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MULTIPLICATIVE INTEGRATION OF INFINITE PRODUCTS

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Introduction. Let G be a complete normed abelian group with norm N_1. Let S be an interval (bounded or otherwise) of real numbers. We propose to study the Stieltjes integral equation

$$h(t) = p + \int_t^a \left[\sum_{k=1}^{\infty} dF_k[h]\right],$$

where p is in G, a is in S, and each F_k is a function on S each value of which is a function from G to G. Our primary tools of investigation will be the works of J. S. MacNerney [**6**; **7**] and their extensions by the author [**4**; **5**]. Our main result, Theorem 4, will show that the equation above can be solved by a product integral of infinite products of solutions for the summands of the integrator. After obtaining our results, we shall specialize them to a linear situation and then show how this specialization allows us to obtain representations for analytic functions having only invertible values in a complex Banach algebra with identity.

Stieltjes integral equations. Let OA^+, OM^+, and E^+ be as in [**6**]. Let H, N_2, and N_3 be as in [**5**]. Let OA be the set to which V belongs only in case V is a function from $S \times S$ to H, and

(i) $V(x, y) + V(y, z) = V(x, z)$ whenever (x, y, z) is in $S \times S \times S$ and $|x - y| + |y - z| = |x - z|$, and

(ii) there is α in OA^+ such that $N_2[V(a, b)] \leqq \alpha(a, b)$ whenever (a, b) is in $S \times S$. If α and V are related as in (ii), then α will be said to dominate V. Let OM be the set to which W belongs only in case W is a function from $S \times S$ to H, and

(i) $W(x, y)W(y, z) = W(x, z)$ whenever (x, y, z) is in $S \times S \times S$ and $|x - y| + |y - z| = |x - z|$, where the multiplication is composition, and

(ii) there is μ in OM^+ such that $N_2[W(a, b) - I] \leqq \mu(a, b) - 1$ whenever (a, b) is in $S \times S$, where I in H is given by $I[p] = p$. If W and μ are related as in (ii), μ will be said to dominate W.

Let OAC^+ (respectively OMC^+) be that subset of OA^+ (respectively OM^+) consisting of all continuous members of OA^+ (respectively OM^+). Let OAC (respectively OMC) be that subset of OA (respectively OM) consisting of all

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those members of OA (respectively OM) dominated by members of OAC^+ (respectively OMC^+). The following theorem is due to MacNerney [**7**].

THEOREM 1. *There is a bijection E from OAC onto OMC such that if V is in OAC and W is in OMC, then* (i), (ii), (iii), (iv), *and* (v) *are equivalent.*

(i) $W = E[V]$.

(ii) $W(a, b)[p] = {}_a\prod^b [I + V][p]$ *whenever* (a, b, p) *is in* $S \times S \times G$.

(iii) $V(a, b)[p] = {}_a\sum^b [W - I][p]$ *whenever* (a, b, p) *is in* $S \times S \times G$.

(iv) *There is* (α, μ) *in* $OAC^+ \times OMC^+$, *with* $\mu = E^+[\alpha]$, *such that*

$$N_3[W(a, b) - I - V(a, b)] \leqq \mu(a, b) - 1 - \alpha(a, b)$$

whenever (a, b) *is in* $S \times S$.

(v) *If* (a, p) *is in* $S \times G$, *and h from S to G is given by* $h(t) = W(t, a)[p]$, *then h has bounded N_1-variation on each compact interval of S and is the only such function such that*

$$h(t) = p + \int_t^a V[h],$$

whenever t is in S.

Remark. The notions of $\prod$ and $\sum$ are to be taken as in [**7**]. The integral in (v) is approximated by sums of the form $\sum^n{}_{k=1} V(s_{2k-2}, s_{2k})[h(s_{2k-1})]$ where $(s_k)^{2n}{}_{k=0}$ is a monotone sequence into S with $s_0 = t$ and $s_{2n} = a$. MacNerney actually showed that E can be defined on all of OA, providing a bijection onto all of OM, and the theorem remains true if the integral in (v) is replaced by the Cauchy right integral. For present purposes, the theorem as stated above will suffice.

