} = \mathcal{L}(\mathcal{H})_h, P = \iota$ and $\mathfrak{g} = 0$. Alternatively \mathfrak{A} could be the real symmetric operators or the quaternionic algebra. In both cases P is unique by Størmer (1979), Lemma 6.1 and hence $P = (\iota + \alpha)/2$ where α is the $*$ -anti-automorphism of $\mathcal{L}(\mathcal{H})$ such that $\alpha(A) = A$ for $A \in \mathfrak{A}$. Consequently

$$\mathfrak{g} = \mathcal{L}(\mathcal{H})^\alpha(-1) = \mathfrak{A}^\perp$$

is B_n, C_n or D_n , by Proposition 1, or if $\dim \mathcal{H} = \infty$ by the definition of B_∞, C_∞ . Conversely assume \mathfrak{g} is a Lie algebra.

We first argue that P must be faithful. Let $r = \text{supp } P$ and assume $r \neq \mathbf{1}$. Let $A = rA(\mathbf{1}-r)$ be nonzero in $\mathcal{L}(\mathcal{H})$. Then $P(A) = 0$ and $A \in \mathfrak{g}$. Since P is self-adjoint $A^* \in \mathfrak{g}$ thus $rA(\mathbf{1}-r)A^*r - (\mathbf{1}-r)A^*rA(\mathbf{1}-r) = [A, A^*] \in \mathfrak{g}$. Now

$$(\mathbf{1}-r)A^*rA(\mathbf{1}-r) \in \mathfrak{g}$$

and consequently $rA(\mathbf{1}-r)A^*r \in \mathfrak{g}$. But P is faithful on $r\mathcal{L}(\mathcal{H})r$ and hence $rA(\mathbf{1}-r)A^*r = 0$ and $rA(\mathbf{1}-r) = 0$ contrary to assumption. Thus $r = \mathbf{1}$ and P is faithful.

Since P is ultra-weakly continuous \mathfrak{A} is a JW-algebra. Let l denote its centre, then $l + il$ is an abelian von Neumann algebra and there is a complete family of normal projections of \mathfrak{A}'' onto $l + il$ and in particular of $\mathfrak{A} + i\mathfrak{A}$ onto $l + il$ (see the proof of $3 \Rightarrow 5$ in Størmer (1972)). Composing these projections with the faithful projection P one obtains a complete family of normal projections of $\mathcal{L}(\mathcal{H})$ onto $l + il$. Thus by Størmer (1972) the centre l is totally atomic.

Next we argue that \mathfrak{A} is a JW-factor. If not there exist two non-zero minimal projections $E \neq F$ in l . Let $[\mathfrak{A}] = P(\mathcal{L}(\mathcal{H}))$ denote the complexification $\mathfrak{A} + i\mathfrak{A}$ of \mathfrak{A} , then $\mathcal{L}(\mathcal{H}) = [\mathfrak{A}] + \mathfrak{g}$, and each $B \in \mathcal{L}(\mathcal{H})$ has a decomposition $B = A + G$ with $A \in \mathfrak{A}$ and $G \in \mathfrak{g}$. But $EAF = 0$ and $P(A \circ G) = A \circ P(G)$ by Størmer (1979), Lemma 4.1. Thus

$$\begin{aligned} P(EGF) &= P(E(EGF) + (EGF)E) \\ &= 2P(E \circ (EGF)) \\ &= 2E \circ P(EGF). \end{aligned}$$

By symmetry

$$P(EGF) = 2F \circ P(EGF)$$

and consequently

$$P(EGF) = 0.$$

Thus $E\mathcal{L}(\mathcal{H})F \subset \mathfrak{g}$. Now assuming $\dim E \leq \dim F$ one can find a partial isometry $V \in E\mathcal{L}(\mathcal{H})F$ such that $VV^* = E$ and $V^*V \leq F$. Since $V \in \mathfrak{g}$, $V^* \in \mathfrak{g}$ and $[V, V^*] \in \mathfrak{g}$. Now by Størmer (1979), Lemma 4.1, $P(EGE) = EP(G)E$ for $G \in \mathfrak{g}$, thus $EgE \subset \mathfrak{g}$ and

$$E = VV^* = E[V, V^*]E \in \mathfrak{g}$$

which is a contradiction. Thus \mathfrak{A} is a JW-factor.

