15

QCD_2, coset models and BRST quantization

15.1 Introduction

In Chapter 9 we realized that the structure of the bosonized non-abelian massless QCD_2 is that of a gauged WZW model with an additional F^2 term of the gauge fields. Apart from the pure gauge term, this is therefore a special form of a two-dimensional coset model, discussed in Section 4.6. This naturally calls for a treatment of the system similar to that for a coset model. Using the form of the gauge fields in terms of scalars f and $\bar{f}$ as $A = if^{-1}\partial f$, $\bar{A} = i\bar{f}\bar{\partial}\bar{f}^{-1}$ with $f(z,\bar{z}), \bar{f}(z,\bar{z}) \in H^c$, the complexification of $H \equiv SU(N_{\rm C})$, leads to a convenient formulation of the model.[1] The main advantage of this approach is that one can then easily decouple the "matter" and the gauge degrees of freedom.

In this chapter we point out that the F^2 term requires a special treatment. The formulation of pure YM theory in terms of the f variables seems naively to contain unexpected "physical" massive color singlet states. This result is obviously neither in accordance with our ideas of the degrees of freedom of the model nor with the lattice and continuum solution of the theory. We show that similar "naive" manipulations in the case of QED_2 do reproduce the Schwinger model results. Using a coupling constant renormalization we show that in the limit of no matter degrees of freedom the coupling constant is renormalized to zero. In this case the unexpected states turn into unphysical massless "BRST" exact states. In the flavored QCD_2 case a similar analysis shows the existence of physical flavorless states of mass $m^2 = \frac{N_{\rm F}}{2\pi}e_c^2$.

This chapter is based on [96].

15.2 The action

The bosonized version of QCD_2 was shown in Chapter 9 to be described by the action,

$$S_{QCD_2} = S_1(u) - \frac{1}{2\pi}\int \mathrm{d}^2z\mathrm{Tr}(iu^{-1}\partial u\bar{A} + iu\bar{\partial}u^{-1}A + \bar{A}u^{-1}Au - A\bar{A}) + \frac{m^2}{2\pi}\int \mathrm{d}^2z : \mathrm{Tr}_G[u+u^{-1}] : + \frac{1}{e_c^2}\int \mathrm{d}^2z\mathrm{Tr}_H\left[F^2\right], \qquad (15.1)$$

[1] This parameterization of the gauge field was previously introduced in [9].

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where $u \in U(N_\mathrm{F} \times N_\mathrm{C})$, $S_k(u)$ is a level k WZW model,

$$S_k(u) = \frac{k}{8\pi}\int \mathrm{d}^2x \mathrm{Tr}(\partial_\mu u \partial^\mu u^{-1}) + \frac{k}{12\pi}\int_B \mathrm{d}^3y \varepsilon^{ijk}\mathrm{Tr}(u^{-1}\partial_i u)(u^{-1}\partial_j u)(u^{-1}\partial_k u), \tag{15.2}$$

A and $\bar{A}$ take their values in the algebra of $H \equiv SU(N_\mathrm{C})$, $F = \bar{\partial}A - \partial\bar{A} + i[A, \bar{A}]$, m^2 equals $m_q\mu C$, where μ is the normal ordering mass and $C = \frac{1}{2}e^\gamma$ with γ, Euler's constant. Apart from the last two terms which correspond to the quark mass term and the YM term, the rest of the action is a level-one $\frac{G}{H}$ coset model with $G = U(N_\mathrm{F} \times N_\mathrm{C})$.

We now introduce the following parameterization for the gauge fields $A = if^{-1}\partial f, \bar{A} = i\bar{f}\bar{\partial}\bar{f}^{-1}$ with $f(z,\bar{z}), \bar{f}(z,\bar{z}) \in SU(N_\mathrm{C})^c$. These type of variables were used frequently in dealing with gauged WZW actions, for instance in computing the effective action of QCD_2 and in the $\frac{G}{G}$ models discussed in Section 4.7. They may be interpreted as Wilson lines along the z and $\bar{z}$ directions. The gauged WZW part of the action, first line of (15.1), takes the form,

$$S_1(u, A) = S_1(fu\bar{f}) - S_1(f\bar{f}).$$

The Jacobian of the change of variables from A to f introduces a dimension $(1,0)$ system of anticommuting ghosts (ρ, χ) in the adjoint representation of H.[2] The WZW part of the action thus becomes

$$S_1(u, A) = S_1(fu\bar{f}) - S_1(f\bar{f}) + \frac{i}{2\pi}\int \mathrm{d}^2z \mathrm{Tr}_H[\rho\bar{D}\chi + \bar{\rho}D\bar{\chi}], \tag{15.3}$$

where $D\chi = \partial\chi - i[A, \chi]$. Our integration variables in the functional integral are $if^{-1}\mathrm{d}f$ and $i\bar{f}\mathrm{d}\bar{f}^{-1}$. This action involves an interaction term of the form $\mathrm{Tr}_H(\bar{\rho}[f^{-1}\partial f, \bar{\chi}])$ and a similar term for ρ, χ. By performing a chiral rotation, like those of Chapter 14, $\bar{\rho} \to f^{-1}\bar{\rho}f$ and $\bar{\chi} \to f^{-1}\bar{\chi}f$ with $\rho \to \bar{f}\rho\bar{f}^{-1}$ and $\chi \to \bar{f}\chi\bar{f}^{-1}$, one achieves a decoupling of the whole ghost system. The price of this is an additional $S^{(H)}_{-2N_\mathrm{C}}(f\bar{f})$ term in the action (here trace over H only) resulting from the corresponding anomaly. This result can be derived by using a non-abelian bosonization of the ghost system. A different bosonization of a (1,0) ghost system was described in Section 6.5.