The following theorem is an easy consequence of earlier work of the present author [**5**, Theorems 5 and 7] (see also [**4**, Theorem 6]).

THEOREM 2. *Let each of V_1 and V_2 be in OAC, with $W_1 = E[V_1]$ and $W_2 = E[V_2]$. Then ${}_a\prod^b W_1W_2[p]$ exists whenever (a, b, p) is in $S \times S \times G$. Furthermore, if M is given by $M(a, b)[p] = {}_a\prod^b W_1W_2[p]$, then $M = E[V_1 + V_2]$.*

Our next theorem is a straightforward extension of Theorem 2.

THEOREM 3. *Suppose that n is a positive integer, $\{V_1, V_2, \ldots, V_n\}$ is a finite subset of OAC, and U in OAC is given by $U = \sum^n{}_{k=1} V_k$. Let $M = E[U]$, and, if k is an integer in $[1, n]$, let $W_k = E[V_k]$. Then*

$$M(a, b)[p] = {}_a\prod^b \left[\prod_{k=1}^n W_k\right][p]$$

whenever (a, b, p) is in $S \times S \times G$.

Before proving Theorem 3, we shall need a lemma. The lemma follows from the second conclusion of [**7**, Lemma 1.1], and we shall not include a proof.

LEMMA 1. *Let n be a positive integer, and suppose that* $\{A_1, \ldots, A_n\}$ *is a subset of H. Suppose that* $\{a_1, \ldots, a_n\}$ *is a set of numbers such that* $N_2[A_k] \leqq a_k$ *whenever k is an integer in* $[1, n]$. *Then*

$$N_3\left[\prod_{k=1}^{n} [I + A_k] - I - \sum_{k=1}^{n} A_k\right] \leqq \prod_{k=1}^{n} [1 + a_k] - 1 - \sum_{k=1}^{n} a_k.$$

Proof of Theorem 3. Choose α in $OA\,C^+$ such that if k is an integer in $[1, n]$ then α dominates V_k. Let (a, b, p) be in $S \times S \times G$. Now clearly, ${}_a\prod^b [\prod^n{}_{k=1} W_k][p]$, if it exists, is given by ${}_a\prod^b [\prod^n{}_{k=1} [I + V_k]][p]$. Let $(s_i)^m{}_{i=0}$ be a chain from a to b. Now

$$N_1\left[\prod_{i=1}^{m}\left[\prod_{k=1}^{n} [I + V_k(s_{i-1}, s_i)]\right][p] - \prod_{i=1}^{m} [I + U(s_{i-1}, s_i)][p]\right]$$

$$\leqq \sum_{j=1}^{m} N_1\left[\prod_{i=1}^{j}\left[\prod_{k=1}^{n} [I + V_k(s_{i-1}, s_i)]\right]\prod_{i=j+1}^{m} [I + U(s_{i-1}, s_i)][p]\right.$$

$$\left.- \prod_{i=1}^{j-1}\left[\prod_{k=1}^{n} [I + V_k(s_{i-1}, s_i)]\right]\prod_{i=j}^{m} [I + U(s_{i-1}, s_i)][p]\right]$$

$$\leqq \sum_{j=1}^{m} \exp[n\alpha(a, s_{j-1})]N_1\left[\prod_{k=1}^{n} [I + V_k(s_{j-1}, s_j)]\right]\left[\prod_{i=j+1}^{m} [I + U(s_{i-1}, s_i)][p]\right]$$

$$- [I + U(s_{j-1}, s_j)]\left[\prod_{i=j+1}^{m} [I + U(s_{i-1}, s_i)][p]\right]$$

$$\leqq N_1[p] \exp[n\alpha(a, b)] \sum_{j=1}^{n} ([1 + \alpha(s_{j-1}, s_j)]^n - [1 + n\alpha(s_{j-1}, s_j)]).$$

The remainder of the proof is now clear.