Next we claim that \mathfrak{A}' is a finite von Neumann algebra and $P|_{\mathfrak{A}'}$ is a trace. Indeed if $B \in \mathfrak{A}'$ then $B = A + G$ with $A \in [\mathfrak{A}]$ and $G \in \mathfrak{g}$. But $P(B) = \lambda \mathbf{1}$ and $\lambda \in \mathbb{C}$ by Størmer (1979), Lemma 4.2. Thus $A = P(B) = \lambda \mathbf{1}$ and $B = G + \lambda \mathbf{1}$. Now let $B' = G' + \lambda' \mathbf{1}$ be another operator in \mathfrak{A}' then

$$[B, B'] = [G, G'] \in \mathfrak{g}$$

and

$$P([B, B']) = 0.$$

So P is a faithful normal trace on \mathfrak{A}' . But if $B' = G' + \lambda' \mathbf{1} \in \mathfrak{A}'$ and $A \in [\mathfrak{A}]$, $G \in \mathfrak{g}$ then

$$\begin{aligned} P(B'(A+G)) &= P(AB' + (G' + \lambda' \mathbf{1})G) \\ &= P(AB' + [G', G] + GG' + \lambda' G) \\ &= P((A+G)B'). \end{aligned}$$

Consequently if ω is any normal state on \mathfrak{A} then $\omega \circ P$ is a normal state containing \mathfrak{A}' in its centralizer and the support of $\omega \circ P$ belongs to $(\mathfrak{A}')^\perp = \mathfrak{A}''$. Let $F = \text{supp } \omega \circ P$ and let \mathcal{M} be the centralizer of $\omega \circ P$ in $\mathcal{L}(\mathcal{H})$. By Tomita's modular theory there exists a faithful normal projection P_ω of $F\mathcal{L}(\mathcal{H})F$ onto MF . Since MF is a finite von Neumann algebra containing $\mathfrak{A}'F$ as a subalgebra there is a faithful normal projection Q_ω of MF onto $\mathfrak{A}'F$ and since MF is of type I $\mathfrak{A}'F$ is of type I (see Tomiyama (1959) or Størmer (1972)). Now P is faithful and hence the states of the form $\omega \circ P$ form a complete set of normal states on $\mathcal{L}(\mathcal{H})$ and their supports are separating for \mathfrak{A}' so that \mathfrak{A}' is of type I. But then \mathfrak{A}'' is of type I and therefore \mathfrak{A} is of type I (Størmer (1968)).

Our next aim is to prove that \mathfrak{A} is reversible. Assume the contrary, then \mathfrak{A} is a spin factor. Let E and $F = \mathbf{1} - E$ be minimal projections in \mathfrak{A} , then EgE is a Lie subalgebra of \mathfrak{g} and P restricted to $E\mathcal{L}(\mathcal{H})E$ is a state ω , that is

$$P(EAE) = \omega(EAE)E.$$

But a state ω on a C^* -algebra \mathcal{B} has a Lie algebra \mathcal{L} as null space if and only if ω is a trace. Indeed if $A, B \in \mathcal{B}$ then $A - \omega(A)\mathbf{1}, B - \omega(B)\mathbf{1} \in \mathcal{L}$. But then

$$[A, B] = [A - \omega(A)\mathbf{1}, B - \omega(B)\mathbf{1}] \in \mathcal{L}$$

and hence

$$\omega([A, B]) = 0$$

for all $A, B \in \mathcal{L}$. In particular $P|_{E\mathcal{L}(\mathcal{H})E}$ is a trace so $\dim \mathcal{H} < +\infty$. Now $P(EAF) = 0$ for all $A \in \mathcal{L}(\mathcal{H})$ by the earlier computation which showed that

$$P(EAF) = 2E \circ P(EAF) = 2F \circ P(EAF).$$

But since P is a trace on both $E\mathcal{L}(\mathcal{H})E$ and $F\mathcal{L}(\mathcal{H})F$ one must have

$$\text{Tr}(A) = \text{Tr}(P(A))$$

for all $A \in \mathcal{L}(\mathcal{H})$. In particular if $G \in \mathfrak{g}$ and $A \in [\mathfrak{A}]$ one has from Effros and Størmer (1979), Lemma 1.1 that

$$\begin{aligned} \text{Tr}(G \circ A) &= \text{Tr}(P(G \circ P(A))) \\ &= \text{Tr}(P(G) \circ P(A)) = 0 \end{aligned}$$

because $P(G) = 0$. Thus $\mathfrak{g} \subseteq \mathfrak{A}^\perp$. But $\dim \mathcal{H} < +\infty$ and $\mathcal{L}(\mathcal{H}) = \mathfrak{g} + [\mathfrak{A}]$ and hence $\mathfrak{g} = \mathfrak{A}^\perp$. But by Proposition 1 this is impossible if \mathfrak{A} is a nonreversible spin factor. Therefore \mathfrak{A} is reversible.