In this language the ghost action takes the form,

$$S_{gh} = S_{N_\mathrm{c}}(l_1, A, \bar{A}) + S_{N_\mathrm{c}}(l_2, A, \bar{A}) + S_{\mathrm{twist}}(l_1) + S_{\mathrm{twist}}(l_2),$$

where l_1 and l_2 are in the adjoint representation and S_{twist} is a twist term given,

$$S_{\mathrm{twist}} = -\frac{N_\mathrm{c}}{2\pi}\int \mathrm{d}^2z \mathrm{Tr}[l\bar{\partial}l^{-1}f^{-1}\partial f]. \tag{15.4}$$

[2] The ghost action was introduced in [8].

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Now using the Polyakov–Wiegmann formula we get,

$$\begin{aligned} S^b_{gh} &= S_{N_c}(fl_1\bar{f}) - S_{N_c}(f\bar{f}) + S_{\rm twist}(l_1) + S_{N_c}(fl_2\bar{f}) - S_{N_c}(f\bar{f}) + S_{\rm twist}(l_2) \\ &= -S_{2N_c}(f\bar{f}) + \frac{i}{2\pi}\int {\rm d}^2z {\rm Tr}_H[\bar{\rho}'\partial\bar{\chi}' + \rho'\bar{\partial}\chi'], \end{aligned} \tag{15.5}$$

where the last line has been transferred back to the ghost language. Notice that unlike the ghost fields in (15.3) the new ghost fields ρ' and χ' are gauge invariant. It is interesting to note that the action given in (15.5) is non-local in terms of the local degrees of freedom A and $\bar{A}$. Note that had we done the right and left rotations separately, we would have got $S_{-2N_c}(f) + S_{-2N_c}(\bar{f})$, which however is not vector gauge invariant, but rather a left–right symmetric scheme.

The full gauge invariant action including the anomaly contribution of the anti-commuting part now reads,

$$\begin{aligned} S_{QCD_2} &= S_1(u) + \frac{1}{e_c^2}\int {\rm d}^2z {\rm Tr}_H\left[F^2\right] + \frac{m^2}{2\pi}\int {\rm d}^2z : {\rm Tr}_G\left[f^{-1}u\bar{f}^{-1} + \bar{f}u^{-1}f\right] : \\ &+ \left[S^{(H)}_{-(N_{\rm F}+2N_{\rm C})}(f\bar{f}) + \frac{i}{2\pi}\int {\rm d}^2z {\rm Tr}_H\left[\bar{\rho}'\partial\bar{\chi}' + \rho'\bar{\partial}\chi'\right]\right]. \end{aligned} \tag{15.6}$$

In deriving eqn. (15.6) we used a redefinition $fu\bar{f} \to u$. This does not require an extra determinant factor. Also, as $S^{(H)}(f\bar{f})$ involves ${\rm Tr}_H$ rather than the Tr_G in $S_1(f\bar{f})$ of eqn. (2.4), a factor of N_f appears. Note that had we introduced the special parameterization of the gauge fields in the fermionic formulation of QCD_2, we would have arrived at the same action after decoupling the fermionic currents from the gauge fields, by performing chiral rotation and then bosonizing the free fermions. Equation (15.6) was derived without paying attention to possible renormalizations. The latter will be treated in Section 15.7.

At this point one may choose a gauge. A convenient gauge choice is $\bar{A} = i\bar{f}\bar{\partial}\bar{f}^{-1} = 0$. Notice that since the underlying space-time is a plane this is a legitimate gauge. The gauge fixed action can be written down using the BRST procedure, namely,

$$\begin{aligned} S_{GF} &= S_{QCD_2} + S^{(gf)} + S^{(gh)} = S_{QCD_2} + \delta_{BRST}(b\bar{A}) = \\ &= S_{QCD_2} + {\rm Tr}_H[B\bar{A}] + {\rm Tr}_H[b\bar{D}c], \end{aligned} \tag{15.7}$$

where S_{GF}, $S^{(gf)}$ and $S^{(gh)}$ are, respectively, the gauge fixed action, the gauge fixing term and the ghost action. The (b, c) fields are yet another $(1, 0)$ ghost system and B is a dimension-one auxiliary field, all in the adjoint representation of $SU(N_C)$. The integration over B introduces a delta function of the gauge choice to the measure of the functional integral. In addition we integrate over the ghosts b and c.