LEMMA 2. *Let* $(A_k)^\infty{}_{k=1}$ *be a sequence into H, and suppose that b is a number such that* $\sum^n{}_{k=1} N_2[A_k] \leqq b$ *whenever n is a positive integer. Then* $\prod^\infty{}_{k=1} [I + A_k]$ *exists in the sense that if p is in G, then* $\lim_{n\to\infty} \prod^n{}_{k=1} [I + A_k][p]$ *exists, and if B from G to G is given by* $B[p] = \lim_{n\to\infty} \prod^n{}_{k=1} [I + A_k][p]$, *then B is in H. Furthermore, if n is a positive integer, then*

$$N_3\left[B - \prod_{k=1}^{n} [I + A_k]\right] \leqq e^b \sum_{j=n}^{\infty} N_2[A_{j+1}].$$

Proof. Let p be in G, let m and n be positive integers, and suppose that $m > n$. Now

$$N_1\left[\prod_{k=1}^{m} [I + A_k][p] - \prod_{k=1}^{n} [I + A_k][p]\right]$$

$$\leqq \sum_{j=n}^{m-1} N_1\left[\prod_{k=1}^{j+1} [I + A_k][p] - \prod_{k=1}^{j} [I + A_k][p]\right]$$

$$\leqq e^b N_1[p] \sum_{j=n}^{m-1} N_2[A_{j+1}].$$

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The remainder of the proof is now clear.

THEOREM 4. *Let* $(V_k)^{\infty}{}_{k=1}$ *be a sequence into* OAC, *let* $(\alpha_k)^{\infty}{}_{k=1}$ *be a sequence into* OAC^+, *suppose that if* k *is a positive integer then* α_k *dominates* V_k, *and suppose that* $\sum^{\infty}{}_{k=1} \alpha_k(a, b)$ *is finite whenever* (a, b) *is in* $S \times S$. *Let* U *in* OAC *be given by* $U(a, b)[p] = \sum^{\infty}{}_{k=1} V_k(a, b)[p]$, *and let* $M = E[U]$. *Then*

$$M(a, b)[p] = {}_a\prod^b \left[\prod_{k=1}^{\infty} [I + V_k]\right][p],$$

whenever (a, b, p) *is in* $S \times S \times G$.

Discussion and proof. The notation in the conclusion merits comment. Let Z be a function from $S \times S$ to H given by $Z(a, b) = \prod^{\infty}{}_{k=1} [I + V_k(a, b)]$, where the infinite product is taken in the sense of Lemma 2. Then the conclusion of Theorem 4 is that $M(a, b)[p] = {}_a\prod^b Z[p]$, whenever (a, b, p) is in $S \times S \times G$. It follows from [7, Corollary 2.5] and Theorem 3 that M is given by

$$M(a, b)[p] = \lim_{n \to \infty} {}_a\prod^b \left[\prod_{k=1}^{n} [I + V_k]\right][p],$$

the convergence being uniform on compact subsets of $S \times S$ and bounded subsets of G. Let β in OAC^+ be given by $\beta(a, b) = \sum^{\infty}{}_{k=1} \alpha_k(a, b)$. Let (a, b) be in $S \times S$, let $(t_j)^m{}_{j=0}$ be a chain from a to b, and let n be a positive integer. Let p be in G. Now