There are now two cases to consider. Either \mathfrak{A}'' is a factor or \mathfrak{A}'' has exactly two nonzero central projections different from $\mathbf{1}$ (Størmer (1968)). Let us first assume \mathfrak{A}'' is a factor. Since \mathfrak{A}'' is of type I we can identify $\mathcal{L}(\mathcal{H})$ with $\mathfrak{A}'' \otimes N$ where N is isomorphic to \mathfrak{A}' and is a type I_n factor with $n < +\infty$. If $n = 1$ then $N = \mathbb{C}$ and $\mathfrak{A}'' = \mathcal{L}(\mathcal{H})$ and P is the unique normal projection of \mathfrak{A}'' onto $[\mathfrak{A}]$ (see Størmer (1979), Proposition 6.2). Either $\mathfrak{g} = 0$ if $[\mathfrak{A}] = \mathcal{L}(\mathcal{H})$, $\mathfrak{g} = B_n$ or D_n if $n \in \mathbb{Z}_+ \cup \{+\infty\}$ and \mathfrak{A} is the real symmetric operators, and $\mathfrak{g} = C_n$ if \mathfrak{A} is the quaternionian irreducible JW-factor. If $n \geq 2$ then Proposition 6.2 of Størmer (1979) establishes that $P = Q \otimes \tau$ where Q is the unique normal projection of \mathfrak{A}'' onto $[\mathfrak{A}]$ and τ is the normalized trace on \mathfrak{A}' . Let $\{e_{ij}; i, j = 1, 2, \dots, n\}$ be a complete set of matrix units in N , then if

$$A = \sum_{i,j=1}^n A_{ij} \otimes e_{ij} \in \mathfrak{A}'' \otimes N$$

one has

$$\begin{aligned} P(A) &= P\left(\sum_{i,j=1}^n A_{ij} \otimes e_{ij}\right) \\ &= \frac{1}{n} Q\left(\sum_{i=1}^n A_{ii}\right). \end{aligned}$$

Thus

$$\mathfrak{g} = \left\{ A; A = \sum_{i,j=1}^n A_{ij} \otimes e_{ij}, \sum_{i=1}^n A_{ii} \in Q^{-1}(0) \right\}$$

and in particular $A \otimes e_{12}, B \otimes e_{21} \in \mathfrak{g}$ for all $A, B \in \mathfrak{U}''$. Thus

$$AB \otimes e_{11} - BA \otimes e_{22} = [A \otimes e_{12}, B \otimes e_{21}] \in \mathfrak{g}$$

and

$$AB - BA = [A, B] \in Q^{-1}(0),$$

that is

$$Q([A, B]) = 0.$$

Hence if ω is any normal state on \mathfrak{U} , $\omega \circ Q$ is a trace on \mathfrak{U}'' and consequently \mathfrak{U}'' is finite-dimensional. In particular Q itself must be the trace and $\mathfrak{g} = A_m$ for some m .

Finally assume \mathfrak{U}'' has exactly two minimal central projections E and F different from $\mathbf{1}$. Since P is faithful $P(E) = \lambda \mathbf{1}$ with $0 < \lambda < 1$ (Størmer (1979), Lemma 4.2) and the map $P_E; E\mathcal{L}(\mathcal{H})E \rightarrow [\mathfrak{U}]E = \mathfrak{U}''E$ defined by