It is interesting to note that the QCD_2 action can be related to a "perturbed" topological $\frac{H}{H}$ coset model. To realize this face of QCD_2 we parameterize u as $ghle^{i\sqrt{\frac{4\pi}{N_{\rm C}N_{\rm F}}}\phi}$ and rewrite (15.7) accordingly. The Polyakov–Wiegmann relation

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implies,

$$S[u] = S[ghl] + \frac{1}{2}\int d^2x\partial_\mu\phi\partial^\mu\phi,$$
$$S[ghl] = S[g] + S[l] + S[h] + \frac{1}{2\pi}\int \mathrm{d}^2x\mathrm{Tr}(g^\dagger\partial_+ gl\partial_- l^\dagger + h^\dagger\partial_+ hl\partial_- l^\dagger). \quad (15.8)$$

Since l is a dimension-zero field with an associated zero central charge we have $S[l] = 0$ and thus,

$$\begin{aligned} S_{GF} &= S_{N_\mathrm{F}}(h) + S^{(H)}_{-(N_\mathrm{F}+2N_\mathrm{C})}(f) + \frac{i}{2\pi}\int \mathrm{d}^2z\mathrm{Tr}_H[\bar\rho\partial\bar\chi + \rho\bar\partial\chi] \\ &+ S_{N_\mathrm{C}}(g) + \frac{1}{2\pi}\int \mathrm{d}^2z[\partial\phi\bar\partial\phi] \\ &+ \frac{m^2}{2\pi}\int \mathrm{d}^2z\mathrm{Tr}_G : \left[f^{-1}ghle^{i\sqrt{\frac{4\pi}{N_\mathrm{C}N_\mathrm{F}}}\phi} + \mathrm{e}^{-i\sqrt{\frac{4\pi}{N_\mathrm{C}N_\mathrm{F}}}\phi}l^{-1}h^{-1}g^{-1}f\right] : \\ &+ \frac{1}{e_c^2}\int \mathrm{d}^2z\mathrm{Tr}_H[(\bar\partial(f^{-1}\partial f))^2]. \end{aligned} \quad (15.9)$$

It is now easy to recognize the first line in the action as the action of $\frac{SU(N_\mathrm{C})}{SU(N_\mathrm{C})}$ topological theory.

It is interesting to note that a WZW term $S_{-2N_C}(f)$ appears in the action even without the introduction of quarks. We therefore digress to an analysis of the pure YM theory in the formulation introduced above.

15.3 Two-dimensional Yang–Mills theory

Pure Yang–Mills theory has attracted much attention recently along the lines of an underlying string theory. Here we restrict our discussion to the 2D Minkowski or Euclidean space-time, where the rich structure of the model on a compact Riemann surface does not show up. In terms of the parameterization introduced in eqn. (15.6) the gauge invariant action of the pure YM theory is,

$$\begin{aligned} S_{YM_2} &= S_{-(2N_\mathrm{C})}(f\bar f) + \tfrac{i}{2\pi}\textstyle\int \mathrm{d}^2z\mathrm{Tr}_H[\bar\rho\partial\bar\chi + \rho\bar\partial\chi] \\ &\quad + \tfrac{1}{e_c^2}\textstyle\int \mathrm{d}^2z\mathrm{Tr}_H[F^2]. \end{aligned} \quad (15.10)$$

Here again we remind the reader that the coupling constant undergoes a multiplicative renormalization. This will be discussed in Section 15.7. Let us first discuss the corresponding equations of motion for f and $\bar f$,

$$\begin{aligned} \delta f &: \bar\partial A - D\bar A + \frac{2}{m_A^2}D\bar D F = \left(1 + \frac{2}{m_A^2}D\bar D\right)F = 0, \\ \delta\bar f &: \partial\bar A - \bar D A - \frac{2}{m_A^2}\bar D D F = -\left(1 + \frac{2}{m_A^2}\bar D D\right)F = 0, \end{aligned} \quad (15.11)$$

where $D = \partial - i[A,\cdot]$, $m_A = e_c\sqrt{\frac{N_\mathrm{C}}{\pi}}$ and $\Box = 2\partial\bar\partial$. In fact these two equations are identical, as $[D,\bar D]F = 0$. The equation is that of a massive gauge field with self interaction. Note that in this approach, unlike the equations that follow from

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varying the action with respect to the gauge fields, one gets two derivatives of F. In deriving the above, it is convenient to remember that,

$$\delta S_{WZW}(f) = \frac{1}{2\pi}\mathrm{Tr}\left\{(f^{-1}\delta f)\bar{\partial}(f^{-1}\partial f)\right\}.$$

The YM action equation (15.10) is obviously invariant under the original gauge transformations,

$$f \to fv(z,\bar{z}) \quad \bar{f} \to v^{-1}(z,\bar{z})\bar{f},$$

with $v \in SU(N_{\mathrm{C}})$. In addition the action is invariant separately under the holomorphic and anti-holomorphic "color" transformations,

$$f \to u(\bar{z})f \quad \bar{f} \to \bar{f}w(z),$$

where $u, w \in SU(N_{\mathrm{C}})$. These are "spurious" transformations since they leave A and $\bar{A}$ invariant. The corresponding holomorphic and anti-holomorphic color currents are,

$$\begin{aligned} \bar{J}^s &= -\tfrac{N_{\mathrm{c}}}{\pi}[i(f\bar{f})\bar{\partial}(f\bar{f})^{-1} - \tfrac{2}{m_A^2} f\bar{D}Ff^{-1}], \\ J^s &= -\tfrac{N_{\mathrm{c}}}{\pi}[i(f\bar{f})^{-1}\partial(f\bar{f}) + \tfrac{2}{m_A^2}\bar{f}^{-1}DF\bar{f}]. \end{aligned} \qquad (15.12)$$