$$\begin{aligned}
&N_1\left[\prod_{j=1}^{m} \left[\prod_{k=1}^{n} [I + V_k(t_{j-1}, t_j)]\right][p] - \prod_{j=1}^{m} \left[\prod_{k=1}^{\infty} [I + V_k(t_{j-1}, t_j)]\right][p]\right] \\
&\quad \leqq \exp[\beta(a, b)] \sum_{j=1}^{m} N_3\left[\prod_{k=1}^{n} [I + V_k(t_{j-1}, t_j)] - \prod_{k=1}^{\infty} [I + V_k(t_{j-1}, t_j)]\right] N_1[p] \\
&\quad \leqq \exp[2\beta(a, b)] \sum_{j=1}^{m} \sum_{k=n}^{\infty} N_2[V_k(t_{j-1}, t_j)] N_1[p] \\
&\quad \leqq \exp[2\beta(a, b)] \sum_{k=n}^{\infty} \alpha_k(a, b) N_1[p].
\end{aligned}$$

Now let $\epsilon > 0$, and find a positive integer n_0 such that if $n > n_0$ then

$$N_1\left[M(a, b)[p] - {}_a\prod^b \left[\prod_{k=1}^{n} [I + V_k]\right][p]\right] < \epsilon/3$$

and

$$N_1[p] \exp[2\beta(a, b)] \sum_{k=n}^{\infty} \alpha_k(a, b) < \epsilon/3.$$

Let n be a positive integer, $n > n_0$, and find a chain s from a to b such that if t refines s, then

$$N_1\left[{}_a\prod^b \left[\prod_{k=1}^{n} [I + V_k]\right][p] - \prod_{j=1}^{m} \left[\prod_{k=1}^{n} [I + V_k(t_{j-1}, t_j)]\right][p]\right] < \epsilon/3.$$

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Now, if t refines s,

$$
\begin{aligned}
N_1\Big[M(a,b)[p] - \prod_{j=1}^{m}\Big[\prod_{k=1}^{\infty}[I + V_k(t_{j-1}, t_j)]\Big][p]\Big]
&\leqq N_1\Big[M(a,b)[p] - {}_a\prod^b \prod_{k=1}^{n}[I + V_k][p]\Big] \\
&\quad + N_1\Big[{}_a\prod^b\Big[\prod_{k=1}^{n}[I + V_k]\Big][p] - \prod_{j=1}^{m}\Big[\prod_{k=1}^{n}[I + V_k(t_{j-1}, t_j)]\Big][p]\Big] \\
&\quad + N_1\Big[\prod_{j=1}^{m}\Big[\prod_{k=1}^{n}[I + V_k(t_{j-1}, t_j)]\Big][p] \\
&\quad - \prod_{j=1}^{m}\Big[\prod_{k=1}^{\infty}[I + V_k(t_{j-1}, t_j)]\Big][p]\Big] \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.
\end{aligned}
$$

This completes the proof.

The linear case. If R is that subset of H to which A belongs only in case $A[p + q] = A[p] + A[q]$ whenever (p, q) is in $G \times G$, then N_2 and N_3 agree on R, and R is a complete normed ring with identity. As was observed by MacNerney in [**6**], the underlying abelian group can be dispensed with in this case. With this in mind, we state the following theorem without proof.

THEOREM 5. *Let R be a complete normed ring with norm N and identity I. Let $(F_n)^\infty{}_{n=1}$ be a sequence, each value of which is a continuous function from S to R with bounded variation on each compact interval of S. Suppose that $\sum^\infty{}_{k=1}[{}_a\sum^b N[dF_k]]$ is finite whenever (a, b) is in $S \times S$, and let F_0 be a continuous function from S to R such that $F_0(a) - F_0(b) = \sum^\infty{}_{k=1}[F_k(a) - F_k(b)]$ whenever (a, b) is in $S \times S$. Let a be in S, and let h be that continuous function from S to R such that*

$$h(t) = I + \int_t^a (dF_0)h,$$

whenever t is in S. Then

$$h(t) = {}_t\prod^a\Big[\prod_{k=1}^{\infty}[I + dF_k]\Big],$$

whenever t is in S.