$$P_E(EAE) = \frac{1}{\lambda} P(EAE)E$$

is a faithful normal projection such that

$$P_E(E) = E.$$

The null space $P_E^{-1}(0)$ is equal to $\mathfrak{g} \cap E\mathcal{L}(\mathcal{H})E$ because $P_E(EAE) = 0$ if and only if $P(EAE) = 0$. Thus $P_E^{-1}(0)$ is a Lie algebra. Since $\mathfrak{U}''E$ is a type I factor it is of the form $\mathcal{L}(\mathcal{X}) \otimes \mathbf{C1}_{\mathcal{M}}$ where $\mathcal{X} \otimes \mathcal{M} = \mathcal{H}$. Furthermore $\mathbf{C1}_{\mathcal{X}} \otimes \mathcal{L}(\mathcal{M}) = (\mathfrak{U}''E)' = \mathfrak{U}'E$; so $P|_{\mathbf{C1}_{\mathcal{X}} \otimes \mathcal{L}(\mathcal{M})}$ is the normalized trace τ . Thus $P_E = \iota \otimes \tau$ by Proposition 6.2 of Størmer (1979) where ι denotes the identity map on $\mathcal{L}(\mathcal{X})$. It follows as in the previous paragraph that either $\dim \mathcal{X} = 1$ or $\dim \mathcal{M} = 1$. If $\dim \mathcal{X} = 1$ then $\mathfrak{U}''E = CE$ and since $\mathfrak{U}''E \simeq \mathfrak{U}''F$ one has $\mathfrak{U}''F = CF$. But \mathfrak{U} is a JW-factor and hence it is not possible that

$$\mathfrak{U}'' = CE \oplus CF.$$

Thus $\dim \mathcal{X} \neq 1$ and one must have $\dim \mathcal{M} = 1$. In this latter case

$$E\mathcal{L}(\mathcal{H})E = \mathfrak{U}''E = [\mathfrak{U}]E$$

and by symmetry $F\mathcal{L}(\mathcal{H})F = \mathfrak{U}''F = [\mathfrak{U}]F$. Since $\mathfrak{U}''E \simeq \mathfrak{U}''F$, $\dim E = \dim F$ and a simple modification of the proof of Lemma 6.3 in Størmer (1979) establishes that

$$P(EAF + FAE) = 0$$

for all $A \in \mathcal{L}(\mathcal{H})$. But then $EAF \in \mathfrak{g}$ and the same argument used to prove that P was faithful allows one to conclude this case is also impossible. Therefore \mathfrak{U}''

cannot have exactly two central projections different from $\mathbf{1}$ and the proof is complete.

COROLLARY 5. *Let \mathcal{H} be an infinite-dimensional complex Hilbert space and P a positive normal identity preserving projection of $\mathcal{L}(\mathcal{H})$ into itself such that $\mathfrak{A} = P(\mathcal{L}(\mathcal{H})_h)$ is a JC-algebra and $P \neq \iota$. Let \mathfrak{g} denote the null space of P .*

The following conditions are equivalent

1. \mathfrak{g} is a Lie algebra
2. \mathfrak{g} is a Lie algebra satisfying

$$\mathfrak{g} \circ \mathfrak{g} \subseteq P(\mathcal{L}(\mathcal{H})).$$

3. *There exists a *-anti-automorphism α of order 2 of $\mathcal{L}(\mathcal{H})$ such that*

$$P = (\iota + \alpha)/2.$$

PROOF. Clearly $3 \Rightarrow 2 \Rightarrow 1$. But condition 1 and Theorem 4 imply that $\mathfrak{g} = B_\infty$ or C_∞ and \mathfrak{A} is irreducible. Thus P is the unique normal projection of $\mathcal{L}(\mathcal{H})$ onto $[\mathfrak{A}]$, by Lemma 6.1 of Størmer (1979), and hence $P = (\iota + \alpha)/2$ and α the *-anti-automorphism of order 2 of $\mathcal{L}(\mathcal{H})$ such that $\mathfrak{A} = \{A; A \in \mathcal{L}(\mathcal{H})_h, \alpha(A) = A\}$.

REMARK. Adopt the assumptions of the above corollary. For \mathfrak{g} to be a Lie algebra it is necessary that \mathfrak{A} is irreducible and $\iota - P$ is a contraction, for example $\|\iota - P\| = \|(\iota - \alpha)/2\| \leq 1$ and it appears that these conditions are sufficient. For example, if P is a projection from $\mathcal{LC}(\mathcal{H})$ into $\mathcal{L}(\mathcal{H})$ and if one assumes that $P(\mathcal{LC}(\mathcal{H})) \subseteq \mathcal{LC}(\mathcal{H})$ then Proposition 7.7 of Arazy and Friedman (1978), and the assumption $P(\mathbf{1}) = \mathbf{1}$, imply that either

$$(P(A) = EAE + (\mathbf{1} - E)A(\mathbf{1} - E)$$

for some projection $E \in \mathcal{L}(\mathcal{H})$ or $P = (\iota + \alpha)/2$ for some *-anti-automorphism of order 2. But in the first case \mathfrak{A} is reducible unless $E = \mathbf{1}$ or 0 and $P = \iota$. Thus only the possibility $P = (\iota + \alpha)/2$ remains.

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