The gauge fixed ($\bar{f} = 1$) action takes the form,

$$\begin{aligned} S_{YM_2} = S_{-(2N_{\mathrm{C}})}(f) &+ \frac{1}{e_c^2}\int \mathrm{d}^2 z \mathrm{Tr}_H[(\bar{\partial}(f^{-1}\partial f))^2] \\ &+ \frac{i}{2\pi}\int \mathrm{d}^2 z \mathrm{Tr}_H[\bar{\rho}\partial\bar{\chi} + \rho\bar{\partial}\chi]. \end{aligned} \qquad (15.13)$$

As is expected the equation of motion at present is just that of eqn. (15.11) after setting $\bar{A} = 0$. Naturally, the action now lacks gauge invariance, nevertheless, it is invariant under the following residual holomorphic transformations,

$$f \to u(\bar{z})f \quad f \to fw(z),$$

with the corresponding holomorphic and anti-holomorphic currents,

$$J_G = -\frac{N_{\mathrm{C}}}{\pi}\left[A + \frac{2}{m_A^2}D(\bar{\partial}A)\right] \quad \bar{J}_G = -\frac{N_{\mathrm{C}}}{\pi}\left[\tilde{A} + \frac{2}{m_A^2}\bar{\tilde{D}}(\partial\tilde{A})\right],$$

and $\tilde{A} = if\bar{\partial}f^{-1}$. Notice that in spite of the similar structure, $\tilde{A}$ is not related to $\bar{A}$ which was set to zero. To better understand the physical picture behind these currents we defer temporarily to the abelian case.

15.4 Schwinger model revisited

Since in the pure Maxwell theory there is no analog to the $(-2N_{\mathrm{C}})$ level WZW term of eqn. (15.10), we study instead the Schwinger model in its bosonized form,

$$S_{(\mathrm{Sch})} = \frac{1}{2\pi}\int \mathrm{d}^2 z\left[\partial X\bar{\partial}X - \sqrt{2}\partial X\bar{A} + \sqrt{2}\bar{\partial}XA + \frac{\pi}{e^2}(\partial\bar{A} - \bar{\partial}A)^2\right].$$

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In analogy to the change of variables in the non-abelian case, we now introduce the following parameterization of the gauge fields $A = \partial\varphi$, $\bar{A} = \bar{\partial}\bar{\varphi}$. In terms of these fields the action takes the form,

$$S_{(\mathrm{Sch})} = \int \frac{\mathrm{d}^2 z}{2\pi} \left\{ [\partial X \bar{\partial} X - \sqrt{2} X \partial\bar{\partial}(\varphi - \bar{\varphi}) + \frac{\pi}{e^2}[\partial\bar{\partial}(\varphi - \bar{\varphi})]^2 + i[\bar{\rho}\partial\bar{\chi} + \rho\bar{\partial}\chi] \right\}.$$

In the gauge $\bar{A} = 0$ and after the field redefinition $\tilde{X} = X + \frac{1}{\sqrt{2}}\varphi$, the action is decomposed into decoupled sectors,

$$\begin{aligned} S_{(\mathrm{Sch})} &= S(\tilde{X}) + S(\varphi) + S_{(\mathrm{ghost})}, \\ S(\tilde{X}) &= \frac{1}{2\pi} \int \mathrm{d}^2 z [\partial\tilde{X}\bar{\partial}\tilde{X}] \quad S(\varphi) = \frac{1}{4\pi} \int \mathrm{d}^2 z \left\{ \frac{2}{\mu^2}[\partial\bar{\partial}\varphi]^2 - \partial\varphi\bar{\partial}\varphi \right\}, \end{aligned} \tag{15.14}$$

where $\mu^2 = \frac{e^2}{\pi}$. The corresponding equations of motion are,

$$\partial\bar{\partial}\left[1 + \frac{2}{\mu^2}\partial\bar{\partial}\right]\varphi = 0, \quad \partial\bar{\partial}\tilde{X} = 0.$$

The invariance under the chiral shifts $\delta\varphi = \epsilon(\bar{z})$ and $\delta\varphi = \epsilon(z)$ are generated by the holomorphically conserved currents,

$$J_G = \partial\varphi + \frac{2}{\mu^2}\partial\bar{\partial}\partial\varphi, \quad \bar{J}_G = \bar{\partial}\varphi + \frac{2}{\mu^2}\partial\bar{\partial}\bar{\partial}\varphi.$$

To handle this type of "hybrid" current we suggest the following decomposition of the massless and massive modes $\varphi = \varphi_1 + \varphi_2$ with,

$$\partial\bar{\partial}\varphi_1 = 0 \quad [2\partial\bar{\partial} + \mu^2]\varphi_2 = 0.$$

In the holomorphic quantization,

$$\Pi = \frac{\delta\mathcal{L}}{\delta(\partial\varphi)} = -\frac{1}{\pi\mu^2}\bar{\partial}\left(\partial\bar{\partial} + \frac{\mu^2}{4}\right)\varphi = \frac{1}{4\pi}\bar{\partial}(\varphi_2 - \varphi_1). \tag{15.15}$$