An application to analytic function theory. Let A be a complex Banach algebra with norm N and identity I. Let U be the open unit disc in the plane, and let HOL be the set to which f belongs only in case f is an analytic function from U to A. Let INV be that subset of HOL to which f belongs only in case $f(0) = I$ and each value of f has multiplicative inverse in A. It has been shown by Hille [**2**] (see also [**3**, Theorem 6.1.3, p. 212]) that if f is in HOL, then

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there is exactly one member F of HOL such that $F(0) = I$ and $F'(z) = f(z)F(z)$ whenever z is in U. Furthermore [**3**, Theorem 6.1.5, p. 213], F is in INV. On the other hand, if F is in INV and f is given by $f(z) = F'(z)F(z)^{-1}$, then $F(0) = I$ and $F'(z) = f(z)F(z)$ whenever z is in U. Thus it is clear that there is a bijection E from HOL onto INV such that if f is in HOL and F is in INV, then $E[f] = F$ if and only if $F'(z) = f(z)F(z)$ whenever z is in U. In our next theorem, we shall apply Theorem 5 to obtain information about the bijection E. It should be noted that our theorem can also be thought of as a representation theorem for members of INV.

THEOREM 6. *Let* (f, F) *be in* E. *Then each of* (i), (ii), *and* (iii) *is true.*

(i) *If* h *is a sequence into* HOL *such that* $h_0 = f$ *and* $h_{n+1}(z) = h_n'(z) + h_n(z)h_0(z)$ *whenever* n *is a positive integer and* z *is in* U, *then* $F(z) = I + \sum_{k=1}^{\infty} (k!)^{-1}z^k h_{k-1}(0)$ *whenever* z *is in* U.

(ii) $F(z) = {}_z\prod^0 \exp[-(ds)f(s)]$ *whenever* z *is in* U, *where the indicated product integrals are path-independent.*

(iii) $F(z) = {}_z\prod^0 [\prod_{k=0}^{\infty} \exp[-(ds)s^k a_k]]$ *whenever* z *is in* U, *where the indicated product integrals are path-independent, and where* $(a_k)_{k=0}^{\infty}$ *is a sequence into* A *such that* $f(z) = \sum_{k=0}^{\infty} z^k a_k$ *whenever* z *is in* U.

Remark. Considerable work has been done involving product integrals and matrix-valued analytic functions. See, for example, Gantmacher [**1**, Chapter 14, Section 7] and Rasch [**8**]. Although he did not write his conclusion in terms of exponentials, Gantmacher essentially proved (ii), in the matrix case, in [**1**, pp. 138–140].

Proof of Theorem 6. Since $F' = h_0 F$, an easy induction shows that $F^{(n)} = h_{n-1}F$ whenever n is a positive integer, so $F^{(n)}(0) = h_{n-1}(0)$ whenever n is a positive integer, and (i) follows. Let z be in U, and let β be a continuous function of bounded variation from $[0, 1]$ to U such that $\beta(0) = 0$ and $\beta(1) = z$. Now, if $0 \leqq t \leqq 1$,

$$F(\beta(t)) = I - \int_t^0 f(\beta(s))F(\beta(s))d\beta(s).$$

Thus Theorem 1 tells us that

$$F(\beta(t)) = {}_t\prod^0 [I - (d\beta)f[\beta]],$$

whenever $0 \leqq t \leqq 1$. In particular,

$$F(z) = F(\beta(1)) = {}_1\prod^0 [I - (d\beta)f[\beta]].$$

It follows from elementary inequalities (see, for example, [**4**, Lemma 4]) that

$$F(z) = {}_1\prod^0 \exp[-(d\beta)f[\beta]].$$

Since this last equation is true for any continuous function β, of bounded variation, from $[0, 1]$ to U, such that $\beta(0) = 0$ and $\beta(1) = z$, it is clear that we have a path-independent product integral, and (ii) follows. Now (iii) follows from Theorem 5 and from the known result that f does have a power series representation. This completes the proof.

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