A unique solution to the commutation relations $[\varphi(z,\bar{z}), \Pi(w,\bar{w})]_{z=w} = i\delta(\bar{z} - \bar{w})$, $[\varphi,\varphi] = 0$ and $[\Pi,\Pi] = 0$ is,

$$\begin{aligned} [\varphi_1(z,\bar{z}), \varphi_1(w,\bar{w})]_{z=w} &= \pi i\epsilon(\bar{z} - \bar{w}) \\ [\varphi_2(z,\bar{z}), \varphi_2(w,\bar{w})]_{z=w} &= -\pi i\epsilon(\bar{z} - \bar{w}) \\ [\varphi_1(z,\bar{z}), \varphi_2(w,\bar{w})]_{z=w} &= 0 \\ \left[\tilde{X}(z,\bar{z}), \tilde{X}(w,\bar{w})\right]_{z=w} &= -\pi i\epsilon(\bar{z} - \bar{w}), \end{aligned} \tag{15.16}$$

where ϵ is the standard antisymmetric step function. Notice that the massless degree of freedom has commutation relations which correspond to a negative metric on the phase space. These relations can also be translated to the following OPEs (choosing the part $\varphi_1(z)$ of φ_1),

$$\begin{aligned} \varphi_1(z)\varphi_1(w) &= \log(z - w), \\ \varphi_2(z,\bar{z})\varphi_2(w,\bar{w}) &= -\log|(z - w)|^2 + O(\mu^2|z - w|^2), \\ \varphi_1(z)\varphi_2(w,\bar{w}) &= O(z - w). \end{aligned} \tag{15.17}$$

It is thus clear that the model is invariant under a $U(1)$ affine Lie algebra of level $k=-1$ since $J_G(z)J_G(w)=\frac{1}{(z-w)^2}$, as $J_G=\partial\varphi_1$ with no contribution from φ_2.

The physical states of the model have to be in the cohomology of the BRST charge. Due to the fact that the current is holomorphically (and the other anti-holomorphically) conserved, it follows that the same property holds for the BRST charge, and thus the space of physical states is an outer product of the cohomology of Q and $\bar{Q}$. The latter are given by,

$$\begin{aligned} Q &= \chi J = \chi(i\partial\tilde{X}+\partial\varphi_1),\\ \bar{Q} &= \bar{\chi}\bar{J} = \bar{\chi}(-i\bar{\partial}\tilde{X}+\bar{\partial}\varphi_1). \end{aligned} \tag{15.18}$$

Expanding the fields $i\partial\tilde{X}$ and $\partial\varphi_1$ in terms of the Laurent modes $\tilde{X}_n$ and $(\tilde{\varphi}_1)_n$ with $[X_n,X_m]=n\delta_{n+m}$ and $[(\varphi_1)_n,(\varphi_1)_m]=n\delta_{n+m}$ we have

$$Q=\sum_n \tilde{\chi}_n\left[\tilde{X}_{-n}-i(\varphi_1)_{-n}\right].$$

Since $J_0=\{Q,\rho_0\}$, physical states have to have a zero eigenvalue of J_0. The general structure of the states in the $\varphi_1,\tilde{X},\rho,\chi$ Fock space is,

$$(\tilde{X}_n)^{n_X}(\varphi_{1\,m})^{n_f}(\chi_k)^{n_\chi}(\rho_l)^{n_\rho}|\text{vac}\,,$$

where obviously n_χ and n_ρ are either 0 or 1. It is straightforward to realize that only the vacuum state and states of the form $(\tilde{X}_0)^{n_X}(\varphi_{1\,0})^{n_f}$ are in the BRST cohomology. Recall that being on the plane we exclude zero modes and thus only the vacuum state remains. Since there is no constraint on the modes of φ_2, the physical states are built solely of φ_2 which are massive modes. This result is identical to the well-known solution of the Schwinger model.

15.5 Back to the YM theory

Equipped with the lesson from the Schwinger model we return now to the YM case and introduce a decomposition of the group element f so that again the gauge currents obey an affine Lie algebra. Let us write $f=f_2f_1$ which implies that,

$$A=if^{-1}\partial f=if_1^{-1}\partial f_1+if_1^{-1}(f_2^{-1}\partial f_2)f_1\equiv J_1+J_2$$

With no loss of generality we take $\bar{\partial}f_1=0$ implying also $\bar{\partial}J_1=0$. Inserting these expressions into J_G of eqn (15.3) one finds,

$$J_G=-\frac{N_C}{\pi}\left[J_1+J_2+\frac{2}{m_A^2}(\partial\bar{\partial}J_2+i[\bar{\partial}J_2,J_1+J_2])\right].$$

If one can consistently require that,

$$J_2+\frac{2}{m_A^2}(\partial\bar{\partial}J_2+i[\bar{\partial}J_2,J_1+J_2])=0,$$

then, in a complete analogy with the abelian case, $J_G = -\frac{N_c}{\pi} J_1$. The latter is an affine current of level $k = -2N_{\rm C}$. One can in fact show that (15.5) can be assumed without a loss of generality. $\bar\partial J_G = 0$ implies that $J_2 + \frac{2}{m_A^2}(\partial\bar\partial J_2 + i[\bar\partial J_2, J_1 + J_2]) = u(z)$, where $u(z)$ is some holomorphic function. We then introduce the shifted currents $\tilde J_2 = J_2 - u(z)$, $\tilde J_1 = J_1 + u(z)$. Now $\bar\partial \tilde J_1 = 0$ as does J_1, and $\tilde J_2$ obeys eqn (15.5) with $\tilde J_1$ replacing J_1. It is easy to check that the shifts in the currents correspond to $f_1 \to v(z) f_1$, $f_2 \to f_1 v(z)^{-1}$ with $u(z) = i f_1^{-1}(v(z)^{-1}\partial v(z)) f_1$.

Note that the equation for J_2 involves a coupling to J_1. This is related to the fact that, unlike the abelian case, one cannot write the action as a sum of decoupled terms which are functions of J_1 and J_2 separately.

Once the color current J_G is expressed in terms of the holomorphic current J_1, the analysis of the space of physical states is directly related to that of the topological $\frac{G}{G}$ model at $k = 0$. The physical states have to be in the cohomologyof the BRST charge, which corresponds to the following holomorphically conserved BRST current,

$$Q(z) = \chi^a\left(J_G^a + \frac{1}{2}J_{gh}^a\right) = -\frac{N_{\rm C}}{\pi}\chi^a\left(\left[A + \frac{2}{m_A^2}D(\bar\partial A)\right]^a + \frac{i}{2}f_{bc}^a\rho^b\chi^c\right).$$

An anti-holomorphic BRST current $\bar Q(z)$ determines the condition for physical states in the analogous manner to Q. From here on we restrict our description to the latter. We define now the zero level affine Lie algebra current,

$$J^a_{(\rm tot)} = J^a_G + J^a_{(gh)} = J^a_G + i, f^a_{bc}\chi_b\rho_c,$$

and the $c = 0$ Virasoro generator T,

$$T(z) = -\frac{1}{N_C} : J_G^a J_G^a : + \rho^a\partial\chi^a,$$

as well as dimension (2,0) fermionic current,

$$G = -\frac{1}{2N_{\rm C}}\rho_a J_G^a,$$

and realize the existence of the "topological coset algebra",

$$\begin{aligned} &T(z) = \{Q, G(z)\}, \quad Q(z) = \left\{Q, j^{\#}(z)\right\}, \quad J^a_{(\rm tot)} = \{Q, \rho^a(z)\}, \\ &\{Q, Q(z)\} = 0, \qquad\qquad \{G, G(z)\} \equiv W(z), \\ &W(z) = \{Q, U(z)\}, \qquad\qquad [W, W(z)] = 0, \end{aligned} \tag{15.19}$$

where $J^{\#} = \chi_a\rho^a$ is "ghost number current",

$$W(z) = \frac{1}{4N_{\rm c}} f_{abc}J_G^a\rho^b\rho^c + \partial\rho^a\rho_a,$$

and,

$$U = \frac{1}{12N_{\rm c}} f_{abc}J_G^a\rho^b\rho^c.$$

A direct consequence is that any physical state has to obey,

$$J_{(\mathrm{tot})}{}^{0}_{0}|\mathrm{phys}> = 0, \quad L_0|\mathrm{phys}> = 0, \quad W_0|\mathrm{phys}> = 0, \tag{15.20}$$

where $J_{(\mathrm{tot})}{}^{i}_{n}$, $\tilde{L}_n$ and W_n are the Laurent modes of $J_{(\mathrm{tot})}{}^{i}$ the Cartan sub-algebra currents, T and W, respectively. In fact the BRST cohomology of the present model is a special case of the set of G/G models.

We therefore refer the reader to those works [9], [200], [229] and present here only the result. On the plane where no ghost zero modes are allowed, the only state in the cohomology is the zero ghost number vacuum state of J_1.

This state can be a tensor-product with oscillators of the massive modes of J_2. Unlike the abelian case, J_G does not commute with J_2 so that in general the J_2 modes are not obviously in the BRST cohomology. However, there is no reason to believe that all the J_2 modes will be excluded by the BRST condition. Those J_2 modes that remain are by definition color singlets.

This result contradicts previous results on YM_2. Usually one believes that pure gluodynamics on the plane is an empty theory since all local degrees of freedom can be gauged away.

15.6 An alternative formulation

To get a better understanding of the subtleties of the Yang–Mills theory when expressed in terms of $A = if^{-1}\partial f$, $\bar{A} = i\bar{f}\bar{\partial}\bar{f}^{-1}$, and for future application, we compare now with another formulation of the theory. A similar approach will be used in the discussion of generalized YM theories in Chapter 16. Consider the following functional integral,

$$\begin{aligned} Z &= \int DAD\bar{A}DBe^{iS(A,\bar{A},B)}, \\ S &= -\int \mathrm{d}^2 z Tr_H[\tfrac{1}{e_c}FB + \tfrac{1}{4}B^2], \end{aligned} \tag{15.21}$$

where B is a pseudoscalar field in the adjoint representation. Obviously the integration over B produces the usual $\mathrm{Tr}[F^2]$ action. It is also easy to realize that the action is invariant under the ordinary gauge symmetry provided that $\delta B = i[\epsilon, B]$. In terms of the f variables after imposing the gauge $\bar{f} = 1$ one finds,

$$S_{YM_2} = S_{-(2N_C)}(f) + \int \mathrm{d}^2 z \mathrm{Tr}_H \left[\left(\frac{i}{e_c}(f^{-1}\partial f)\bar{\partial}B \right) - \frac{1}{4}B^2 \right] + S^{(gh)},$$

where $S^{(gh)} = \frac{i}{2\pi}\int \mathrm{d}^2 z \mathrm{Tr}_H[\bar{\rho}\partial\bar{\chi} + \rho\bar{\partial}\chi]$. One should again bear in mind that the coupling constant undergoes a multiplicative renormalization. This will be discussed in the next section. Using Polyakov–Wiegmann we get,

$$\begin{aligned} S_{YM_2}(B) = \Gamma_{2N_C}(B) &- \tfrac{1}{4}\mathrm{Tr}_H[B^2] \\ &- S_{(2N_C)}(vf) + S^{(gh)}, \end{aligned} \tag{15.22}$$

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where $\Gamma_k(B) = S_k(v)$ with $\frac{1}{e_c}\bar{\partial}B = \frac{2N_{\rm C}}{2\pi}(iv\bar{\partial}v^{-1})$. The second line in (15.22) is a $c = 0$ "topological system". Since the underlying Minkowski space-time does not admit zero modes we can safely integrate over the corresponding fields. We can further pass from functional integrating over B to $iv\bar{\partial}v^{-1}$. This involves the insertion of $\det\frac{\bar{D}}{\bar{\partial}}$ which will introduce a $\Gamma_{-2N_{\rm C}}(B)$ term with no additional ghost terms. The functional integral (16.1) thus takes the final form,

$$Z = \int D[v]\mathrm{e}^{-i\left(\frac{1}{4}\int \mathrm{d}^2 z \mathrm{Tr}_H[B^2]\right)}.$$

It is thus clear that in the present formulation there is no trace of the massive "physical modes" discussed in the previous section.

15.7 The resolution of the puzzle

Encouraged by the result of the last section, we proceed now to reexamine the steps that led to the unexpected massive modes in the pure YM_2 theory. In particular, we would like to check whether in addition to the implementation of proper determinants there is no coupling constant renormalization that has to be invoked when passing to the quantum theory expressed in the f variables. For this purpose we turn on again the matter degrees of freedom. We introduce $N_{\rm F}$ quarks in the fundamental color representation and explore the behavior of the system in the limit $N_{\rm F} \to 0$. Recall that the action of this model is given in eqn. (15.9). Starting actually from eqn. (15.1), taking the massless limit, writing A in terms of f in the action but still with A as an integration variable, and using the formulation presented in the previous section, the path integral of the colored degrees of freedom now reads

$$\begin{aligned} Z^{(\mathrm{col})} &= \int [DA][DB][Dh]\mathrm{e}^{iS^{(\mathrm{col})}} \\ S^{(\mathrm{col})} &= S_{N_{\rm F}}(h) + \frac{N_{\rm F}}{2\pi}\int \mathrm{d}^2 z \mathrm{Tr}_H[h\bar{\partial}h^{-1}f^{-1}\partial f] \\ &\quad + \int \mathrm{d}^2 z \mathrm{Tr}_H\left[\left(\frac{i}{e_c}(f^{-1}\partial f)\bar{\partial}B\right) - \frac{1}{4}B^2\right], \end{aligned} \tag{15.23}$$

where we have also gone from u to h as in Section 15.2.

It was found out that quantum consistency imposes finite renormalization on the coupling constant of the current-gauge field interaction.[3] This renormalization is expressed in the following equality,

$$\begin{aligned} Z(\bar{J}) &\equiv \int DA\mathrm{e}^{i\left[S_k(f) + \frac{1}{2\pi}\int \mathrm{d}^2 z \mathrm{Tr}_H[i(f^{-1}\partial f)\bar{J}]\right]} \\ &= \int Df\mathrm{e}^{i[S_{k-2N_{\rm C}}(f) + \frac{e(-k)}{2\pi}\int \mathrm{d}^2 z \mathrm{Tr}_H[(f^{-1}\partial f)\bar{J}]}\int D(gh)\mathrm{e}^{iS^{(gh)}} \\ &= \mathrm{e}^{i\Gamma_{-k+2N_{\rm C}}\left[\left(\frac{e(-k)}{-k+2N_{\rm C}}\right)\bar{J}\right]}\int D(gh)\mathrm{e}^{iS^{(gh)}}, \end{aligned} \tag{15.24}$$

[3] The finite renormalization of the coupling was introduced by D. Kutasov in [146].

where k is an arbitrary level and $\Gamma_k(L) = S_k(w)$ for $L = iw\bar{\partial}w^{-1}$. The renormalization factor $e(k)$ has to satisfy $\frac{e(-k-2N_\mathrm{C})}{e(k)} = \frac{k}{k+2N_\mathrm{C}}$. In addition it is clear from eqn. (15.24) that it has to be singular at the origin. It can be shown that $e(k)$ takes the form $e(k) = \sqrt{\frac{k+2N_\mathrm{C}}{k}}$. Implementing this renormalization in our case, eqn. (15.23) takes the form,

$$S^{(\mathrm{col})} = S_{N_\mathrm{F}}(\tilde{h}) + S_{-(N_\mathrm{F}+2N_\mathrm{C})}(f) + \int \mathrm{d}^2 z \mathrm{Tr}_H \left[\left(i\sqrt{\frac{N_\mathrm{F}+2N_\mathrm{C}}{N_\mathrm{F}}}\frac{1}{e_c}(f^{-1}\partial f)\bar{\partial}B\right) - \frac{1}{4}B^2\right] + S^{(gh)}, \tag{15.25}$$

where $\tilde{h} = fh$. After integrating the auxiliary field B the action becomes,

$$S^{(\mathrm{col})} = S_{N_\mathrm{F}}(\tilde{h}) + S_{-(N_\mathrm{F}+2N_\mathrm{C})}(f) + \int \mathrm{d}^2 z \mathrm{Tr}_H \left[\left(\frac{N_\mathrm{F}+2N_\mathrm{C}}{e_c^2 N_\mathrm{F}}\right)\right][\bar{\partial}(f^{-1}\partial f)]^2] + S^{(gh)}. \tag{15.26}$$

It is now straightforward to realize that the equation of motion which follows from the variation with respect to f is that of eqn. (15.11) where now $m_A = e_c\sqrt{\frac{N_\mathrm{F}}{2\pi}}$. Thus, the coupling constant renormalization turns the massive modes into massless ones in the case of pure YM theory ($N_\mathrm{F} = 0$). Notice that to reach this conclusion it is enough to use the fact that $e(k)$ has to be singular at $k = 0$ and the explicit expression of $e(k)$ is really not needed. Following the arguments presented in Chapter 5, it is clear that these states that became massless are not in the BRST cohomology and thus not in the physical spectrum.

A somewhat similar derivation of the triviality of the model in the $N_\mathrm{F} = 0$ limit is the following. We integrate in eqn. (15.25) over the ghost fields and over f, using again the coupling constant renormalization, and find,

$$Z = \int D[v] \mathrm{e}^{-\left\{iS_{N_\mathrm{F}}(v) + \frac{e_c^2 N_\mathrm{F}^2}{4}\int \mathrm{d}^2 z \mathrm{Tr}_H[B^2(v)]\right\}}.$$

It is now clear that the action vanishes at $N_\mathrm{F} = 0$ and hence again, on trivial topology, the theory is empty. Notice, however, that the implementation of renormalization modifies also the result of the previous section.

The final conclusion is that in both methods one finds that indeed the pure YM theory has an empty space of physical states as of course is implied by the original formulation in terms of A. We have demonstrated that in this formulation it follows only after taking subtleties of renormalization into account.

15.8 On bosonized QCD_2

To resolve the puzzle of the YM theory we were led to analyze the color and flavor sectors of QCD_2. The full bosonized QCD_2 includes in addition the baryon number degrees of freedom. The corresponding action is given by eqns. (15.6), (15.7) or by eqn. (15.9). In the past the low-lying baryonic spectrum in the strong coupling limit $\frac{m_q}{e_c} \to 0$ was extracted using a semi-classical quantization. In this chapter our analysis was based on switching off the mass term, $m_q = 0$. This limit cannot be treated by the semi-classical approach, as the soliton solution is not there for $m_q = 0$. In our case here one finds a decoupled WZW action for the flavor degrees of freedom $S_{N_C}(g)$ and a decoupled free field action for the baryon degree of freedom, in addition to the action of the colored degrees of freedom which is given in eqn. (15.26) or eqn. (15.7). The general structure of a physical state in this case is that of a tensor product of g and ϕ with the colored degrees of freedom f, h and the ghosts. The structure of QCD_2 which emerged from the semi-classical quantization for $m_q \neq 0$ involves g and ϕ only. In our case here the f colored degrees of freedom acquire mass $m_A = e_c\sqrt{\frac{N_F}{2\pi}}$ while the h degrees of freedom remain massless. In the limit $e_c \to \infty$ the f degrees of freedom decouple. It is thus clear that one has to introduce the mass term which couples the three sectors. The massless limit of QCD_2 can then be derived by taking the limit $m_q \to 0$ after solving for the physical states. Indeed, it was shown in the limit of $e_c \to \infty$ that turning on $m_q \neq 0$ results in a hadronic spectrum where the flavor representation and the baryon number were correlated. The analysis of the spectrum of the massive multi-flavor QCD_2 in the approach of this work remains to be worked out.

15.9 Summary and discussion

In this chapter we have analyzed 2D YM and QCD theories using a special parametrization of the gauge fields in terms of group elements. In the $m_q = 0$ case it enabled us to decouple the matter and gauge degrees of freedom. However, this formulation led, in a naive treatment, to unexpected massive modes. Even though we did not present a full solution of the theory we had reasonable arguments to believe that the BRST projection would not exclude these modes. The fact that a similar approach to QED_2 reproduced the known results of the Schwinger model, enhanced the puzzling phenomenon. Eventually, we showed that a coupling constant renormalization, renders the unexpected massive modes into massless un-physical states. The benefit of this detective work is the appearance of "physical" massive states in massless QCD_2.